Dynamical Systems and Deep Learning-Exercises 1 (solutions)

1. Find the dominant term and the smallest big O complexity of the following expressions as $x \to \infty$:

- $98x \log x - 23x^{1.1}$.
  Solution: $-23x^{1.1}$ is dominant. $O(x^{1.1})$
- $7x^2 - \frac{4x^3}{\log x}$.
  Solution: $-\frac{4x^3}{\log x}$ is dominant. $O(\frac{x^3}{\log x})$.

2. Decide whether each statement below is true or false as $x \to \infty$ and prove your assertion:

- $(x^5 + 12x^4 - 3x + 2)/(3 + x^5) = o(1)$.
  Solution: False. Since

$$\lim_{x\to\infty} (x^5 + 12x^4 - 3x + 2)/(3 + x^5) = 1 \neq 0.$$

- $3x^2 - 4x + 5 \sim x^2$.
  Solution: False. Since $(3x^2 - 4x + 5)/x^2 = 3 - 4/x + 5/x^2 \to 3 \neq 1$ as $x \to \infty$.
- $x^{-1.2} - 5x^{-2} \sim x^{-1.2}$.
  Solution: True. Since $(x^{-1.2} - 5x^{-2})/x^{-1.2} = 1 - 5/x^{0.8} \to 1$ as $x \to \infty$.
- $f_1 = O(g_1)$ and $f_2 = O(g_2) \Rightarrow f_1 f_2 = O(g_1 g_2)$.
  Solution: True. Suppose $M_i > 0$ and $x_i \in \mathbb{R}$ satisfy $|f_i(x)| \leq M_i|g_i(x)|$ for $x > x_i$ where $i = 1, 2$. Then for $M = M_1 M_2$ and $x_0 = \max\{x_1, x_2\}$, we have $|f_1(x) f_2(x)| = |f_1(x)||f_2(x)| \leq M_1|g_1(x)| \times M_2|g_2(x)| = M|g_1(x) g_2(x)|$ for $x > x_0$.

- $f_1 = o(g)$ and $f_2 = o(g) \Rightarrow f_1 + f_2 = o(g)$ and $f_1 f_2 = o(g)$.

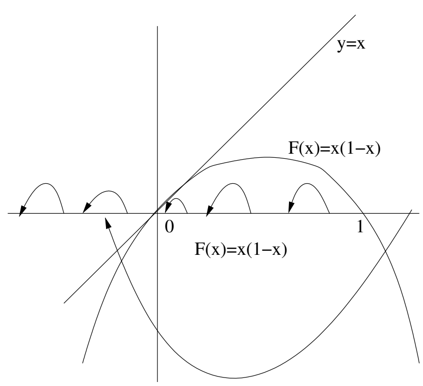
  Solution: The first statement is true since limits add up:

  $$\lim_{x\to\infty}(f_1(x)+f_2(x))/g(x) = \lim_{x\to\infty} f_1(x)/g(x) + \lim_{x\to\infty} f_2(x)/g(x) = 0+0 = 0.$$

  But the second is false, eg, $f_1(x) = f_2(x) = x$ and $g(x) = x^2$.

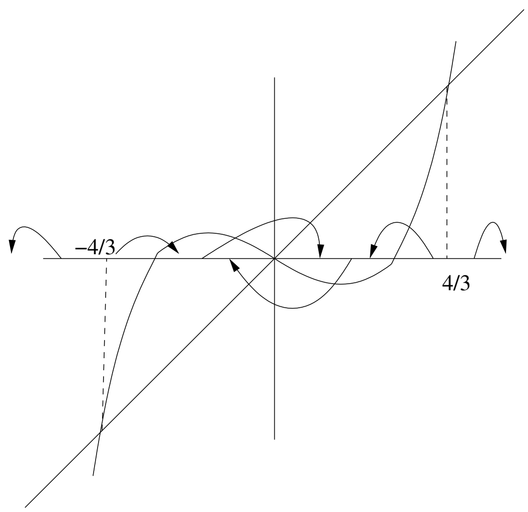
3. Determine the type of the fixed points of the map $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x(1-x)$ and sketch its phase portrait.

Solution: We have $F'(x) = 1 - 2x$. For fixed points we solve $F(x) = x$ i.e., $x - x^2 = x$. There is a unique fixed point $x = 0$ with $F'(0) = 1$, i.e., 0 is non-hyperbolic. It is attracting locally on the right, in fact $[0, 1]$ is its basin of attraction, and repelling on the left.



4. Find all fixed points of $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x^3 - \frac{7}{9}x$ and determine whether they are attracting, repelling or neither. Sketch the phase portrait of the map.

Solution: Note $F'(x) = 3x^2 - 7/9$. And $F(x) = x$ gives three fixed points: $x = 0$ is an attracting fixed point ($|F'(0)| = 7/9 < 1$). There are two repelling fixed points at $\pm 4/3$ (($|F'(4/3)| = |F'(-4/3)| = 41/9 > 1$). Orbits of points in $(-4/3, 4/3)$ converge to 0, those in $(4/3, \infty)$ tend to ∞ and those in $(-\infty, -4/3)$ tend to $-\infty$.



5. Find all fixed points of $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x^3 - 2x$ and determine their nature. Show that F has a period orbit $\{1, -1\}$ of period 2.
(*) What is the type of this periodic orbit?

Solution: There are three fixed points at $x = 0$, $x = \pm\sqrt{3}$ which are all repelling ($|F'(0)| = |-2| = 2 > 1$, $|F'(\pm\sqrt{3})| = 7 > 1$). The pair of points 1 and -1 are mapped to each other, so they form a periodic orbit of period 2, which is non-hyperbolic since by the chain rule: $(F^2)'(1) = F'(F(1))F'(1) = F'(-1)F'(1) = 1 \times 1 = 1$. This periodic orbit is however **weakly** attracting on both sides. This can be seen by graphical analysis near 1 and -1.

Alternatively, put $G = F \circ F$ which will satisfy $G(1) = 1$, $G(-1) = -1$

and $G'(1) = G'(-1) = 1$. By the Taylor series expansion:

$$G(1+\delta) = G(1) + G'(1)\delta + G''(1)\delta^2/2 + G'''(1)\delta^3/(2\times 3) + \cdots$$

Therefore, all we need to do is to obtain the first non-zero higher derivative of G (beyond the first derivative $G'(1)$) at 1: We have $G(x) = (x^3-2x)^3-2(x^3-2x) = x^9-6x^7+12x^5-10x^3+4x$ and thus $G'(x) = 9x^8-42x^6+60x^4-30x^2+4$ with $G'(1) = G'(-1) = 1$ and $G''(x) = 72x^7 - 252x^5 + 240x^3 - 60x$ with $G''(1) = G''(-1) = 0$. Finally, $G'''(x) = 7\times 72x^6 - 5\times 252x^4 + 3\times 240x^2 - 60$ with $G'''(1) = G'''(-1) = -96 < 0$. Therefore, the Taylor series expansion near 1 gives: $G(1+\delta) \approx 1 + \delta - 96\delta^3$ which shows that 1 (and -1) are attracting fixed points of G, and thus the orbit $\{1, -1\}$ is attracting.