Complex Systems- Exercises 2 (solutions)

1. Find all fixed points of $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x^3 - 2x$ and determine their nature. Show that F has a period orbit $\{1, -1\}$ of period 2. What is the type of this periodic orbit?

Solution: There are three fixed points at $x = 0$, $x = \pm\sqrt{3}$ which are all repelling ($|F'(0)| = |-2| = 2 > 1$, $|F'(\pm\sqrt{3})| = 7 > 1$). The pair of points 1 and -1 are mapped to each other, so they form a periodic orbit of period 2, which is non-hyperbolic since by the chain rule: $(F^2)'(1) = F'(F(1))F'(1) = F'(-1)F'(1) = 1 \times 1 = 1$. This periodic orbit is however **weakly** attracting on both sides. This can be seen by graphical analysis near 1 and -1.

Alternative, put $G = F \circ F$ which will satisfy $G(1) = 1$, $G(-1) = -1$ and $G'(1) = G'(-1) = 1$ and obtain the first non-zero higher derivative of G at 1: We have $G(x) = (x^3 - 2x)^3 - 2(x^3 - 2x) = x^9 - 6x^7 + 12x^5 - 10x^3 + 4x$ and thus $G'(x) = 9x^8 - 42x^6 + 60x^4 - 30x^2 + 4$ with $G'(1) = G'(-1) = 1$ and $G''(x) = 72x^7 - 252x^5 + 240x^3 - 60x$ with $G''(1) = G''(-1) = 0$. Finally, $G'''(x) = 7 \times 72x^6 - 5 \times 252x^4 + 3 \times 240x^2 - 60$ with $G'''(1) = G'''(-1) = -96 < 0$. Therefore, the Taylor series expansion near 1 gives: $G(1+\delta) \approx 1 + \delta - 96\delta^3$ which shows that 1 (and -1) are attracting fixed points of G, and thus the orbit $\{1, -1\}$ is attracting.

2. Find the explicit form of

(i) the maps f_1, f_2, f_3 in the generation of the Sierpinky triangle, and

(ii) the maps f_1, f_2, f_3, f_4 in the generation of the Koch curve.

Solution: (i) An easy way to do these two problems is to note that in the complex plane $z = x + iy$ ($i = \sqrt{-1}$),

- scaling by factor $r > 0$ is given by $z \mapsto rz$
- rotation by θ is given by $z \mapsto e^{i\theta}z$ and
- translation by $a = a_1 + ia_2$ is given by $z \mapsto z + a$.

In both cases (i) and (ii), each f_n is a composition of a scaling, a rotation and a translation:

For (i), we get $f_1(z) = z/2$, $f_2(z) = z/2 + 1/2$ and $f_3(z) = z/2 + \frac{1}{4} + \frac{\sqrt{3}i}{4}$. For (ii), we get $f_1(z) = z/3$, $f_2(z) = e^{\pi i/3}z/3 + 1/3$, $f_3(z) = e^{-\pi i/3}z/3 + 1/2 + \sqrt{3}i/6$ and $f_4(z) = z/3 + 2/3$.

3. Find the fixed points of each of the maps f_1, f_2, f_3, f_4 in the generation of the Koch curve. What points of $\{1, 2, 3, 4\}^{\mathbb{N}}$ correspond to these points? Do the same for the Sierpinski triangle and the Cantor set.

Solution: Use the solution above. For example, for the Koch curve, the fixed point of $f_2 : z \mapsto \frac{z}{3}e^{i\frac{\pi}{3}} + \frac{1}{3}$ is obtained by solving $f_2(z) = z$ which has unique solution $z = \frac{1}{3-e^{i\pi/3}}$. The fixed point of f_k ($k = 1, 2, 3, 4$) corresponds to k^{ω}. For the Sierpinski triangle, the three fixed points are the three vertices of the triangle. For the Cantor set, they are 0 and 1.

4. Consider the Cantor set C and its generating sequence $\langle I_n \rangle_{n\geq 0}$. Find $d_H(I_n, C)$ in $\mathcal{P}(\mathbb{R})$.

Solution: The point of I_0 with the supremum distance from C is its midpoint $1/2$; it has equal distance from the points $1/3$ and $2/3$ of C (the ones on either side of the large gap in the middle), namely $1/2 - 1/3 = 1/6 = d_H(I_0, C)$. The points of I_1 with the supremum distance from C are the midpoints of the two parts of I_1; their distance from C is $1/3 \times 1/6 = 1/18$ because of the scaling factor $1/3$. Thus, the general value is $d_H(I_n, C) = \frac{1}{6\times 3^n}$.

5. Find the attractor of the IFS $\{f_1, f_2\}$ in $\mathbb{R}$, where, for $x \in \mathbb{R}$, $f_1(x) = ax$ and $f_2(x) = (1 - a)x + a$, with $0 < a < 1$.

Solution: The fixed points of f_1 and f_2 are respectively at 0 and 1. In fact, $[0, 1]$ is a trapping region for both f_1 and f_2 and a simple check show that $[0, 1]$ itself is the unique fixed point of f, the attractor of the IFS.

6. Repeat question 5 with f_1, f_2 given by $f_1(x) = 0$ and $f_2(x) = \frac{2}{3}x + \frac{1}{3}$.

Solution: The fixed points of f_1 and f_2 are again respectively at 0 and 1. The interval $[0, 1]$ is a trapping region. Iteration of f on $[0, 1]$ gives:

$$f([0,1]) = \{0\} \cup [\tfrac{1}{3}, 1]$$
$$f(\{0\} \cup [\tfrac{1}{3}, 1]) = \{0\} \cup \{\tfrac{1}{3}\} \cup [\tfrac{5}{9}, 1]$$
$$f(\{0, \tfrac{1}{3}\} \cup [\tfrac{5}{9}, 1]) = \{0, \tfrac{1}{3}, \tfrac{5}{9}\} \cup [\tfrac{19}{27}, 1]$$

Note that $0 = 1-1$, $\frac{1}{3} = 1 - \frac{2}{3}$, $\frac{5}{9} = 1 - \frac{4}{9}$, and $\frac{19}{27} = 1 - \frac{8}{27}$. Thus, the general term is

$$f^n([0,1]) = \{1 - (\tfrac{2}{3})^0, \dots, 1 - (\tfrac{2}{3})^{n-1}\} \cup [1 - (\tfrac{2}{3})^n, 1]$$

which can be verified by induction. The intersection of all the sets $f^n([0, 1])$ is

$$\{1 - (\tfrac{2}{3})^n \mid n \geq 0\} \cup \{1\},$$

7. Describe the attractor of the IFS, $f_0, f_1, f_2 : \mathbb{R} \to \mathbb{R}$ with $f_j \mapsto \frac{x}{4} + \frac{3j}{8}$ $(j = 1, 2, 3)$ and find its similarity dimension.

Solution: The attractor is a Cantor set obtained by: (i) start from $[0, 1]$, (ii) remove two symmetrically placed subintervals of length equal to $1/8$ of the original interval, and (ii) repeat the removing scheme for each remaining subinterval for ever. The attractor is strictly self-similar: made up of 3 copies of itself scaled by $1/4$. so its similarity dimension is $\log 3/\log 4$.

8.

(i) Show that if $g : \mathbb{R}^m \to \mathbb{R}^m$ has contracting factor $s < 1$, then the closed ball with centre $u \in \mathbb{R}^m$ and of radius $\|u - g(u)\|/(1 - s)$ is a trapping region for g, i.e., is mapped by g into itself.

(ii) Given an IFS, $f_1, f_2, \dots, f_N : \mathbb{R}^m \to \mathbb{R}^m$ such that f_i has contractivity factor s_i, find a closed ball centred at u which is mapped by f into itself, where $f : \mathcal{P}(\mathbb{R}^m) \to \mathcal{P}(\mathbb{R}^m) : A \mapsto \bigcup_{i=1}^N f_i[A]$.

(iii)* When $m = 2$, $N = 2$, and the maps $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(1 \le i \le 2)$ are both affine, explain how you would find $u \in \mathbb{R}^2$ and $R \ge 0$ such that the closed ball $C(u, R)$ is the smallest trapping disk for f?

Solution:

(i) Suppose $\|x - u\| \le (\|u - g(u)\|)/(1 - s)$. Then by the triangular inequality:

$$|g(x) - u\| \le \|g(x) - g(u)\| + \|g(u) - u\|$$
$$\le s\|x - u\| + \|g(u) - u\| \le \frac{s}{1-s}\|g(u) - u\| + \|g(u) - u\|$$
$$= (\|u - g(u)\|)/(1 - s).$$

(ii) Put $R(u) = \max_{1 \le 0 \le N} R_i(u)$ where $R_i(u) = (\|u - f_i(u)\|)/(1 - s_i)$.

(iii)* First note that **the** contractivity factor s of a map $g : X \to X$ of a metric space is $s = \inf\{t : d(g(x), g(y)) \le td(x, y)\}$. The contractivity factor of an affine map is the contractivity factor of its linear part, which is equal to the ℓ_2 norm of the matrix representing this linear part in the standard coordinates, i.e., the largest singular value of the matrix representing the linear part in the standard coordinates. (Recall that the singular values of a matrix A are the positive square roots of the eigenvalues of AA^T where A^T is the transpose of A.)

Now, suppose for $i = 1, 2$, the map $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ has, in matrix notation in the standard coordinates, the following action:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k_i \\ l_i \end{pmatrix}.$$

We compute the contractivity factors s_i of f_i using the above method. Then for a point $u = (x, y)$, we compute

$$R_i^2(x, y) = (\|u - f_i(u)\|^2)/(1 - s_i)^2 = \frac{(a_i x + b_i y + k_i - x)^2 + (c_i x + d_i y + l_i - y)^2}{(1 - s_i)^2}$$

We need to find $\max\{R_i^2(x, y) : x, y \in \mathbb{R}^2 \text{ and } i = 1, 2\}$ and the values of x and y where this max is attained. This we can do using the Lagrange's multiplier method by finding the minimum value of

$$z(x, y, \lambda) = R_1^2(x, y) + \lambda(R_2^2(x, y) - R_1^2(x, y)),$$

for x, y, λ. Geometrically, $z_i(x,y) = R_i^2(x,y)$ is an elliptic cone with vertex at the fixed point of f_i on the (x,y)-plane for each $i = 1,2$. The two cones z_1 and z_2 intersect at a curve and the minimum value R is obtained where this curve is a minimum.

9*. Show that if $F : [a,b] \to \mathbb{R}$ is differentiable and its derivative F' is continuous at a fixed point x_0 of F, then x_0 is attracting (repelling) if $|F'(x_0)| < 1$ ($|F'(x_0)| > 1$).

Solution: Assume first that $|F'(x_0)| < 1$. The continuity of F' at x_0 implies that there exists some $\delta > 0$ such that $|F'(x)| < k$ for $x_0 - \delta < x < x_0 + \delta$ where $k = (|F'(x_0)| + 1)/2$ (simply put $\epsilon = (1 - |F'(x_0)|)/2$ in the definition of continuity of F' at x_0). Now, by the mean value theorem applied to $[x_0, x]$ (or to $[x, x_0]$) where $|x_0 - x| < \delta$, there exists $x^* \in (x_0, x)$ (or $x^* \in (x, x_0)$) such that $F(x) - F(x_0) = F'(x^*)(x - x_0)$. Thus, since $k < 1$, we obtain:

$$|F(x) - F(x_0)| = |F'(x^*)||x - x_0| < k\delta < \delta.$$

Remembering that $F(x_0) = x_0$, we get:

$$|F(x) - x_0| < k\delta < \delta.$$

Thus, we can recursively replace x with $F(x)$ to obtain for any positive integer n:

$$|F^n(x) - x_0| < k^n\delta < \delta,$$

which shows that $F^n(x) \to x_0$ as $n \to \infty$ for $x \in (x_0 - \delta, x_0 + \delta)$, since $0 < k < 1$.

In case, $|F'(x_0)| > 1$, we obtain for any $x \neq x_0$ with $|x - x_0| < \delta$:

$$|F^n(x) - x_0| > k^n|x - x_0|,$$

if $F^{n-1}(x) \in (x_0 - \delta, x_0 + \delta)$. So $F^n(x) \notin (x_0 - \delta, x_0 + \delta)$ for the least n with $k^n|x - x_0| > \delta$, i.e., for

$$n = \lceil \frac{\log(\delta/|x - x_0|)}{\log k} \rceil.$$