Complex Systems- Exercises 3
Attractors and Chaos

1. Show that the tail map sends open balls to open balls, i.e. show that for any
   \(x \in \Sigma^N\) and any integer \(n \geq 1\), we have:
   \[\sigma[O(x, 1/2^n)] = O(\sigma x, 1/2^{n-1}).\]

2. Check that the tail map is continuous. (Hint: Show that the pre-image of any
   open ball is an open set.)

3. Check that the tail map satisfies the following:
   - Sensitive to initial conditions (with \(\delta = 1/2\)).
   - Topologically transitive (show that \(\forall\) open \(U \neq \emptyset, \exists n. \sigma^n(U) = \Sigma^N\)).
   - Its periodic orbits are dense in \(\Sigma^N\).
   - It has a dense orbit.

4. Suppose \(g : Y \rightarrow Y\) is a dynamical system with a semi-conjugacy:

   \[
   \begin{array}{ccc}
   \Sigma^N & \xrightarrow{\sigma} & \Sigma^N \\
   h & \downarrow & h \\
   Y & \xrightarrow{g} & Y
   \end{array}
   \]

   From such a semi-conjugacy, show that we can deduce the following results
   about \(g\) from the corresponding properties of \(\sigma\):
(i) $g$ is topologically transitive.

(ii) Periodic orbits of $g$ are dense.

(iii) $g$ has a dense orbit.

(*) 5. Consider $Q_d : \mathbb{R} \to \mathbb{R} : x \mapsto x^2 + d$ Show that for $d < 1/4$, the map $Q_d$ is conjugate via a linear map of type $L : x \mapsto \alpha x + \beta$ to $F_c : \mathbb{R} \to \mathbb{R} : x \mapsto cx(1-x)$ for a unique $c > 1$.

Hint: Find $\alpha, \beta, c$ in terms of $d$ such that:

$$F_c \circ L = L \circ Q_d$$