Complex Systems- Exercises 3
Attractors and Chaos

1. Show that the tail map sends open balls to open balls, i.e. show that for any $x \in \Sigma^{\mathbb{N}}$ and any integer $n \geq 1$, we have:

$$\sigma[O(x, 1/2^n)] = O(\sigma x, 1/2^{n-1}).$$

2. Check that the tail map is continuous. (Hint: Show that the pre-image of any open ball is an open set.)

3. Check that the tail map satisfies the following:

- Sensitive to initial conditions (with $\delta = 1/2$).
- Topologically transitive (show that $\forall$ open $U \neq \emptyset$. $\exists n$. $\sigma^n(U) = \Sigma^{\mathbb{N}}$).
- Its periodic orbits are dense in $\Sigma^{\mathbb{N}}$.
- It has a dense orbit.

4. Suppose $g : Y \to Y$ is a dynamical system with a semi-conjugacy:

$$\begin{array}{ccc} \Sigma^{\mathbb{N}} & \xrightarrow{\sigma} & \Sigma^{\mathbb{N}} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Y \end{array}$$

From such a semi-conjugacy, show that we can deduce the following results about g from the corresponding properties of σ:

(i) g is topologically transitive.

(ii) Periodic orbits of g are dense.

(iii) g has a dense orbit.

(*) 5. Consider $Q_d : \mathbb{R} \to \mathbb{R} : x \mapsto x^2 + d$ Show that for $d < 1/4$, the map Q_d is conjugate via a linear map of type $L : x \mapsto \alpha x + \beta$ to $F_c : \mathbb{R} \to \mathbb{R} : x \mapsto cx(1-x)$ for a unique $c > 1$.

Hint: Find α, β, c in terms of d such that:

$$F_c \circ L = L \circ Q_d$$