1. Suppose $P \in \mathbb{R}^{n \times n}$ is a stochastic matrix.

   (i) Show that the 2-step transition matrix $P^{(2)} = P \circ P = P^2$ is a stochastic matrix.

   (ii) By using induction, show that $P^n$ is a stochastic matrix for any positive integer $n$.

2. Find the communicating classes of the stochastic matrix

   \[
   P = \begin{pmatrix}
   1/2 & 1/2 & 0 & 0 \\
   1/2 & 1/2 & 0 & 0 \\
   1/3 & 1/6 & 1/6 & 1/3 \\
   0 & 0 & 0 & 1
   \end{pmatrix}
   \]

   on the set of states \{1, 2, 3, 4\} and decide if $P$ is irreducible or not.

3. Suppose $0 < p, q < 1$ and consider

   \[
   P = \begin{pmatrix}
   p & 1 - p \\
   1 - q & q
   \end{pmatrix}
   \]

   • Check that $P$ has an eigenvalue 1 and an eigenvalue $\lambda$ with $|\lambda| < 1$. Determine the stationary distribution $\pi$ of $P$.

   • By taking the two left eigenvectors of $P$ as the new basis of $\mathbb{R}^2$, show that given any initial probability vector $p$, we have $\lim_{n \to \infty} p P^n = \pi$.

4. Show that $\pi P = \pi \iff \pi(aI + (1 - a) P) = \pi$, for $0 < a < 1$, where $I \in \mathbb{R}^{n \times n}$ is the identity matrix.

5. Show that if $\pi$ satisfies the detailed balanced condition for a stochastic matrix $P$, then it is a stationary distribution.
6. Rewrite the stochastic updating rule for the stochastic Hopfield network to obtain the probability of flipping:

\[
\Pr(x_i \rightarrow -x_i) = \frac{1}{1 + \exp(\Delta E/T)},
\]

where \(\Delta E = E' - E\) is the change in energy.

7. Show that, with respect to the transition matrix for flipping nodes in a stochastic Hopfield network, the distribution

\[
\Pr(x) = \frac{\exp(-E(x)/T)}{Z},
\]

satisfies the detailed balanced condition.

8. Suppose we have a stochastic Hopfield network with \(N\) nodes and \(q\) is the uniform distribution on the nodes, i.e., \(q(i) = 1/N\) for \(1 \leq i \leq N\). Check that the following probabilistic transition rule is an example of Gibbs sampling:

- At each point in time, select a node \(i\) with probability \(q(i)\);
- flip the value \(x_i\) of \(i\) with probability:

\[
\Pr(x_i \rightarrow -x_i) = \frac{1}{1 + \exp(\Delta E/T)},
\]

where \(\Delta E = E' - E\) is the change in energy.