# Dynamical Systems and Deep Learning - Exercises 3

1. 1. Suppose $P \in \mathbb{R}^{n\times n}$ is a stochastic matrix.

   (i) Show that the 2-step transition matrix $P^{(2)} = P \circ P = P^2$ is a stochastic matrix.

   (ii) By using induction, show that $P^n$ is a stochastic matrix for any positive integer $n$.

2. Find the communicating classes of the stochastic matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/3 & 1/6 & 1/6 & 1/3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

on the set of states $\{1, 2, 3, 4\}$ and decide if $P$ is irreducible or not.

3. Suppose $0 < p, q < 1$ and consider

$$P = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix} \tag{2}$$

- Check that $P$ has an eigenvalue 1 and an eigenvalue $\lambda$ with $|\lambda| < 1$. Determine the stationary distribution $\pi$ of $P$.
- By taking the two left eigenvectors of $P$ as the new basis of $\mathbb{R}^2$, show that given any initial probability vector $p$ we have $\lim_{n\to\infty} pP^n = \pi$.

4. Show that $\pi P = \pi \iff \pi(aI + (1-a)P) = \pi$, for $0 < a < 1$, where $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

5. Show that if $\pi$ satisfies the detailed balanced condition for a stochastic matrix $P$, then it is a stationary distribution.

6. Rewrite the stochastic updating rule for the stochastic Hopfield network to obtain the probability of flipping:

$$\text{Pr}(x_i \to -x_i) = \frac{1}{1 + \exp(\Delta E/T)}, \tag{3}$$

where $\Delta E = E' - E$ is the change in energy.

7. Show that, with respect to the transition matrix for flipping nodes in a stochastic Hopfield network, the distribution

$$\text{Pr}(x) = \frac{\exp(-E(x)/T)}{Z}, \tag{4}$$

satisfies the detailed balanced condition.

8. Suppose we have a stochastic Hopfield network with $N$ nodes and $q$ is the uniform distribution on the nodes, i.e., $q(i) = 1/N$ for $1 \leq i \leq N$. Check that the following probabilistic transition rule is an example of Gibbs sampling:

- At each point in time, select a node $i$ with probability $q(i)$;
- flip the value $x_i$ of $i$ with probability:

$$\text{Pr}(x_i \to -x_i) = \frac{1}{1 + \exp(\Delta E/T)},$$

  where $\Delta E = E' - E$ is the change in energy.