Complex Systems- Exercises 3 (solutions)

1. 1. Suppose $P \in \mathbb{R}^{n \times n}$ is a stochastic matrix.

   (i) Show that the 2-step transition matrix $P^{(2)} = P \circ P = P^2$ is a stochastic matrix.

   (ii) By using induction, show that $P^n$ is a stochastic matrix for any positive integer $n$.

Solution: (i) We need to check that all entries of $P^2$ are non-negative and the sum of all entries in every row is 1. Since all entries of $P$ are non-negative we have $(P^2)_{ij} = \sum_{k=1}^{n} P_{ik} P_{kj} \geq 0$. Also for every row with index $i$:

   $$\sum_{j=1}^{n} (P^2)_{ij} = \sum_{j=1}^{n} \sum_{k=1}^{n} P_{ik} P_{kj} = \sum_{k=1}^{n} (\sum_{j=1}^{n} P_{ik} P_{kj})$$

   $$\sum_{k=1}^{n} (P_{ik} \sum_{j=1}^{n} P_{kj}) = \sum_{k=1}^{n} (P_{ik} \cdot 1) = \sum_{k=1}^{n} P_{ik} = 1.$$

(ii) We need to check that if $P^k$ for a positive integer $k$ is stochastic, so is $P^{k+1}$. Let $Q = P^k$. Then $P^{k+1} = Q \circ P$ and the proof follows exactly as in (i) with the first instance of $P$ in $P^2 = P \circ P$ replaced with $Q$.

2. Find the communicating classes of the stochastic matrix

   $$P = \begin{pmatrix}
   1/2 & 1/2 & 0 & 0 \\
   1/2 & 1/2 & 0 & 0 \\
   1/3 & 1/6 & 1/6 & 1/3 \\
   0 & 0 & 0 & 1
   \end{pmatrix} \tag{1}$$

   on the set of states $\{1, 2, 3, 4\}$ and decide if $P$ is irreducible or not.

Solution: The communicating classes are the following sets:

- $\{1, 2\}$, since 1 and 2 communicate with each other but no other state can be accessed from them,
• \{3\}, since 3 cannot be accessed from any other state, and,
• \{4\}, since no other state can be accessed from 4.

Thus, \(P\) is not irreducible since it has more than one communicating class.

3. Suppose \(0 < p, q < 1\) and consider

\[
P = \begin{pmatrix} p & 1 - p \\ 1 - q & q \end{pmatrix}
\]

- Check that \(P\) has an eigenvalue 1 and an eigenvalue \(\lambda\) with \(|\lambda| < 1\). Determine the stationary distribution \(\pi\) of \(P\).
- By taking the two left eigenvectors of \(P\) as the new basis of \(\mathbb{R}^2\), show that given any initial probability vector \(p\) we have \(\lim_{n \to \infty} pP^n = \pi\).

**Solution:** (i) The solution of \(\det(P - \lambda I) = 0\) leads to \(\lambda^2 - \lambda(p + q) + p + q - 1 = 0\). Clearly 1 is an eigenvalue corresponding to left eigenvector \(\pi\) with \(\pi P = \pi\), i.e., \(P^T \pi^T = \pi^T\). A simple calculation shows that

\[
\pi = \left(\frac{1 - q}{2 - p - q}, \frac{1 - p}{2 - p - q}\right)
\]

which is the stationary distribution. The other eigenvalue is \(\lambda = p + q - 1\) since the product of the two eigenvalues is the constant term in the discriminant. By summing the two inequalities \(0 < p < 1\) and \(0 < q < 1\) and subtracting 1 from all sides we get \(-1 < \lambda < 1\).

(ii) Let \(x = (1, -1)\) be the eigenvector corresponding to \(\lambda = p + q - 1\). Then we can write \(p = a\pi + bx\) where \(a, b\) are real numbers. Then we have

\[
pP^n = (a\pi + bx)P^n = a\pi P^n + bx P^n = a\pi + b\lambda^n x
\]

as \(n \to \infty\), we have \(\lambda^n \to 0\). Thus we get

\[
\lim_{n \to \infty} pP^n = a\pi
\]

Since \(pP^n\) is a probability vector for all \(n \in \mathbb{N}\), it follows that \(a = 1\) as required. Alternatively, we can argue as follows. Since \(\pi\) and \(p\) are both probability vectors,
the sum of the two components of \(p - \pi\) is zero and thus \(p - \pi = c(1, -1) = cx\) for some real number \(c\). Therefore,

\[
p^n - \pi = (p - \pi) P^n = c x P^n = c \lambda^n x \to 0,
\]
as \(n \to \infty\).

4. Show that \(\pi P = \pi \iff \pi(aI + (1-a)P) = \pi\), for \(0 < a < 1\), where \(I \in \mathbb{R}^{n \times n}\) is the identity matrix.

**Solution:** Let \(\pi P = \pi\). Then,

\[
\pi(aI + (1-a)P) = a\pi + (1-a)\pi P = a\pi + (1-a)\pi = \pi.
\]

On the other hand, if \(\pi(aI + (1-a)P) = \pi\), then rewriting the equation we obtain:

\[
a\pi + (1-a)\pi P = \pi \implies (1-a)\pi = (1-a)\pi P,
\]

from which the result follows after dividing by \(1-a > 0\).

5. Show that if \(\pi\) satisfies the detailed balanced condition for a stochastic matrix \(P\), then it is a stationary distribution.

**Solution:** Suppose

\[
\pi_i P_{ij} = \pi_j P_{ji}, \text{ for } 1 \leq i, j \leq N
\]

Then

\[
(\pi P)_i = \sum_{j=1}^{n} \pi_j P_{ji} = \sum_{j=1}^{n} \pi_i P_{ij} = \pi_i \sum_{j=1}^{n} P_{ij} = \pi_i
\]

6. Rewrite the stochastic updating rule for the stochastic Hopfield network to obtain the probability of flipping:

\[
Pr(x_i \to -x_i) = \frac{1}{1 + \exp(\Delta E/T)}, \tag{3}
\]

where \(\Delta E = E' - E\) is the change in energy.

**Solution:** We have:

\[
Pr(x_i) = \frac{1}{1 + \exp(-2h_i x_i/T)}
\]
Note also from Exercise 2(ii) in sheet 2 that when \( x_i \to -x_i \), we have:
\[
\Delta E = E' - E = 2h_ix_i
\]
Thus, when we have \( x_i \to -x_i \)
\[
\Pr(x_i \to -x_i) = \Pr(-x_i|x_i) = \Pr(-x_i|\Delta E = 2h_ix_i) = \frac{1}{1 + \exp(2h_ix_i/T)} = \frac{1}{1 + \exp(\Delta E/T)}
\]
7. Show that, with respect to the transition matrix for flipping nodes in a stochastic Hopfield network, the distribution
\[
\pi(x) = \Pr(x) = \frac{\exp(-E(x)/T)}{Z},
\]
(4)
satisfies the detailed balanced condition.

**Solution:** Assume \( E(x_i) \) and \( E(-x_i) \) denote the energies of the network when node \( i \) has values \( x_i \) and \( -x_i \) respectively while all other nodes keep their values unchanged. Then, using the result of Exercise 5, we have:
\[
\Pr(x_i)\Pr(x_i \to -x_i) = \frac{e^{-E(x_i)/T}}{Z} \frac{1}{1 + e^{(E(-x_i)-E(x_i))/T}}
\]
\[
= \frac{1}{Z} \cdot \frac{1}{e^{E(x_i)/T} + e^{E(-x_i)/T}} = \frac{e^{-E(-x_i)/T}}{Z} \frac{1}{1 + e^{(E(x_i)-E(-x_i))/T}} = \Pr(-x_i)\Pr(-x_i \to x_i).
\]
8. Suppose we have a stochastic Hopfield network with \( N \) nodes and \( q \) is the uniform distribution on the nodes, i.e., \( q(i) = 1/N \) for \( 1 \leq i \leq N \). Check that the following probabilistic transition rule is an example of Gibbs sampling:
- At each point in time, select a node \( i \) with probability \( q(i) \);
- flip the value \( x_i \) of \( i \) with probability:
\[
\Pr(x_i \to -x_i) = \frac{1}{1 + \exp(\Delta E/T)},
\]
where \( \Delta E = E' - E \) is the change in energy.

**Solution:** By Exercise 7, we know that \( \pi(x) = \frac{\exp(-E(x)/T)}{Z} \) is the stationary distribution of the stochastic network and the conditional probability distribution for flipping a node is as given in the present problem. Therefore, by the definition of Gibbs sampling we indeed have an example of it here.