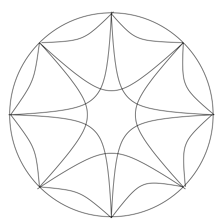
Complex Systems- Exercises 4 (solutions)

1. Show that the clustering coefficient for a one dimensional lattice with periodic boundary condition (i.e., a circle), as for example in the figure below, can be computed to be

$$C = \frac{3(z-2)}{4(z-1)}$$

which tends to $3/4$ as $z \to \infty$. (Here, $z \ll N$ and z is assumed to be even so that every vertex has $z/2$ connections with its neighbours on one side and $z/2$ connections on the other side.)



Solution: Let $z = 2k$. Consider a vertex A placed say at the origin 0. It is connected with its neighbours at $-k, -k+1, \ldots, -2, -1, 1, 2, \ldots, k-1, k$. There are a total of $2k(2k-1)/2 = k(2k-1)$ pairs of such neighbours. We now count the connections between these neighbours starting from $-k$ on the left and moving one vertex at a time to the right. Each vertex $-k, -(k-1), \ldots, -2, -1, 1$ (i.e., a total of $k+1$ vertices) is connected to $k-1$ neighbours of A (remember that in this count A itself is excluded). This gives $(k+1)(k-1)$ edges. Vertices starting from 2 and moving to rightward to k will have, in addition to those already counted, respectively, $k-2, k-3, \ldots, 1, 0$ other connections, i.e,

$$\sum_{n=0}^{k-2} n = (k-2)(k-1)/2.$$

Therefore there are a total of

$$(k+1)(k-1)+(k-2)(k-1)/2 = (k-1)(2k+2+k-2)/2 = 3k(k-1)/2$$

connections between A's neighbours.

Thus, the clustering coefficient is $C = \frac{3k(k-1)/2}{k(2k-1)} = \frac{3(k-1)}{2(2k-1)} = \frac{3(z-2)}{4(z-1)}$.

2. Find the average distance in a (non-periodic) one dimensional lattice of length ℓ with $z = 2$ and obtain its asymptotic behaviour as $\ell \to \infty$.

Solution: Let the lattice be represented by the points $0, 1, 2, \ldots, \ell-1, \ell$ on the real line. The number of unordered pairs of points is $N = \ell(\ell+1)/2$. Considering these pairs in their order from left to right as

$$(0,1), (0,2), \ldots, (0,\ell), (1,2), (1,3), \ldots, (1,\ell), (2,3), (2,4), \ldots, (2,\ell),$$

$$(3,4), \ldots, (3,\ell), \ldots, \ldots, (\ell-1,\ell).$$

The length of the edges with these pairs of vertices in the above order is

$$1, 2, \ldots, \ell, 1, 2, \ldots, \ell-1, 1, 2, \ldots, \ell-2, \ldots, \ldots, 1.$$

Therefore, the sum of the lengths of these edges is

$$S = \sum_{n=1}^{\ell} \frac{n(n+1)}{2} = \frac{1}{2}(\sum_{n=1}^{\ell} n + n^2) =$$

$$\frac{1}{2}(\frac{\ell(\ell+1)}{2} + \frac{\ell(\ell+1)(2\ell+1)}{6}) = \frac{\ell(\ell+1)(\ell+2)}{6}.$$

The average length is thus

$$\frac{S}{N} = (\ell(\ell+1)(\ell+2)/6)/(\ell(\ell+1)/2) = (\ell+2)/3 \sim \ell/3,$$

as $\ell \to \infty$.

3. We can equivalently define a random graph by its size N and its total number of edges n.

(i) What is the total number of possible graphs with this specification?

(ii) Find z and p (as defined in the notes) in terms of N and n.

(iii) Starting with the definition of a random network as in the notes, find the expected value $\langle n \rangle$ of the number of edges n.

Solution: (i) There are $N(N-1)/2$ possible edges if we have N vertices, so the answer is:

$$\binom{N(N-1)/2}{n} = \frac{M!}{n!(M-n)!},$$

where $M = N(N-1)/2$.

(ii) Since there are $2n$ end-points for n edges, we have: $z = 2n/N$. On the other hand, there are $N(N-1)/2$ pairs of distinct vertices, so $p = n/(N(N-1)/2) = 2n/(N(N-1))$.

(iii) The probability of an edge between two vertices is $p = z/(N-1)$. Thus, $\langle n \rangle = p(N(N-1)/2) = Nz/2$.

4. Find the expected value and the second moment of the degree of vertices

$$\langle k \rangle = \sum_{k=1}^{\infty} kP(k) = \sum_{k=1}^{\infty} k2^{-k},$$

$$\langle k^2 \rangle = \sum_{k=1}^{\infty} k^2 P(k) = \sum_{k=1}^{\infty} k^2 2^{-k},$$

for the random growing network, where $P(k) = 2^{-k}$. Hence, find z_2/z_1 and discuss percolation transition for this network.

Hint: Evaluate $\langle k \rangle = 2\langle k \rangle - \langle k \rangle$ and $\langle k^2 \rangle = 2\langle k^2 \rangle - \langle k^2 \rangle$.

Solution:

$$\langle k \rangle = 2\langle k \rangle - \langle k \rangle = \sum_{k=1}^{\infty} k2^{-(k-1)} - \sum_{k=1}^{\infty} k2^{-k}$$

$$= \sum_{k=0}^{\infty} (k+1)2^{-k} - \sum_{k=1}^{\infty} k2^{-k} = \sum_{k=0}^{\infty} 2^{-k} = 2.$$

$$\langle k^2 \rangle = 2\langle k^2 \rangle - \langle k^2 \rangle = \sum_{k=1}^{\infty} k^2 2^{-(k-1)} - \sum_{k=1}^{\infty} k^2 2^{-k}$$

$$= \sum_{k=0}^{\infty} (k+1)^2 2^{-k} - \sum_{k=1}^{\infty} k^2 2^{-k} = \sum_{k=0}^{\infty} (2k+1) 2^{-k} = 2 \sum_{k=1}^{\infty} k 2^{-k} + \sum_{k=0}^{\infty} 2^{-k}$$

$$= 2\langle k \rangle + 2 = 2 \times 2 + 2 = 6.$$

Thus,

$$z_2/z_1 = \frac{\langle k^2 \rangle}{\langle k \rangle} - 1 = \frac{6}{2} - 1 = 2 > 1.$$

It follows, as we had clearly expected, that we are above the percolation threshold and thus there will be a giant cluster.