Complex Systems- Exercises 4 (solutions)

1. Show that the clustering coefficient for a one dimensional lattice with periodic boundary condition (i.e., a circle), as for example in the figure below, can be computed to be

\[ C = \frac{3(z-2)}{4(z-1)} \]

which tends to \(3/4\) as \(z \to \infty\). (Here, \(z \ll N\) and \(z\) is assumed to be even so that every vertex has \(z/2\) connections with its neighbours on one side and \(z/2\) connections on the other side.)

Solution: Let \(z = 2k\). Consider a vertex \(A\) placed say at the origin \(0\). It is connected with its neighbours at \(-k, -k + 1, \ldots, -2, -1, 1, 2, \ldots, k - 1, k\). There are a total of \(2k(2k - 1)/2 = k(2k - 1)\) pairs of such neighbours. We now count the connections between these neighbours starting from \(-k\) on the left and moving one vertex at a time to the right. Each vertex \(-k, -(k - 1), \ldots, -2, -1, 1\) (i.e., a total of \(k + 1\) vertices) is connected to \(k - 1\) neighbours of \(A\) (remember that in this count \(A\) itself is excluded). This gives \((k + 1)(k - 1)\) edges. Vertices starting from 2 and moving to rightward to \(k\) will have, in addition to those already counted, respectively, \(k - 2, k - 3, \ldots, 1, 0\) other connections, i.e,

\[ \sum_{n=0}^{k-2} n = (k - 2)(k - 1)/2. \]
Therefore there are a total of 
\[(k + 1)(k - 1) + (k - 2)(k - 1)/2 = (k - 1)(2k + 2 + k - 2)/2 = 3k(k - 1)/2\]
connections between A's neighbours.

Thus, the clustering coefficient is
\[C = \frac{3k(k-1)/2}{k(2k-1)} = \frac{3(k-1)}{2(2k-1)} = \frac{3(z-2)}{4(z-1)}.\]

2. Find the average distance in a (non-periodic) one dimensional lattice of length \(\ell\) with \(z = 2\) and obtain its asymptotic behaviour as \(\ell \to \infty\).

**Solution:** Let the lattice be represented by the points 0, 1, 2, \ldots, \(\ell - 1\), \(\ell\) on the real line. The number of unordered pairs of points is \(N = \ell(\ell+1)/2\). Considering these pairs in their order from left to right as

\[(0, 1), (0, 2), \ldots, (0, \ell), (1, 2), (1, 3), \ldots, (1, \ell), (2, 3), (2, 4), \ldots, (2, \ell),\]
\[(3, 4), \ldots, (3, \ell), \ldots, \ldots, (\ell - 1, \ell).\]

The length of the edges with these pairs of vertices in the above order is

\[1, 2, \ldots, \ell, 1, 2, \ldots, \ell - 1, 1, 2, \ldots, \ell - 2, \ldots, \ldots, 1.\]

Therefore, the sum of the lengths of these edges is

\[S = \sum_{n=1}^{\ell} \frac{n(n + 1)}{2} = \frac{1}{2} \left( \sum_{n=1}^{\ell} n + n^2 \right) = \frac{1}{2} \left( \frac{\ell(\ell + 1)}{2} + \frac{\ell(\ell + 1)(2\ell + 1)}{6} \right) = \frac{\ell(\ell + 1)(\ell + 2)}{6}.\]

The average length is thus

\[\frac{S}{N} = \frac{(\ell(\ell + 1)(\ell + 2)/6) / (\ell(\ell + 1)/2)}{(\ell(\ell + 1)/2)} = (\ell + 2)/3 \sim \ell/3,\]
as \(\ell \to \infty\).

3. We can equivalently define a random graph by its size \(N\) and its total number of edges \(n\).

(i) What is the total number of possible graphs with this specification?
(ii) Find \( z \) and \( p \) (as defined in the notes) in terms of \( N \) and \( n \).

(iii) Starting with the definition of a random network as in the notes, find the expected value \( \langle n \rangle \) of the number of edges \( n \).

**Solution:** (i) There are \( \frac{N(N - 1)}{2} \) possible edges if we have \( N \) vertices, so the answer is:

\[
\binom{N(N - 1)/2}{n} = \frac{M!}{n!(M - n)!},
\]

where \( M = \frac{N(N - 1)}{2} \).

(ii) Since there are \( 2n \) end-points for \( n \) edges, we have: \( z = \frac{2n}{N} \). On the other hand, there are \( \frac{N(N - 1)}{2} \) pairs of distinct vertices, so \( p = \frac{n}{\frac{N(N - 1)}{2}} = \frac{2n}{N(N - 1)} \).

(iii) The probability of an edge between two vertices is \( p = \frac{z}{(N - 1)} \). Thus, \( \langle n \rangle = p\langle N(N - 1)/2 \rangle = Nz/2 \).

4. Find the expected value and the second moment of the degree of vertices

\[
\langle k \rangle = \sum_{k=1}^{\infty} kP(k) = \sum_{k=1}^{\infty} k2^{-k},
\]

\[
\langle k^2 \rangle = \sum_{k=1}^{\infty} k^2 P(k) = \sum_{k=1}^{\infty} k^2 2^{-k},
\]

for the random growing network, where \( P(k) = 2^{-k} \). Hence, find \( \frac{z_2}{z_1} \) and discuss percolation transition for this network.

**Hint:** Evaluate \( \langle k \rangle = 2\langle k \rangle - \langle k \rangle \) and \( \langle k^2 \rangle = 2\langle k^2 \rangle - \langle k^2 \rangle \).

**Solution:**

\[
\langle k \rangle = 2\langle k \rangle - \langle k \rangle = \sum_{k=1}^{\infty} k2^{-(k-1)} - \sum_{k=1}^{\infty} k2^{-k}
\]

\[
= \sum_{k=0}^{\infty} (k + 1)2^{-k} - \sum_{k=1}^{\infty} k2^{-k} = \sum_{k=0}^{\infty} 2^{-k} = 2.
\]
\[ \langle k^2 \rangle = 2\langle k^2 \rangle - \langle k^2 \rangle = \sum_{k=1}^{\infty} k^2 2^{-(k-1)} - \sum_{k=1}^{\infty} k^2 2^{-k} \]

\[ = \sum_{k=0}^{\infty} (k + 1)^2 2^{-k} - \sum_{k=1}^{\infty} k^2 2^{-k} = \sum_{k=0}^{\infty} (2k + 1)2^{-k} = 2 \sum_{k=1}^{\infty} k2^{-k} + \sum_{k=0}^{\infty} 2^{-k} \]

\[ = 2\langle k \rangle + 2 = 2 \times 2 + 2 = 6. \]

Thus,

\[ \frac{z_2}{z_1} = \frac{\langle k^2 \rangle}{\langle k \rangle} - 1 = \frac{6}{2} - 1 = 2 > 1. \]

It follows, as we had clearly expected, that we are above the percolation threshold and thus there will be a giant cluster.