Complex Systems- Exercises 5

1. Consider a network with some arbitrary degree distribution $\{k_i\}$ for $i = 1, \ldots, N$.

(i) Show that for a given node i, the clustering coefficient C_i (i.e., the fraction of neighbours of i that are connected to each other) is given by:

$$C_i = \frac{2E_i}{k_i(k_i - 1)},$$

where E_i is the total number of edges that exist between the k_i neighbours of i.

(ii) Using (i) above, check the average clustering coefficient for a random graph.

Solution: (i) The number of pairs of neighbouring vertices is $k_i(k_i - 1)/2$. Thus, the fraction of the connected neighbours is

$$C_i = \frac{E_i}{k_i(k_i - 1)/2} = \frac{2E_i}{k_i(k_i - 1)}.$$

(ii) For a random graph, we have $E_i = \frac{z(z-1)}{2}p$ and $k_i = z$. Thus,

$$C = \frac{2E_i}{k_i(k_i - 1)} = \frac{2\frac{z(z-1)}{2}p}{z(z-1)} = p.$$

2. Show that the average clustering coefficient C of a network with a given degree distribution is given by:

$$C = \frac{1}{N\langle k \rangle}\left(\frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle}\right)^2.$$

Solution In fact, C is the probability that two neighbours of the two neighbouring vertices i and j are connected. But the average number of neighbours of each of these two vertices away from the other is

$$K = \left(\frac{\langle k^2 \rangle}{\langle k \rangle} - 1 \right).$$

Thus, the average total number of unordered pairs of stubs consisting of neighbours of i and j away from each other is $K^2/2$. Since the total number of edges is $N\langle k \rangle/2$, we obtain by division:

$$C = \frac{1}{N\langle k \rangle} \left(\frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} \right)^2.$$