Complex Systems- Exercises 6 (solutions)

1. Assume we have three boolean networks all with $N = 3$, $K = 1$ and a cyclic linkage tree

$$f_1 : \sigma_2 \mapsto \sigma_1, \quad f_2 : \sigma_3 \mapsto \sigma_2, \quad f_3 : \sigma_1 \mapsto \sigma_3.$$ 

The coupling functions $f_i$ ($i = 1, 2, 3$) for the three networks, labelled (i), (ii) and (iii), are given as follows:

(i) $f_1 = f_2 = f_3 = \text{identity}.$

(ii) $f_1 = f_2 = f_3 = \text{negation}.$

(iii) $f_1 = f_2 = \text{negation}$ and $f_3 = \text{identity}.$

For each of the three networks, determine all cycles and their attraction basins.

**Solution.** The cycles for the three networks are given below. In each case the basin of attraction coincides with the cycle.
Consider the network with four boolean variables $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ ($N = 4$) below. The boolean variable $\sigma_1$ has one controlling variable, namely $\sigma_2$, i.e., $K_1 = 1$, while $\sigma_2$ has three controlling variables, namely $\sigma_1, \sigma_2, \sigma_3$, i.e., $K_2 = 3$. For others, we have $K_3 = 1$ and $K_4 = 2$.

Assume that the four boolean functions $f_i : \{0, 1\}^{K_i} \rightarrow \{0, 1\}$ with $1 \leq i \leq 4$ are all generalised XOR-functions, i.e., their output is 1, respectively 0, if the number of input 1’s is odd, respectively even. (i) Find all cycles of the network. (ii) Let $\Sigma_0 = 0001$ and $\hat{\Sigma}_0 = 1001$. Find the Hamming distance $h(\Sigma_t, \hat{\Sigma}_t)$ for all $t \geq 0$.

Solution. (i) There are four cycles as in the diagram below.

(ii) $h(\Sigma_t, \hat{\Sigma}_t) = \begin{cases} 
1 & t = 0 \pmod{5} \\
2 & t = 1 \pmod{5} \\
4 & t = 2 \pmod{5} \\
3 & t = 3 \pmod{5} \\
2 & t = 4 \pmod{5}
\end{cases}$