Quantum Computing - Exercises 1

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1. Show that for complex numbers x and y we have:
$|x+y|^2 = |x|^2 + |y|^2 + 2\mathrm{Re}(x^*y)$.

2. Generalize the following notions and properties given for the vector space $\mathbb{R}^2$ in the notes to $\mathbb{R}^d$.

- Define linear independence and linear dependence of vectors in $\mathbb{R}^d$.
- What is the least integer n such that any set of n vectors in $\mathbb{R}^d$ will be linearly dependent?
- What is a basis of $\mathbb{R}^d$? How many linearly independent vectors it takes to get a basis for $\mathbb{R}^d$?
- Define the notion of an *orthonormal* basis for $\mathbb{R}^d$. What would be the *standard* basis of $\mathbb{R}^d$?

3. Generalize the following notions and properties given for the vector space $\mathbb{C}^2$ in the notes to $\mathbb{C}^d$.

- Define the norm $\|w\|$ of a vector and the innerproduct of two vectors w_1 and w_2 in $\mathbb{C}^d$. What is the dual of a vector w in $\mathbb{C}^d$ and what can it be identified with?
- Define linear independence and linear dependence of vectors in $\mathbb{C}^d$.
- What is the least integer n such that any set of n vectors in $\mathbb{C}^d$ will be linearly dependent?
- What is a basis of $\mathbb{C}^d$? How many linearly independent vectors it takes to get a basis for $\mathbb{C}^d$?
- Define the notion of an *orthonormal* basis for $\mathbb{C}^d$. What would be the *standard* basis of $\mathbb{C}^d$?

4. Show that the two vectors w_1 and w_2 in $\mathbb{C}^4$ with $w_1^T = \frac{1}{2}(1, 1, 1, 1)$ and $w_2^T = \frac{1}{2}(1, -1, 1, -1)$ are orthogonal unit vectors. Find vectors w_3 and w_4 such that the collection $\{w_1, w_2, w_3, w_4\}$ forms an orthonormal basis for $\mathbb{C}^4$.

5. Show that any qubit can be expressed as

$$|\psi\rangle = e^{i\gamma}\left(\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right),$$

for real numbers γ, θ and ϕ. The phase factor $e^{i\gamma}$ has no observational effect and can be dropped.

6. Show that there is a one to one correspondence between qubits

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$

and the points on the unit sphere in $\mathbb{R}^3$, called the *Bloch sphere*, with θ and ϕ as the spherical coordinates of a point of the sphere; see the figure below.