Domain theory and Continuous Data Types

Exercises 1

1. Prove that any *finite* poset with a least element is a cpo.

2. Prove that for any set A, $(\mathcal{P}(A), \subseteq)$, the set of all subsets of A ordered by inclusion, is a cpo.

3. Prove that for any set Σ, $(\mathsf{Str}(\Sigma), \sqsubseteq)$, where $\sqsubseteq$ is the prefix ordering, is a cpo. (You must first check that it is a poset).

4. Are there chains in $(\mathsf{Str}(\Sigma), \sqsubseteq)$ which have a unique *upper bound*? How about chains with more than one upper bound? Classify all the chains according to the number of upper bounds they have.

5. In a poset, define two chains $\langle a_i \rangle_{i \geq 0}$ and $\langle b_i \rangle_{i \geq 0}$ to be *cofinal* if

(i) $\forall i \geq 0 \, \exists j \geq 0. [a_i \sqsubseteq b_j]$ and

(ii) $\forall i \geq 0 \, \exists j \geq 0. [b_i \sqsubseteq a_j]$.

Draw a picture of two cofinal chains. Prove that if two chains are cofinal and one of them has a lub then the other has the same lub.

6. Which points on the Koch curve have two addresses? Locate as far as you can the place of the points on the Koch curve whose addresses are $\overline{21} = 21212121\ldots$, $2\overline{1} = 21111\ldots$ and $1\overline{4} = 144444\ldots$.

7. Find the explicit form of the maps f_1, f_2, f_3, f_4 in the generation of the Koch curve.

[Hint: In the complex plane $z = x + iy$ $(i = \sqrt{-1})$,

(i) scaling by factor $r > 0$ is given by $z \mapsto rz$

(ii) rotation by θ is given by $z \mapsto e^{i\theta} z$ and

(ii) translation by $a = a_1 + ia_2$ is given by $z \mapsto z + a$.

Each f_n is a composition of a scaling, a rotation and a translation.]

Note:

The following exercises, each labelled by a $*$, require more mathematical background. There will be such exercises in each tutorial, which will cover some of the topological and categorical aspects of domain theory. These exercises are *not* an examinable part of the course, but should be attempted by students with more advanced mathematical training as well as those who are interested enough to take the challenge.

8*. Let $(P, \sqsubseteq)$ be a poset. A non-empty subset $A \subseteq P$ is *directed* if for any pair of elements $a, b \in A$, there exists $c \in A$ with $a, b \sqsubseteq c$. A *directed complete partial order* or a *dcpo* is a poset in which every directed subset has a lub.

- Check that every dcpo with bottom is a cpo.

Let $(D, \sqsubseteq)$ be a dcpo. For $a, b \in D$, we say *a approximates b* or *a* is *way-below b* and write $a \ll b$, if for all directed subsets $A \subseteq D$,

$$b \sqsubseteq \bigsqcup A \ \Rightarrow\ \exists c \in A.\, a \sqsubseteq c.$$

Show that

- $a \ll b \ \Rightarrow\ a \sqsubseteq b$
  $a \sqsubseteq b \ll c \ \Rightarrow\ a \ll c$
  $b \ll c \sqsubseteq d \ \Rightarrow\ b \ll d.$
- $\ll$ is anti-symmetric and transitive.

9*. Show that $([0,1], \leq)$, where $\leq$ is the usual ordering on the unit interval, is a dcpo. Characterise the way-below relation.

10*. A subset $B \subseteq D$ of a dcpo D is a *base* of D if for any element $a \in D$, the set $\{b \in B \mid b \ll a\}$ is directed and has lub a. A dcpo is *(ω-)continuous* if it has a (countable) base. Show that if D is continuous, then D is a base of itself. Check that $([0,1], \leq)$ is ω-continuous. Does it have a unique countable base?