Quantum Computing - Exercises 10

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1.

(i) Prove that $|E(t)\rangle = e^{-i\mathcal{H}t/\hbar}|E(0)\rangle$ satisfies the Schrödinger equation for any time independent $\mathcal{H}$.

(ii) Show that the operator $U(t) = e^{-i\mathcal{H}t/\hbar}$ is unitary.

(iii) For a unitary matrix U, define $\log U$, show that $-i\log U$ is Hermitian and deduce that $U = \exp(iK)$ for some Hermitian K.

2. With the definitions and notations given on page 128 of the lecture notes, consider the projection operator P_S, the Hermitian matrix A, with eigenvalues λ_i and the corresponding operators P_i.

(i) Show that $P_S \circ P_S = P_S$ and that any projection operator P is in fact projection into some subspace of $\mathbb{C}^n$.

(ii) Check that P_j is the projection to the eigenspace corresponding to eigenvalue λ_j.

(iii) Show that $A = \sum_{j=1}^{k} \lambda_j P_j$.

3. Let $\{v_{jm} | 1 \leq m \leq n_j,\ 1 \leq j \leq k\}$ be the collection of the eigenvectors of a Hermitian matrix A on $\mathbb{C}^n$, with the eigenvectors v_{jm} for $1 \leq m \leq n_j$ giving an orthonormal basis for the eigenspace corresponding to eigenvalue λ_j. Let $|\psi\rangle = \sum_{jm} \alpha_{jm} |v_{jm}\rangle$ be an arbitrary state. Show that the probability of obtaining λ_j is $p(\lambda_j) = \sum_{m=1}^{n_j} |\alpha_{jm}|^2$.

4. Find the energy levels and the energy eigenstates of the quantum systems with the following Hamiltonians:

(i) Pauli-Y: $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

(ii) Hadamard: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(ii) CNOT:$\frac{1}{2}(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|)$.

In each case determine the probabilities of measuring the energy levels and the expected value of the Hamiltonian for an arbitrary state.

5. Show that, at time t, the average values of the x, y and z-component of spin, given respectively by the Hermitian matrices

$$I^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad I^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

are as follows:

$$\begin{aligned} \langle\psi(t)|I^x|\psi(t)\rangle &= a\cos(\omega_0 t + \phi) \\ \langle\psi(t)|I^y|\psi(t)\rangle &= -a\sin(\omega_0 t + \phi) \\ \langle\psi(t)|I^z|\psi(t)\rangle &= \tfrac{1}{2}(|c_0(0)|^2 - |c_1(0)|^2), \end{aligned}$$

where $c_0(0)c_1^*(0) = ae^{i\phi}$ with real a and ϕ.

6.

(i) Show that under the action of a resonant field the average values of the spin components at time t for the state which is initially in the ground state are given by:

$$\langle I^x\rangle(t) = 0, \ \langle I^y\rangle(t) = \frac{1}{2}\sin\Omega t, \ \langle I^z\rangle(t) = \frac{1}{2}\cos\Omega t.$$

Hence, the average value of the z-component goes from $1/2$ at $t = 0$ to $-1/2$ at $t = \pi/\Omega$.

(ii) Show that one can construct, starting with the ground state, an equally weighted superposition of the ground and the excited states with a pulse of the resonant field applied for a suitable time interval. This therefore implements a fair coin.