Quantum Computing - Solutions 10

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1.

(i) Prove that $|E(t)\rangle = e^{-i\mathcal{H}t/\hbar}|E\rangle$ satisfies the Schrödinger equation for any time independent $\mathcal{H}$.

(ii) Show that the operator $U(t) = e^{-i\mathcal{H}t/\hbar}$ is unitary.

(iii) For a unitary matrix U, define $\log U$, show that $-i \log U$ is Hermitian and deduce that $U = \exp(iK)$ for some Hermitian K.

Solution: We have:

(i)

$$|E(t)\rangle = e^{-i\mathcal{H}t/\hbar}|E\rangle = |E\rangle + \frac{(-i\mathcal{H}t/\hbar)^1}{1!}|E\rangle + \cdots + \frac{(-i\mathcal{H}t/\hbar)^n}{n!}|E\rangle + \cdots.$$

Differentiating the RHS term by term, we get:

$$\frac{d|E(t)\rangle}{dt} = (-i\mathcal{H}/\hbar)|E\rangle + (-i\mathcal{H}/\hbar)\frac{(-i\mathcal{H}/\hbar)^1}{1!}|E\rangle + \cdots + (-i\mathcal{H}/\hbar)\frac{(-i\mathcal{H}t/\hbar)^n}{n!}|E\rangle + \cdots$$

$$= (-i\mathcal{H}/\hbar)(|E\rangle + \frac{(-i\mathcal{H}t/\hbar)^1}{1!}|E\rangle + \cdots + \frac{(-i\mathcal{H}t/\hbar)^n}{n!}|E\rangle + \cdots)$$

$$= (-i\mathcal{H}/\hbar)e^{-i\mathcal{H}t/\hbar}|E\rangle = (-i\mathcal{H}/\hbar)|E(t)\rangle.$$

(ii) We have:

$$U(t) = e^{-i\mathcal{H}t/\hbar} = I + \frac{(-i\mathcal{H}t/\hbar)^1}{1!} + \cdots + \frac{(-i\mathcal{H}t/\hbar)^n}{n!} + \cdots.$$

Hence, taking the adjoint of the RHS term by term and noting that $\mathcal{H}$ is self-adjoint, we get:

$$(U(t))^\dagger = (e^{-i\mathcal{H}t/\hbar})^\dagger = I^d ag + \frac{(i\mathcal{H}^\dagger t/\hbar)^1}{1!} + \cdots + \frac{(i\mathcal{H}^\dagger t/\hbar)^n}{n!} + \cdots$$

$$= e^{i\mathcal{H}t/\hbar}.$$

But from the definition of the exponential of an operator (or matrix) L:

$$e^L| = I + \frac{L^1}{1!} + \frac{L^2}{2!} + \cdots + \frac{L^n}{n!} + \cdots.$$

Hence, in general we have:

$$e^L e^{-L} = I + (\frac{L^1}{1!} - \frac{L^1}{1!}) + (\cdots) + \cdots = I$$

as all nontrivial powers of L cancel out. It follows that in particular: $e^{-i\mathcal{H}t/\hbar}e^{i\mathcal{H}t/\hbar} = I$ and hence $e^{-i\mathcal{H}t/\hbar}$ is unitary.

(iii) Suppose first that U is diagonal. Then $U_{nj} = e^{i\theta_n}$ if $j = n$ and is zero otherwise. Then $(\log U)_{nj} = i\theta_n$ if $j = n$ and is zero otherwise. Note that $-i(\log U)_{nj} = \theta_n$ if $j = n$ and is zero otherwise, and hence $-i\log U$ is Hermitian, as it is real and symmetric self-transpose. In general, there exists a change of basis given by matrix B such that $BUB^{-1} = V$ is diagonal. We define: $\log U = B \log V B^{-1}$. We then have $-i\log U = B(-i\log V)B^{-1}$ which is Hermitian since $(-i\log V)$ is Hermitian and B is unitary and $(B(-i\log V)B^{-1})^\dagger = (B^{-1})^\dagger(-i\log V)^\dagger B^\dagger = B(-i\log V)B^{-1}$.

2. With the definitions and notations given on page 128 of the lecture notes, consider the projection operator P_S, the Hermitian matrix A, with eigenvalues λ_i and the corresponding operators P_i.

(i) Show that $P_S \circ P_S = P_S$ and that any projection operator P is in fact projection into some subspace of $\mathbb{C}^n$.

(ii) Check that P_j is the projection to the eigenspace corresponding to eigenvalue λ_j.

(iii) Show that $A = \sum_{j=1}^k \lambda_j P_j$.

Solution:

(i) Since $P_S = \sum_{m=1}^t |w_m\rangle\langle w_m|$ and w_m form an orthonormal basis for the subspace S, we get:

$$P_S \circ P_S = \sum_{m,n=1}^{t} |w_m\rangle\langle w_m|w_n\rangle\langle w_n|$$

$$= \sum_{m=1}^{t} |w_m\rangle\langle w_m| = P_S.$$

If $P : \mathbb{C}^n \to \mathbb{C}^n$ is is any projection operator, let $S = \{P(v)| \in \mathbb{C}^n\}$. Then S is a subspace, i.e. it is closed under addition and scalar multiplication by complex numbers: If $u_1, u_2 \in S$ then there exist $v_1, v_2 \in \mathbb{C}^n$ such that $P(v_i) = u_i$, $i = 1, 2$. Hence, $P(v_1 + v_2) = P(v_1) + P(v_2) = u_1 + u_2 \in S$. Similarly if $u \in S$ and $c \in \mathbb{C}$ then there exists $v \in \mathbb{C}^n$ with $P(v) = u$. Hence $P(cv) = cP(v) = cu$ and therefore $cu \in S$. Suppose $S \subseteq \mathbb{C}^n$ is a t-dimensional subspace

of $\mathbb{C}^n$ with orthonormal basis $|w_m\rangle$ $(1 \leq m \leq t)$. Then, we check that $P = P_S = \sum_{m=1}^{t} |w_m\rangle\langle w_m| : \mathbb{C}^n \to \mathbb{C}^n$. Let $v \in S$, with $v = P(u)$ then since $P_S \circ P_s = P_s$ and $P \circ P = P$, we get $P(v) = P^2(u) = P(u) = v = P_S(v)$. On the other hand, let $v \in S^\perp = \{u \in \mathbb{C}^n | \forall w \in S.\ \langle u|w\rangle = 0\}$. Then, $P_S|v\rangle = \sum_{m=1}^{t} |w_m\rangle\langle w_m|v\rangle = 0$. Also if $P|v\rangle = |v'\rangle$ then $\langle v'|Pv\rangle = \langle Pv'|v\rangle = \langle v'|v\rangle = 0$. Hence, P and P_S agree on S and on $S^\perp$. But if $u \in \mathbb{C}^n$, then $P(u) \in S$ and we check that $u - P(u) \in S^\perp$: For any $w \in S$ we have: $\langle u - P(u)|w\rangle = \langle u|w\rangle - \langle P(u)|w\rangle = \langle u|w\rangle - \langle u|P(w)\rangle = \langle u|w\rangle - \langle u|w\rangle = 0$. Hence, any $u \in \mathbb{C}^n$ can be written as $u = (u - P(u)) + P(u) = v + w$ with $v \in S^\perp$ and $w \in S$. Since P and P_S are both linear and agree on S and $S^\perp$ we conclude that $P = P_S$.

(ii) We have: $P_j = \sum_{m=1}^{n_j} |v_{jm}\rangle\langle v_{jm}| : \mathbb{C}^n \to \mathbb{C}^n$. Let S be the subspace of the eigenvectors corresponding to eigenvalue λ_j. For $v \in S$ we have $P_j(v) = v$ whereas for $v \in S^\perp$ we have $P_j(v) = 0$. Since any $u \in \mathbb{C}^n$ can be written as $u = P_S(u) + (u - P_S(u)$ with $P_S(u) \in S$ and $u - P_S(u) \in S^\perp$, it follows that $S = \{P_j(u)|u \in \mathbb{C}^n\}$. Hence, $P_j = P_S$.

(iii) Since $\{v_{jm}|1 \leq m \leq n_j,\ 1 \leq j \leq k\}$ form an orthonormal basis for $\mathbb{C}^n$, we only need to check that A and $\sum_{j=1}^{k} \lambda_j P_j$ have the same action on these basis vectors. But $Av_{jm} = \lambda_j v_{jm} = \sum_{j=1}^{k} \lambda_j P_j(v_{jm}$.

3. Let $\{v_{jm}|1 \leq m \leq n_j,\ 1 \leq j \leq k\}$ be the collection of the eigenvectors of a Hermitian matrix A on $\mathbb{C}^n$, with the eigenvectors v_{jm} for $1 \leq m \leq n_j$ giving an orthonormal basis for the eigenspace corresponding to eigenvalue λ_j. Let $|\psi\rangle = \sum_{jm} \alpha_{jm}|v_{jm}\rangle$ be an arbitrary state. Show that the probability of obtaining λ_j is $p(\lambda_j) = \sum_{m=1}^{n_j} |\alpha_{jm}|^2$.

Solution: By definition and orthonormality of $\{v_{jm}|1 \leq m \leq n_j,\ 1 \leq j \leq k\}$ we have:

$$\sqrt{p(\lambda_j)} = \langle\psi|P_j|\psi\rangle$$

$$= \sum_{il} \alpha_{il}^* \langle |v_{il}| P_j \sum_{km} \alpha_{km}|v_{km}\rangle$$

$$= \sum_{il} \alpha_{il}^* \langle |v_{il}| \sum_{km} \alpha_{jm}|v_{jm}\rangle$$

$$= \sum_{m=1}^{n_j} |\alpha_{jm}|^2.$$

4. Find the energy levels and the energy eigenstates of the quantum systems with the following Hamiltonians:

(i) Pauli-Y: $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

(ii) Hadamard: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(ii) CNOT:$\frac{1}{2}(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|)$.

In each case determine the probabilities of measuring the energy levels and the expected value of the Hamiltonian for an arbitrary state.

Solution:

(i) Eigenvalues are -1 with unit eigenvector $\frac{1}{\sqrt{2}}\binom{i}{1}$ and 1 with unit eigenvector $\frac{1}{\sqrt{2}}\binom{1}{i}$. Projection to the first eigenvector is $\frac{1}{2}\binom{i}{1}(-i,1) = \frac{1}{2}\binom{1 \quad i}{-i \quad 1}$, and the projection to the second is $\frac{1}{2}\binom{1}{i}(1,-i) = \frac{1}{2}\binom{1 \quad -i}{i \quad 1}$. Given a state $\binom{a}{b}$, we have: $p(-1) = \frac{1}{2}(a^*, b^*)\binom{1 \quad i}{-i \quad 1}\binom{a}{b} = |a|^2 + ia^*b - iab^* + |b|^2$ and $p(1) = \frac{1}{2}(a^*, b^*)\binom{1 \quad -i}{i \quad 1}\binom{a}{b} = |a|^2 - ia^*b + iab^* + |b|^2$. The average value is: $\langle Y\rangle = -1p(-1) + 1p(1) = -(|a|^2 + ia^*b - iab^* + |b|^2) + |a|^2 - ia^*b + iab^* + |b|^2 = 2(-ia^*b + iab^*)$.

(ii) The eigenvalues are $\sqrt{2}$ and $-\sqrt{2}$ corresponding to unit eigenvectors $c\begin{pmatrix}1\\ \sqrt{2}-1\end{pmatrix}$ (where $c^{-1} = \sqrt{4-2\sqrt{2}}$) and $d\begin{pmatrix}1\\ -1-\sqrt{2}\end{pmatrix}$ (where $d^{-1} = \sqrt{4+2\sqrt{2}}$) respectively. If $\binom{a}{b}$ is an arbitrary state, then $p(\sqrt{2}) = \frac{1}{4-2\sqrt{2}}(a,b)\begin{pmatrix}1 & \sqrt{2}-1\\ \sqrt{2}-1 & 3-2\sqrt{2}\end{pmatrix}\binom{a}{b}$. And $p(-\sqrt{2}) = \frac{1}{4+2\sqrt{2}}(a,b)\begin{pmatrix}1 & \sqrt{2}-1\\ \sqrt{2}-1 & 3-2\sqrt{2}\end{pmatrix}\binom{a}{b}$. The average value is $\langle H\rangle = \sqrt{2}p(\sqrt{2}) - \sqrt{2}p(-\sqrt{2})$.

(iii) The eigenvalues are 1 with eigenvectors $(1,0,0,0)^T, (0,1,0,0)^T, (0,0,1,1)^T$ and -1 with eigenvector $(0,0,1,-1)^T$. Projection to the eigenspace of -1 is $P_{-1} = \frac{1}{2}(0,0,1,-1)(0,0,1,-1)^T$ and projection to the eigenspace of 1 is $P_1 = C + \frac{1}{2}(0,0,1,-1)(0,0,1,-1)^T$, where C is the matrix of CNOT. Given a state $(a,b,c,d)^T$, we have: $p(-1) = (a,b,c,d)P_{-1}(a,b,c,d)^T$ and $p(1) = (a,b,c,d)P_1(a,b,c,d)^T$. Finally, the average value is $p(1) - p(-1)$.

5. Show that, for the state $|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$ with

$$c_0(t) = c_0(0)e^{i\omega_0 t/2}, \qquad c_1(t) = c_1(0)e^{-i\omega_0 t/2},$$

the average values of the x, y and z-component of spin, given respectively by the Hermitian matrices

$$I^x = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad I^y = \frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad I^z = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

are as follows:

$$\begin{aligned}\langle\psi(t)|I^x|\psi(t)\rangle &= a\cos(\omega_0 t + \phi)\\ \langle\psi(t)|I^y|\psi(t)\rangle &= -a\sin(\omega_0 t + \phi)\\ \langle\psi(t)|I^z|\psi(t)\rangle &= \tfrac{1}{2}(|c_0(0)|^2 - |c_1(0)|^2),\end{aligned}$$

where $c_0(0)c_1^*(0) = ae^{i\phi}$ with real a and ϕ.

Solution:

$$I^x|\psi(t)\rangle = I^x(c_0(0)e^{i\omega_0 t/2}|0\rangle + c_1(0)e^{-i\omega_0 t/2}|1\rangle)$$
$$= \frac{1}{2}c_0(0)e^{i\omega_0 t/2}|1\rangle + \frac{1}{2}c_1(0)e^{-i\omega_0 t/2}|0\rangle.$$

Hence,

$$\langle I^x\rangle = \langle\psi_(t)|I^x|\psi_(t)\rangle$$
$$= (c_0(0)^* e^{i\omega_0 t/2}\langle 0| + c_1(0^*)e^{-i\omega_0 t/2}\langle 1|)\frac{1}{2}c_0(0)e^{i\omega_0 t/2}|1\rangle + \frac{1}{2}c_1(0)e^{-i\omega_0 t/2}|0\rangle$$
$$= \frac{1}{2}(c_0(0)^* c_1(0))e^{-i\omega_0 t/2} + c_1(0)^* c_0(0))e^{i\omega_0 t/2}$$
$$= a\cos(\omega_0 t + \phi).$$

$\langle I^y\rangle$ and $\langle I^z\rangle$ are computed in a similar way.

6.

(i) Show that under the action of a resonant field the average values of the spin components at time t for the state which is initially in the ground state are given by:

$$\langle I^x\rangle(t) = 0,\ \langle I^y\rangle(t) = \frac{1}{2}\sin\Omega t,\ \langle I^z\rangle(t) = \frac{1}{2}\cos\Omega t.$$

Hence, the average value of the z-component goes from $1/2$ at $t = 0$ to $-1/2$ at $t = \pi/\Omega$.

(ii) Show that one can construct, starting with the ground state, an equally weighted superposition of the ground and the excited states with a pulse of the resonant field applied for a suitable time interval. This therefore implements a fair coin.

Solution: (i) Repeating the solution of the previous problem (Problem 5) for:

$$c_0(t) = c_0(0)\cos\frac{\Omega t}{2} + ic_1(0)\sin\frac{\Omega t}{2},$$

$$c_1(t) = ic_0(0)\sin\frac{\Omega t}{2} + c_1(0)\cos\frac{\Omega t}{2},$$

we get with the initial conditions $c_0(0) = 1$ and $c_1(0) = 0$:

$$\langle I^x\rangle(t) = \frac{1}{2}(c_0(0)^* c_1(0) + c_1(0)^* c_0(0)) = 0$$

$$\langle I^y\rangle(t) = \frac{1}{2}(c_0(0)c_1(0)^* - c_1(0)c_0(0)^*) = \frac{1}{2}\sin\Omega t.$$

$$\langle I^z \rangle(t) = \frac{1}{2}(|c_0(0)|^2 - |c_1(0)|^2) = \frac{1}{2}\cos \Omega t.$$

(ii) Recall that the general solution is:

$$c_0(t) = c_0(0)\cos\frac{\Omega t}{2} + ic_1(0)\sin\frac{\Omega t}{2},$$

$$c_1(t) = ic_0(0)\sin\frac{\Omega t}{2} + c_1(0)\cos\frac{\Omega t}{2}.$$

If initially the system is in the ground state, i.e. $c_0(0) = 1$ and $c_1(0) = 0$, then at time t:

$$c_0(t) = \cos\frac{\Omega t}{2}, \qquad c_1(t) = i\sin\frac{\Omega t}{2}.$$

If we choose $t = \pi/(2\Omega)$, then we get

$$c_0(\pi/(2\Omega)) = \cos\frac{\Omega\pi/(2\Omega)}{2} = \cos\pi/4 = \sqrt{2}/2$$

$$c_1(t) = i\sin\frac{\Omega\pi/(2\Omega)}{2} = \sin\pi/4 = \sqrt{2}/2.$$

Hence, a $\pi/2$-pulse produces an equally weighted superposition of the two basis vectors.