Quantum Computing, Solutions 1

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1. (i) Show that for complex numbers x and y we have:
$|x+y|^2 = |x|^2 + |y|^2 + 2\mathrm{Re}(x^*y)$.

(ii) Compare x^*x with x^2 for a complex number x and explain why the inner product of two complex vectors $v, w \in \mathbb{C}^d$ is defined as $v^\dagger w$ rather than $v^T w$.

Solution:(i) Let $x = x_1 + ix_2$ and $y = y_1 + iy_2$. Then $x+y = x_1+y_1+i(x_2+y_2)$. Hence,

$$|x+y|^2 = (x_1+y_1)^2+(x_2+y_2)^2 = (x_1^2+x_2^2)+(y_1^2+y_2^2)+2(x_1y_1+x_2y_2) = |x|^2+|y|^2+2\mathrm{Re}(x^*y),$$

since $\mathrm{Re}(x^*y) = \mathrm{Re}[(x_1 - iy_1)(x_2 + iy_2)] = x_1x_2 + y_1y_2$.

(ii) x^*x is always a non-negative real number, the lenght of the complex number x, whereas x^2 is in general a complex number. Since we would like (v, v) to give us the length of the complex vector v, we need to use $v^\dagger$ in the definition of the inner product $(v, w) = v^\dagger w$ to get we a non-neagtive real number.

2. Generalize the following notions and properties given for the vector space $\mathbb{R}^2$ in the notes to $\mathbb{R}^d$.

(i) Define linear independence and linear dependence of vectors in $\mathbb{R}^d$.

(ii) What is the least integer n such that any set of n vectors in $\mathbb{R}^d$ will be linearly dependent?

(iii) What is a basis of $\mathbb{R}^d$? How many linearly independent vectors it takes to get a basis for $\mathbb{R}^d$?

(iv) Define the notion of an *orthonormal* basis for $\mathbb{R}^d$. What would be the *standard* basis of $\mathbb{R}^d$?

Solution:

(i) Same definition as in $\mathbb{R}^2$: the vectors

$$\{v_i \in \mathbb{R}^d \mid i = 1, 2, \ldots, k\}$$

are *linearly independent* if whenever

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0$$

for $a_i \in \mathbb{R}$ $(i = 1, \ldots, k)$ then $a_i = 0$ for all $i = 1, \ldots, k$. Otherwise they are called linearly dependent.

(ii) $d + 1$.

(iii) A basis of $\mathbb{R}^d$ is any set of linearly independent vectors in $\mathbb{R}^d$ such that any vector in $\mathbb{R}^d$ can be expressed as a linear combination of vectors in the set. Any d linearly independent vectors in $\mathbb{R}^d$ form a basis.

(iv) A basis $\{v_i \mid 1 \le i \le d\}$ is an orthonormal basis if $v_i \cdot v_i = 1$, for $1 \le i \le d$ and $v_i \cdot v_j = 0$ for $i \ne j$. The standard basis of $\mathbb{R}^d$ consists of the d vectors v_i, $1 \le i \le d$, such that all entries of the column vector v_i are 0 except the ith entry which is 1.

3. Generalize the following notions and properties given for the vector space $\mathbb{C}^2$ in the notes to $\mathbb{C}^d$.

(i) Define the norm $\|w\|$ of a vector and the innerproduct of two vectors w_1 and w_2 in $\mathbb{C}^d$. What is the dual of a vector w in $\mathbb{C}^d$ and what can it be identified with?

(ii) Define linear independence and linear dependence of vectors in $\mathbb{C}^d$.

(iii) What is the least integer n such that any set of n vectors in $\mathbb{C}^d$ will be linearly dependent?

(iv) What is a basis of $\mathbb{C}^d$? How many linearly independent vectors it takes to get a basis for $\mathbb{C}^d$?

(v) Define the notion of an *orthonormal* basis for $\mathbb{C}^d$. What would be the *standard* basis of $\mathbb{C}^d$?

Solution:

(i) The norm of a vector w with $w^T = (w_1, w_2, \ldots, w_d)$ is $\|w\| = \sqrt{\sum_{j=1}^{d} |w_j|^2}$. The innerproduct of the two vectors w_1 and w_2 with $w_1^T = (w_{11}, w_{12}, \ldots, w_{1d})$ and $w_2^T = (w_{21}, w_{22}, \ldots, w_{2d})$ is $(w_1 \cdot w_2) = \sum_{j=1}^{d} w_{1j}^* w_{2j}$. The dual of w is the linear map $L_w : \mathbb{C}^d \to \mathbb{C}$ defined by $L_w(v) = (w, v) = w^\dagger v$ and can be identified with $w^\dagger$.

(ii) Exactly as in $\mathbb{C}^2$.

(iii) $d + 1$.

(iv) A basis of $\mathbb{C}^d$ is any set of linearly independent vectors in $\mathbb{C}^d$ such that any vector in $\mathbb{C}^d$ can be expressed as a linear combination of vectors in the set. Any d linearly independent vectors in $\mathbb{C}^d$ form a basis.

(v) A basis $\{w_i \mid 1 \leq i \leq d\}$ is is an orthonormal basis if $(w_i, w_i) = 1$, for $1 \leq i \leq d$ and $(w_i, w_j) = 0$ for $i \neq j$.

4. Show that the two vectors w_1 and w_2 in $\mathbb{C}^4$ with $w_1^T = \frac{1}{2}(1,1,1,1)$ and $w_2^T = \frac{1}{2}(1,-1,1,-1)$ are orthogonal unit vectors. Find vectors w_3 and w_4 such that the collection $\{w_1, w_2, w_3, w_4\}$ forms an orthonormal basis for $\mathbb{C}^4$.

Solution: $\|w_1\| = \sqrt{\frac{1^2+1^2+1^2+1^2}{4}} = 1$ and $\|w_2\| = \sqrt{\frac{1^2+(-1)^2+1^2+(-1)^2}{4}} = 1$. Also $(w_1, w_2) = \frac{1-1+1-1}{4} = 0$. To find $w_3 = (a,b,c,d)$ and $w_4 = (e,f,g,h)$ we need to solve:

(i) $(w_1, w_3) = a+b+c+d = 0$, (ii) $(w_1, w_4) = e+f+g+h = 0$, (iii) $(w_2, w_3) = a-b+c-d = 0$, (iv) $(w_2, w_4) = e-f+g-h = 0$, (v) $(w_3, w_4) = ae+bf+cg+dh = 0$, (vi) $\|w_3\|^2 = a^2+b^2+c^2+d^2 = 1$ and (vii) $\|w_4\|^2 = e^2+f^2+g^2+h^2 = 1$.

These are seven equations with eight unknowns. Therefore, there are an infinite number of solutions. By inspection, we see that $w_3 = \frac{1}{\sqrt{2}}(1,0,-1,0)$ and $w_4 = \frac{1}{\sqrt{2}}(0,1,0,-1)$ is one particularly simple solution.

5. Show that any qubit can be expressed as

$$|\psi\rangle = e^{i\gamma}\left(\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right),$$

for real numbers γ, θ and ϕ. The phase factor $e^{i\gamma}$ has no observational effect and can be dropped.

Solution: Recall that any complex number z can be written as $z = re^{i\psi}$ where $r = |z|$ and ψ is the angle from the positive real axis to z measured anti-clockwise in $\mathbb{C}$. Consider the general qubit $\alpha|0\rangle + \beta|1\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$. Let $\alpha = e^{i\gamma}\cos(\theta/2)$ where

(i) $\cos(\theta/2) = |\alpha|^2 \leq 1$ for some θ with $0 \leq \theta \leq \pi$, and,

(ii) γ is the angle that α makes with the positive real axis in $\mathbb{C}$ as in Figure 1.

Then, $|\beta| = \sqrt{1-|\alpha|^2} = \sqrt{1-(\cos(\theta/2))^2} = \sin(\theta/2)$. Suppose $\beta = e^{i\lambda}\sin(\theta/2)$, where λ is the angle that β makes with the positive real axis in $\mathbb{C}$. Then $\lambda = \gamma + \phi$ where ϕ is the angle between α and β in $\mathbb{C}$.

6. Show that there is a one to one correspondence between qubits

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$

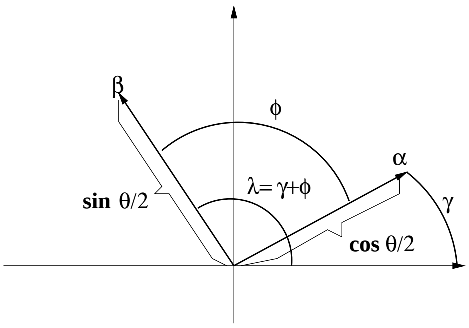

Figure 1: Representation of a qubit

and the points on the unit sphere in $\mathbb{R}^3$, called the *Bloch sphere*, with θ and ϕ as the spherical coordinates of a point of the sphere; see Figure 2.

Solution: The crucial point is to realize from Exercise 5 that for a qubit $\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ the angle θ is restricted to $0 \leq \theta \leq \pi$. On the one hand the spherical angle θ that a point on the unit sphere in $\mathbb{R}^3$ makes with the z axis also satisfies $0 \leq \theta \leq \pi$. On the other hand, the angle ϕ in the qubit $\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ is in the range $0 \leq \phi < 2\pi$ as is the angle the projection of a unit vector in the Bloch sphere on the xy-plane makes with the x-axis. Hence, up to a global phase factor, there is a one-to-one correspondence between qubits $\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ and points on the Bloch sphere.

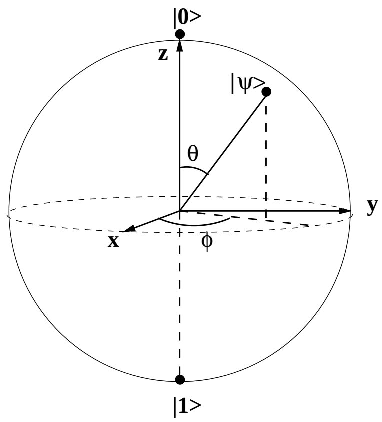

Figure 2: The Bloch sphere