Domain theory and Fractals- Exercises 1 - Solutions

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1. Let

$$a_0 \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq a_3 \sqsubseteq \ldots$$

be a chain in a finite poset. Since there are only a finite number of elements in the poset, this chain must be eventually constant, i.e. there exists $i \geq 0$ such that $a_j = a_i$ for all $j \geq i$, so that the chain is of the form

$$a_0 \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq a_3 \sqsubseteq \ldots \sqsubseteq a_i \sqsubseteq a_i \sqsubseteq a_i \sqsubseteq \ldots.$$

Then a_i is the lub of the chain. Hence, all chains have lubs.

2. It is easy to check that $\subseteq$ is indeed reflexive, anti-symmetric and transitive. The least element is the empty set. Assume, we have a chain

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq B_3 \subseteq B_4 \subseteq \ldots,$$

with $B_i \subseteq A$ for all $i \geq 0$. We claim that the lub of the chain is $\bigcup_i B_i = \{x \mid x \in B_i \text{ for some } i\}$. Clearly, $B_n \subseteq \bigcup_i B_i$ for all $n \geq 0$, so $\bigcup_i B_i$ is an upper bound for the chain. If C is any upper bound for the chain, we have $B_i \subseteq C$ for all $i \geq 0$. Hence $\bigcup_i B_i \subseteq C$. Therefore $\bigcup_i B_i$ is the lub of the chain.

3. First we check that the prefix ordering $\sqsubseteq$ gives us a poset. Let $a, b, c \in \mathsf{Str}(\Sigma)$. Recall that $a \sqsubseteq b$ iff $b = al$ for some $l \in \mathsf{Str}(\Sigma)$ and that ϵ is the empty sequence. We have:

- reflexive: $a \sqsubseteq a$ since $a = a\epsilon$.
- anti-symmetric: If $a \sqsubseteq b$ and $b \sqsubseteq a$, we get $b = al$ for some $l \in \mathsf{Str}(\Sigma)$ and, hence, $al \sqsubseteq a$, which implies $l = \epsilon$. Therefore, $b = a\epsilon = a$.
- transitive: If $a \sqsubseteq b \sqsubseteq c$, then $c = bl$ and $b = ak$, for some $l, k \in \mathsf{Str}(\Sigma)$. Hence, $c = akl = am$ (where $m = kl$). Therefore, $a \sqsubseteq c$.

Clearly, ϵ is the least element. It remains to show that any chain has a lub. A chain $\langle a_i \rangle_{i \geq 0}$ in $(\mathsf{Str}(\Sigma), \sqsubseteq)$ can be of two types:

(i) There exists $n \geq 0$, such that the maximum length of a_i, for $i \geq 0$ is n. Then the chain is eventually constant and the recurring sequence, which is the sequence of length n, is the lub of the chain.

(ii) For all $n \geq 0$ there exists $i_n \geq 0$ such that the sequence a_{i_n} has length at least n and hence we have $a_{i_n} = a_{i_n 1} a_{i_n 2} \ldots a_{i_n n} \ldots$. Then the lub of the chain will be $a = a_{i_1 1} a_{i_2 2} a_{i_3 3} \ldots$.

4. Assume Σ is non-empty. A chain whose lub is an infinite sequence has a unique upper bound which is its lub (since if a is an infinite sequence $a \sqsubseteq b$ implies $a = b$). A chain whose lub is a finite sequence has infinitely many upper bounds (if a is a finite sequence and the lub of the chain, then we have for example $a \sqsubseteq ak \sqsubseteq akk \sqsubseteq akkk \sqsubseteq \ldots$, for any $k \in \Sigma$, so that a, ak, akk, $\ldots$ are all upper bounds for the chain).

5. Suppose $\langle a_i \rangle_{i \geq 0}$ and $\langle b_i \rangle_{i \geq 0}$ are cofinal and that $\langle a_i \rangle_{i \geq 0}$ has lub a say. We show that a is the lub of $\langle b_i \rangle_{i \geq 0}$.

- For any $i \geq 0$ there is some $j \geq 0$ with $b_i \sqsubseteq a_j$. Since $a_j \sqsubseteq a$ we get $b_i \sqsubseteq a$ for all $i \geq 0$. So a is an upper bound for $\langle b_i \rangle_{i \geq 0}$.
- Suppose b is any upper bound for $\langle b_i \rangle_{i \geq 0}$. Then b is an upper bound for $\langle a_i \rangle_{i \geq 0}$ as well, since for any $i \geq 0$ there is some $j \geq 0$ with $a_i \sqsubseteq b_j \sqsubseteq b$. Hence, $a \sqsubseteq b$, since a is the lub of $\langle a_i \rangle_{i \geq 0}$.

6. Any point common between two segments in each polygon E_n, i.e. any joint in E_n, is on the Koch curve and has two addresses. These points have labels $a1\bar{4} = a2\bar{1}$, $a2\bar{4} = a3\bar{1}$ and $a3\bar{4} = a4\bar{1}$ for any finite sequence a.

7. In the complex plane we get:

- $f_1 : z \mapsto \frac{z}{3}$.
- $f_2 : z \mapsto \frac{z}{3} e^{i\frac{\pi}{3}} + \frac{1}{3}$.
- $f_3 : z \mapsto \frac{z}{3} e^{-i\frac{\pi}{3}} + (\frac{1}{2} + i\frac{\sqrt{3}}{6})$.
- $f_4 : z \mapsto \frac{z}{3} + \frac{2}{3}$.