Quantum Computing - Exercises 2

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1.

(i) Show that composition of two linear maps is linear.

(ii) Show that if $L_1, L_2 : \mathbb{C}^2 \to \mathbb{C}^2$ are linear, so is $\alpha_1 L_1 + \alpha_2 L_2 : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$(\alpha_1 L_1 + \alpha_2 L_2)(v) = \alpha_1 L_1(v) + \alpha_2 L_2(v).$$

(iii) Show that for $|\psi\rangle, |\phi\rangle \in \mathbb{C}^2$, the map $|\psi\rangle\langle\phi| : \mathbb{C}^2 \to \mathbb{C}^2$ defined, using the bracket notation, by $|\psi\rangle\langle\phi|(|x\rangle) = \langle\phi|x\rangle|\psi\rangle$ is linear.

2. Show that the matrix representation of the NOT gate in the basis $|0\rangle$, $|1\rangle$ is given by $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. What is its matrix representation in the basis $|+\rangle$, $|-\rangle$?

3.

(i) Define the notion of a linear map $L : \mathbb{C}^d \to \mathbb{C}^d$.

(ii) Let $\{w_1, w_2, \cdots, w_d\}$ be the standard basis of $\mathbb{C}^d$, i.e. w_j is a d-column vector all whose entries are 0 except the jth entry which is 1. Suppose $L(w_j) = \sum_{k=1}^{d} a_{kj} w_k$. Let $A = (a_{kj})$ be the induced matrix. What is $L(v)$ in terms of A if $v = \sum_{k=1}^{d} v_k w_k$ where $v_k \in \mathbb{C}$ are the components of $v \in \mathbb{C}^d$ with respect to the standard basis?

(iii) Extend the corresponding results for the change of basis from $\mathbb{C}^2$ to $\mathbb{C}^d$.

4.

(i) Check that a matrix is unitary iff its columns (or its rows) form an orthonormal basis.

(ii) Check that the matrix for a change in the computational basis is unitary.

(iii) Show that $(AB)^\dagger = B^\dagger A^\dagger$ and deduce that if a linear map has a unitary matrix representation, then its matrix representation in any computational basis is unitary.

5. Show that

$$\langle u|Av\rangle = \langle A^\dagger u|v\rangle.$$

Deduce that a matrix M is unitary iff it preserves all inner products, i.e. iff $\langle Mu|Mv\rangle = \langle u|v\rangle$ for all $u, v \in \mathbb{C}^2$.

6. Show that any unitary matrix U can be expressed as

$$U = \begin{pmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\gamma/2 & -e^{i(\alpha-\beta/2+\delta/2)}\sin\gamma/2 \\ e^{i(\alpha+\beta/2-\delta/2)}\sin\gamma/2 & e^{i(\alpha+\beta/2+\delta/2)}\cos\gamma/2 \end{pmatrix}$$

where α, β, δ and γ are real numbers.

7. Verify the output, up to a global phase, of the following quantum network.