Domain theory and Continuous Data Types
Exercises 2

1. What is the simplest example of a poset with $\perp$ which is not a cpo? Can you think of other examples?

2. In $(\mathcal{P}(\mathbb{N}), \subseteq)$, what is the least upper bound of the chain $P_1 \subseteq P_2 \subseteq P_3 \subseteq P_4 \subseteq \cdots$ where P_n is the set of the first n prime numbers?

3. For any set A, consider the poset $(\mathcal{P}(A), \supseteq)$, i.e. subsets of A ordered by *superset* inclusion. A chain then will look like $B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$. Do chains in $(\mathcal{P}(A), \supseteq)$ have lubs? Is $(\mathcal{P}(A), \supseteq)$ a cpo?

4. In $(\mathsf{Str}(\{0,1\}), \sqsubseteq)$ can you find two cofinal chains which have no common elements?

5. Locate as far as you can the position of points with the following labels in the Sierpinski triangle: $\overline{2}$, $13\overline{2}$, $\overline{123}$, $\overline{31}$.

6. What points of the real line correspond to the points of the Cantor set with the following labels: $11\overline{0}$, $010\overline{1}$, $\overline{1101}$.

7. Find the fixed points of each of the maps f_1, f_2, f_3, f_4 in the generation of the Koch curve. What points of $\mathsf{Str}(\{1,2,3,4\})$ correspond to these points? Do the same for the Sierpinski triangle and the Cantor set.

8. Consider the Cantor set C and its generating sequence $\langle I_n \rangle_{n \geq o}$. Find $d_H(I_n, C)$ in $\mathcal{P}(\mathbb{R})$. Do the same for the Sierpinski triangle and its generating sequence in $\mathcal{P}(\mathbb{R}^2)$.

9. Show that the function d defined on sequences $a = a_0a_1a_2\ldots$ and $b = b_0b_1b_2\ldots$ in $\mathsf{Str}(\Sigma)$ as follows is a distance function:

$$d(a,b) = \begin{cases} 0 & \text{if } a = b \\ \frac{1}{2^n} & \text{otherwise} \end{cases}$$

where n is the least integer such that $a_n \neq b_n$ (or one of them is undefined).

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Recall that a *topology* on a space X is given by a set of subsets of X, called the *open* subsets, which is closed under infinite unions and finite intersections. (Therefore, the empty set $\emptyset$, being the empty union, and X itself, being the empty intersection, are always open.) A *closed* subset is one whose complement is open.

10*. Let $(D, \sqsubseteq)$ be a dcpo. The *Scott* topology on D is defined by characterising its open sets as follows. A subset $O \subseteq D$ is open if

- O is upward closed, i.e. $x \in O \;\&\; x \sqsubseteq y \;\Rightarrow\; y \in O$; and
- whenever $A \subseteq D$ is a directed subset with $\bigsqcup A \in O$, then $A \cap O \neq \emptyset$.

Consider the Scott topology on D.

(i) Show that for any $x \in D$, the subset $\{y \in D \mid y \not\sqsubseteq x\}$ is open.

(ii) Characterise the closed subsets.

(iii) What are the (Scott) open subsets of $(\mathbb{N} \cup \{\infty\}, \leq)$, $([0,1], \leq)$ and $(\mathsf{Str}(\Sigma), \sqsubseteq)$?

11*. A topology on a space X is said to be T_0 if for any pair of distinct points $x, y \in X$, there exists an open set containing one but not the other. Show that the Scott topology on any dcpo is T_0.