Quantum Computing - Solutions 2

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1.

(i) Show that composition of two linear maps is linear.

(ii) Show that if $L_1, L_2 : \mathbb{C}^2 \to \mathbb{C}^2$ are linear, so is $\alpha_1 L_1 + \alpha_2 L_2 : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$(\alpha_1 L_1 + \alpha_2 L_2)(v) = \alpha_1 L_1(v) + \alpha_2 L_2(v).$$

(iii) Show that for $|\psi\rangle, |\phi\rangle \in \mathbb{C}^2$, the map $|\psi\rangle\langle\phi| : \mathbb{C}^2 \to \mathbb{C}^2$ defined, using the bracket notation, by $|\psi\rangle\langle\phi|(|x\rangle) = \langle\phi|x\rangle|\psi\rangle$ is linear.

Solution:

(i) Let $L : \mathbb{C}^2 \to \mathbb{C}^2$ and $L' : \mathbb{C}^2 \to \mathbb{C}^2$ be linear maps. Then $L \circ L' : \mathbb{C}^2 \to \mathbb{C}^2$ and $L \circ L'(a_1 v_1 + a_2 v_2) = L(L'(a_1 v_1 + a_2 v_2)) = L(a_1 L'(v_1) + a_2 L'(v_2)) = a_1 L(L'(v_1)) + a_2 L(L'(v_2)) = a_1 L \circ L'(v_1) + a_2 L \circ L'(v_2)$.

(ii) This follows step by step from the definitions:
$(\alpha_1 L_1 + \alpha_2 L_2)(a_1 v_1 + a_2 v_2) = \alpha_1 L_1(a_1 v_1 + a_2 v_2) + \alpha_2 L_2(a_1 v_1 + a_2 v_2) = (\alpha_1 a_1 L_1(v_1) + \alpha_1 a_2 L_1(v_2)) + (\alpha_2 a_1 L_2(v_1) + \alpha_2 a_2 L_2(v_2)) = a_1(\alpha_1 L_1 + \alpha_2 L_2)(v_1) + a_2(\alpha_1 L_1 + \alpha_2 L_2)(v_2)$.

(iii) This is a consequence of the linearity of the dual $\langle\phi|$ of the vector $|\phi\rangle$:

$|\psi\rangle\langle\phi|(a|x\rangle + b|y\rangle) = (a\langle\phi|x\rangle + b\langle\phi|y\rangle)|\psi\rangle = a|\psi\rangle\langle\phi|(|x\rangle) + b|\psi\rangle\langle\phi|(|y\rangle)$.

2. Show that the matrix representation of the NOT gate in the basis $|0\rangle$, $|1\rangle$ is given by $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. What is its matrix representation in the basis $|+\rangle$, $|-\rangle$?

Solution: We have $|0\rangle \mapsto |1\rangle$ and $|1\rangle \mapsto |0\rangle$. Hence, $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the matrix representation. We have $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Hence, $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$. Therefore, the matrix for the change of basis is $B = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. We have $B^{-1} = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. Hence the matrix representation of NOT in the new basis is: $BXB^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

3.

(i) Define the notion of a linear map $L : \mathbb{C}^d \to \mathbb{C}^d$.

(ii) Let $\{w_1, w_2, \ldots, w_d\}$ be the standard basis of $\mathbb{C}^d$, i.e. w_j is a d-column vector all whose entries are 0 except the jth entry which is 1. Suppose $L(w_j) = \sum_{k=1}^{d} a_{kj} w_k$. Let $A = (a_{kj})$ be the induced matrix. What is $L(v)$ in terms of A if $v = \sum_{k=1}^{d} v_k w_k$ where $v_k \in \mathbb{C}$ are the components of $v \in \mathbb{C}^d$ with respect to the standard basis?

(iii) Extend the corresponding results for the change of basis from $\mathbb{C}^2$ to $\mathbb{C}^d$.

Solution:

(i) A map $L : \mathbb{C}^d \to \mathbb{C}^d$ is linear if $L(a_1 v_1 + a_2 v_2) = a_1 L(v_1) + a_2 L(v_2)$ for all $v_1, v_2 \in \mathbb{C}^d$ and all $a_1, a_2 \in \mathbb{C}$.

(ii) Av.

(iii) Suppose $w_j \in \mathbb{C}^d$ (for $j = 1, \ldots, d$) is the standard basis and $v_j \in \mathbb{C}^d$ (for $j = 1, \ldots, d$) with $w_j = \sum_i b_{ij} v_i$. Then the matrix $B = (b_{ij})$ is the matrix for the change of basis. The vector with coordinates $(a_1, \ldots, a_n)^T$, with respect to the standard basis w_j, will have coordinates $B(a_1, \ldots, a_n)^T$ wrt the v_j basis. If a linear map has matrix representation A wrt the w_j basis, then it will have matrix representation BAB^{-1} wrt the v_j basis.

4.

(i) Check that a matrix is unitary iff its columns (or its rows) form an orthonormal basis.

(ii) Check that the matrix for a change in the computational basis is unitary.

(iii) Show that $(AB)^\dagger = B^\dagger A^\dagger$ and deduce that if a linear map has a unitary matrix representation, then its matrix representation in any computational basis is unitary.

Solution:

(i) If U is unitary then $U^\dagger U = I$ and $UU^\dagger = I$ where I is the identity matrix. The first equality tells us that the columns of U form an orthonormal basis and the second equality says that the rows of U form an orthonormal basis. Conversely, suppose that the columns of U form an orthonormal basis. Then $U^\dagger U = I$ and hence $U^\dagger = U^{-1}$. Similarly if the rows of U form an orthonormal basis. Then $UU^\dagger = I$ and hence $U^\dagger = U^{-1}$.

(ii) Let $\delta_{mi} = 1$ if $m = i$ and 0 otherwise. Suppose $|w_j\rangle \in \mathbb{C}^d$ (for $j = 1, \ldots, d$) and $|v_j\rangle \in \mathbb{C}^d$ (for $j = 1, \ldots, d$) are computational basis with $B = (b_{ij})$ as the matrix for the change of basis: $|w_i\rangle = \sum_k b_{ki}|v_k\rangle$. Then $\langle w_m| = \sum_t b^*_{tm}\langle v_t|$. Hence, $\delta_{mi} = \langle w_m|w_i\rangle = \sum_{k,t} b^*_{tm} b_{ki}\langle v_t|v_k\rangle = \sum_{k,t} b^*_{tm} b_{ki}\delta_{tk} = \sum_k b^*_{km} b_{ki}$. This shows that the columns of B form an orthonormal basis. Hence, B is unitary.

(iii) First note that $(AB)^T = B^T A^T$ since $((AB)^T)_{ij} = (AB)_{ji} = \sum_m A_{jm}B_{mi} = \sum_m B_{mi}A_{jm} = \sum_m (B^T)_{im}(A^T)_{mj} = (B^T A^T)_{ij}$. Furthremore, since for complex numbers x and y we have $(xy)^* = x^* y^*$, it follows that $(AB)^\dagger = ((AB)^T)^* = (B^T A^T)^* = (B^T)^*(A^T)^* = B^\dagger A^\dagger$.

Now assume a linear map has a unitary matrix representation U, i.e. $U^\dagger = U^{-1}$. Suppose we have a different basis with B the matrix of the change of basis. Then the matrix representation in the new basis is $BUB^{-1} = BUB^\dagger$ since the matrix for a change in basis is unitary by part (ii). We have: $(BUB^\dagger)^\dagger = (B^\dagger)^\dagger U^\dagger B^\dagger = BU^{-1}B^\dagger$. The result follows as the latter is the inverse of BUB^{-1}.

5. Show that

$$\langle u|Av\rangle = \langle A^\dagger u|v\rangle.$$

Deduce that a matrix M is unitary iff it preserves all inner products, i.e. iff $\langle Mu|Mv\rangle = \langle u|v\rangle$ for all $u, v \in \mathbb{C}^2$.

Solution: $\langle u|Av\rangle = \sum_i u^*_i (Av)_i = \sum_i u^*_i \sum_j A_{ij}v_j = \sum_i \sum_j (A^T)_{ji} u^*_i v_j = \sum_i \sum_j ((A^T)^*)^*_{ji} u^*_i v_j = \sum_i \sum_j (A^\dagger)^*_{ji} u^*_i v_j = \sum_j ((A^\dagger u)_j)^* v_j = \langle A^\dagger u|v\rangle$.

Now, if M is unitary then $\langle Mu|Mv\rangle = \langle M^\dagger Mu|v\rangle = \langle u|v\rangle$. On the other hand, suppose $\langle Mu|Mv\rangle = \langle u|v\rangle$ for all $u, v \in \mathbb{C}^2$, then $\langle u|M^\dagger Mv\rangle = \langle u|v\rangle$ for all $u, v \in \mathbb{C}^2$. Suppose $(M^\dagger M)_{ii} \neq 1$ for some i, then $\langle i|M^\dagger M|i\rangle = \langle i|(M^\dagger M)_{ii}|i\rangle = (M^\dagger M)_{ii} \neq 1 = \langle i|i\rangle$. Hence, $(M^\dagger M)_{ii} = 1$ for all i. Similarly, suppose $(M^\dagger M)_{ij} \neq 0$ for some $i \neq j$, then $\langle i|M^\dagger M|j\rangle = \langle i|(M^\dagger M)_{ij}|j\rangle = (M^\dagger M)_{ij} \neq 0 = \langle i|j\rangle$. Hence, $(M^\dagger M)_{ij} = 0$ for all $i \neq j$.

6. Show that any unitary matrix U can be expressed as

$$U = \begin{pmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\gamma/2 & -e^{i(\alpha-\beta/2+\delta/2)}\sin\gamma/2 \\ e^{i(\alpha+\beta/2-\delta/2)}\sin\gamma/2 & e^{i(\alpha+\beta/2+\delta/2)}\cos\gamma/2 \end{pmatrix}$$

where α, β, δ and γ are real numbers.

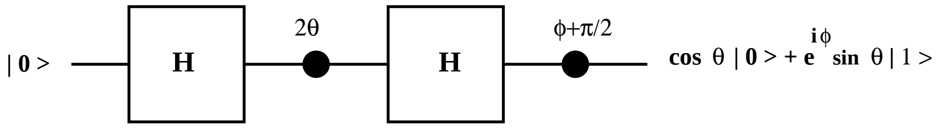
Solution: Let $U = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$. Since the rows (respectively, the columns) must form unit vectors, we can put:

$$a = e^{iA}\cos\frac{\gamma}{2},\ b = e^{iB}\sin\frac{\gamma}{2},\ c = -e^{iC}\sin\frac{\gamma}{2},\ d = e^{iD}\cos\frac{\gamma}{2}.$$

Now, the two columns must be orthogonal, hence: $-A + C = -B + D$. Now let $\alpha = (A + D)/2$, $\beta = D - C$ and $\delta = -A + C = -B + D$. Then, we get:

$$\begin{array}{ll} 2A = 2\alpha - \delta - \beta & 2C = 2\alpha - \beta + \delta \\ 2D = 2\alpha + \beta + \delta & 2B = 2D - 2\delta = 2\alpha + \beta - \delta. \end{array}$$

7. Verify the output, up to a global phase, of the following quantum network.



Solution:

$$\begin{array}{llll} H: & |0\rangle & \mapsto & \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right). \\ 2\theta: & \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) & \mapsto & \left(\begin{smallmatrix}1 & 0\\0 & e^{2i\theta}\end{smallmatrix}\right)\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\e^{2i\theta}\end{smallmatrix}\right) = e^{i\theta}\left(\begin{smallmatrix}e^{-i\theta}\\e^{i\theta}\end{smallmatrix}\right). \\ H: & \left(\begin{smallmatrix}e^{-i\theta}\\e^{i\theta}\end{smallmatrix}\right) & \mapsto & \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}e^{-i\theta}+e^{i\theta}\\e^{-i\theta}-e^{i\theta}\end{smallmatrix}\right) = \sqrt{2}\left(\begin{smallmatrix}\cos\theta\\-i\sin\theta\end{smallmatrix}\right). \\ \phi+\pi/2: & \left(\begin{smallmatrix}\cos\theta\\-i\sin\theta\end{smallmatrix}\right) & \mapsto & \left(\begin{smallmatrix}\cos\theta\\-ie^{i(\phi+\pi/2)}\sin\theta\end{smallmatrix}\right) = \left(\begin{smallmatrix}\cos\theta\\e^{i\phi}\sin\theta\end{smallmatrix}\right). \end{array}$$