Domain theory and Continuous Data Types// Exercises 3

1. Prove or disprove by giving a counterexample: For any two non-empty bounded subsets $A, B \subseteq \mathbb{R}^n$, we have $d_H(A, B) = 0$ only if $A = B$.

2. Given any chain $a_0 \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq \ldots$ in $(\mathsf{Str}(\Sigma), \sqsubseteq)$ find $d(\bigsqcup_{i\geq 0} a_i, a_j)$ for each j.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n = 1, 2, 3)$. Suppose $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ is a decreasing sequence of subsets. Show that $f(\bigcap_{i\geq 0} A_i) \subseteq \bigcap_{i\geq 0} f(A_i)$.

4. Find the attractor of the IFS $\{f_1, f_2\}$ in $\mathbb{R}$, where, for $x \in \mathbb{R}$, $f_1(x) = ax$ and $f_2(x) = (1-a)x + a$, with $0 < a < 1$.

5. Repeat question 4 with f_1, f_2 given by $f_1(x) = 0$ and $f_2(x) = \frac{2}{3}x + \frac{1}{3}$.

6. Describe the approximation sequence in the generation of the Koch curve starting with a fixed point of one of its generating maps. What happens if you start with a set consisting of two fixed points? Do the same for the Sierpinski triangle and the Cantor set.

7. Show that, given an IFS $\{f_1, f_2, \ldots, f_N\}$ such that f_i has a contracting factor s_i, the map $f : \mathcal{P}(\mathbb{R}^m) \to \mathcal{P}(\mathbb{R}^m)$ with $f(A) = \bigcup_{1\leq i\leq N} f_i(A)$ has a contracting factor $s = \max_{1\leq i\leq N} s_i$.

8*. Let $\mathcal{H}^n$ be the set of all non-empty bounded closed subsets of $\mathbb{R}^n$ and let $A \in \mathcal{H}^n$, say a large closed ball around the origin. Suppose $\mathcal{H}^n(A) = \{B \in \mathcal{H}^n \mid B \subseteq A\}$.

(i) Show that $(\mathcal{H}^n(A), \supseteq)$ is a dcpo.

(ii) Show that for $B, C \in \mathcal{H}^n(A)$, we have

$$B \ll C \iff \exists r > 0 \text{ with } B \supseteq C_r$$

where C_r is the r-parallel body of C as in the notes.

(iii) Show that $(\mathcal{H}^n(A), \supseteq)$ is a continuous dcpo.

(iv) Show that $(\mathcal{H}^n(A), \supseteq)$ is in fact ω-continuous.