Quantum Computing - Exercises 4

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1. Show that if for a single qubit unitary operation U we have $U = V^2$ where V is another single qubit unitary operation, then the double controlled U gate, called $\mathsf{C}^2(U)$ gate, can be implemented as in Figure 1. Show that for $V = (1-i)(I+iX)/2$ this implements the Toffoli gate.

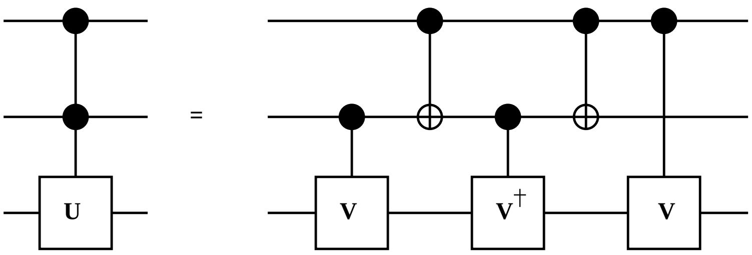

Figure 1: Implementation of $C^2(U)$

2. Check that $XYX = -Y$ and $XZX = -Z$; then show that $XR_y(\theta)X = R_y(-\theta)$ and $XR_z(\theta)X = R_z(-\theta)$.

3. Assuming that the the single qubit operation U is given as in the lecture notes (page 58) by $U = e^{i\alpha}AXBXC$ with $ABC = I$, show that the network in Figure 2 implements the controlled U gate, $\mathsf{C}(\mathsf{U})$, using the single qubit gate X the phase gate α and the single qubit gates A, B and C.

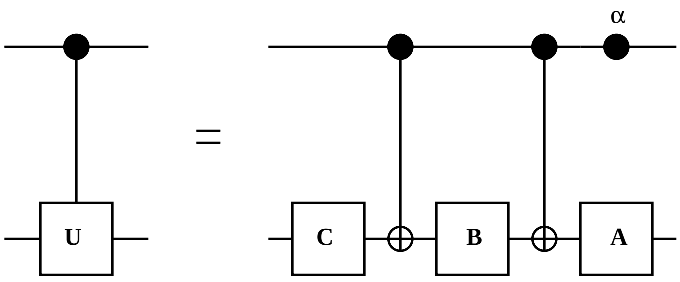

Figure 2: Implementation of $C(U)$

4. Show that the $\mathsf{C^n}(\mathsf{U})$ gate, where U is single qubit unitary operation, can be implemented using $n-1$ work qubits with input $|0\rangle$ as in the network in

Figure 3, depicted for $n = 5$. Deduce that any such $\mathsf{C^n(U)}$ gate can be implemented by $O(n)$ single qubit and CNOT gates.

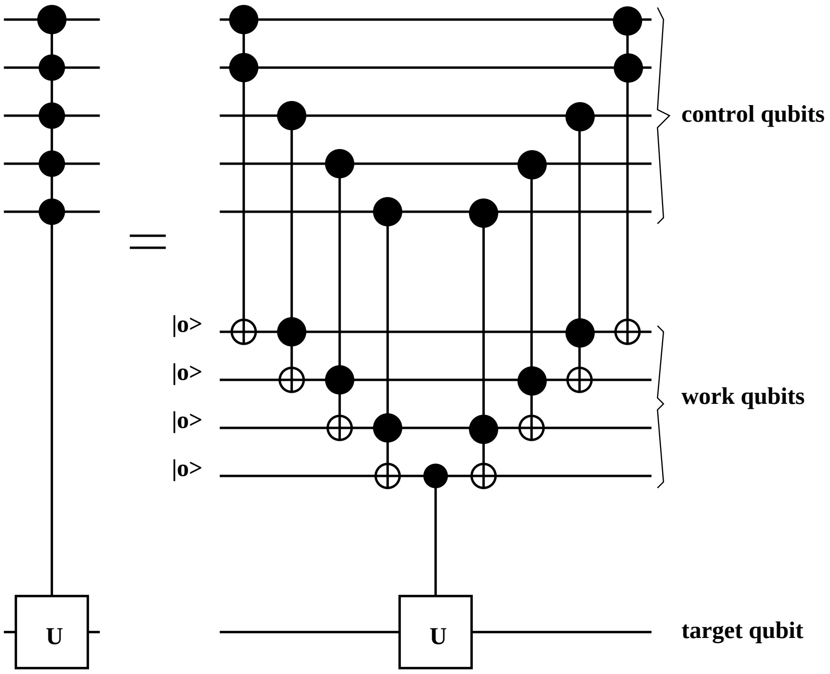

Figure 3: Implementation of $C^n(U)$

5. Let U be the two-level matrix

$$U = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & d \end{pmatrix}$$

Then the the non-trivial levels correspond to $s = 000$ and $t = 111$.

(i) Check that if we use the Gray Code sequence $000 \to 001 \to 011 \to 111$, then from Algorithm 4.9 in the notes (page 54), we obtain the network in Figure 4.

(ii) Check directly that the network in Figure 4 actually implements U.

(iii) What is the matrix for the change of basis corresponding to the two controlled gates in the network in Figure 4?

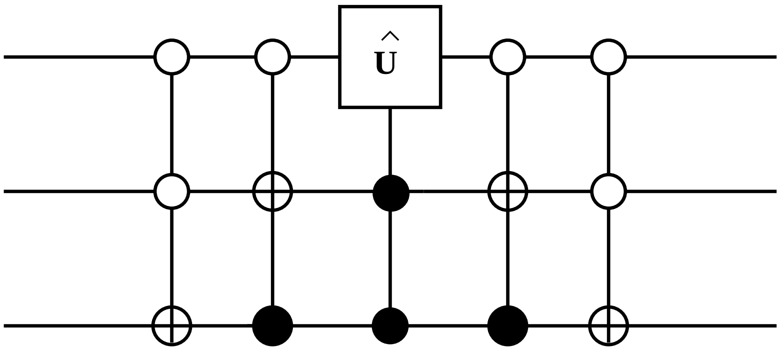

Figure 4: Network for a two-level unitary operation