Domain theory and Continuous Data Types
Exercises 4

1. Let $f : D \to D$ be a continuous function on the cpo D. Show that the set of fixed points of f, with the order inherited from D, is itself a cpo.

2. Let $\mathsf{Bool}_\perp$ be the poset:

$$\begin{tikzcd} \mathsf{tt} & & \mathsf{ff} \\ & \perp \arrow[ul] \arrow[ur] & \end{tikzcd}$$

For each of the following functions $f : \mathsf{Str}(\mathsf{Bool}_\perp) \to \mathsf{Bool}_\perp$ determine if f is (a) monotone, (b) continuous.

$$f(a) = \begin{cases} \mathsf{tt} & \text{if } a \text{ contains at least one } \mathsf{tt} \\ \mathsf{ff} & \text{otherwise} \end{cases}$$

$$f(a) = \begin{cases} \mathsf{tt} & \text{if } a \text{ contains at least one } \mathsf{tt} \\ \perp & \text{otherwise} \end{cases}$$

$$f(a) = \begin{cases} \mathsf{tt} & \text{if } a \text{ contains infinitely many } \mathsf{tt}\text{'s} \\ \perp & \text{otherwise} \end{cases}$$

3. Consider the following recursive definition of a subset $S \subseteq \Sigma^*$ (with $E \in S$ and $a, b \in \Sigma$):

$$E \;=\; \varepsilon \mid ab \mid EEE.$$

Construct a continuous function $f : (\mathcal{P}(\Sigma^*), \subseteq) \to (\mathcal{P}(\Sigma^*), \subseteq)$ such that S is the least fixed point of f. Does f have any other fixed points?

4. Let $f \in D \to E$ and $g \in E \to F$ be continuous functions on cpo's. Show that the composition $g \circ f : D \to F$ defined by $g \circ f(d) = g(f(d))$ is again a continuous function.

5. Check that the function F in the definition of the factorial function (Example 3.8 page 54) is continuous.

6. Show that fix is a *uniform* fixed point operator, i.e. check that if $f : D \to D$ and $g : E \to E$ are continuous functions on cpo's D and E and if $h : D \to E$ is a strict continuous function with $h \circ f = g \circ h$, then $\mathsf{fix}(g) = h(\mathsf{fix}(f))$.

7. Show that equality of real numbers is semi-decidable, in the sense that if two real numbers are not equal then this can be verified in finite time. How about $\leq$ and $<$?

8. Show that the composition of two lft's is represented by the product of the two matrices representing them.

9*. A map $f : D \to E$ of dcpo's D and E is said to be *continuous* if it is monotone and preserves lubs of directed sets. f is said to be *Scott-continuous* if it is continuous with respect to the Scott topologies on D and E, i.e. if $f^{-1}(O) \subseteq D$ is a Scott open subset of D for all Scott open subsets $O \subseteq E$ of E. Show that f is continuous iff it is Scott continuous. (Therefore, the two notions of continuity coincide.)