Quantum Computing - Solutions 5

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1. Check that the output of the network in Figure 1 is indeed as in the table on its left up to the normalization factor $1/\sqrt{2}$.

in	**out**
|00>	**(|00> + |11>)**
|01>	**(|01> + |10>)**
|10>	**(|00> – |11>)**
|11>	**(|01> – |10>)**

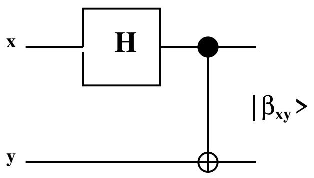

Figure 1: The Bell or EPR states

Solution:

$$
\begin{aligned}
|00\rangle &\mapsto \tfrac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \mapsto \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\\
|01\rangle &\mapsto \tfrac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \mapsto \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\\
|10\rangle &\mapsto \tfrac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \mapsto \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\\
|11\rangle &\mapsto \tfrac{1}{\sqrt{2}}(|01\rangle - |11\rangle) \mapsto \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}
$$

2. In the quantum teleportation network of Figure 2 (see page 61 in the notes), measurement of the first two qubits by Alice will collapse Bob's qubit as follows:

$$00 \mapsto |\psi_3(00)\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$01 \mapsto |\psi_3(01)\rangle = \alpha|1\rangle + \beta|0\rangle$$

$$10 \mapsto |\psi_3(10)\rangle = \alpha|0\rangle - \beta|1\rangle$$

$$11 \mapsto |\psi_3(11)\rangle = \alpha|1\rangle - \beta|0\rangle$$

Alice communicates her two bits mn with Bob over a classical channel. Bob will then send his qubit through the circuit $X^n Z^m$ where

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Check that the final result $|\psi_4\rangle$ is indeed the state $|\psi_4\rangle = |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

Solution:

$$
\begin{aligned}
&X^0 Z^0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\\
&X^1 Z^0 \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = X \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\\
&X^0 Z^1 \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = Z \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\\
&X^1 Z^1 \begin{pmatrix} -\beta \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}
\end{aligned}
$$

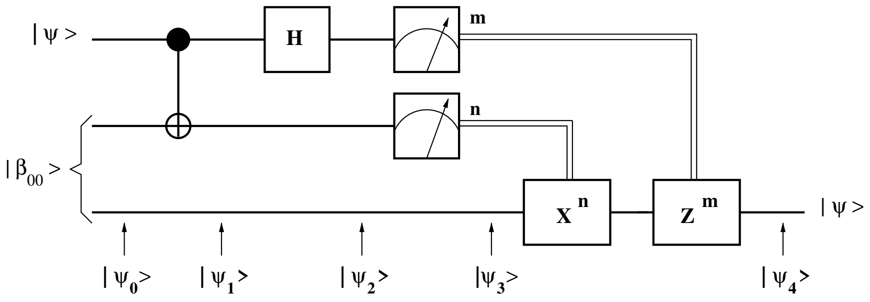

Figure 2: Quantum teleportation

3. Let $f : \{0,1\} \to \{0,1\}$ be a Boolean function which outputs a bit for each input bit and let $U_f : |x, y\rangle \mapsto |x, y \oplus f(x)\rangle$. Show that U_f is unitary.

Solution: We have shown in general in Section 4.4 that such a transformation is unitary. Here is a direct proof. The matrix for U_f is easily seen to be:

$$\begin{pmatrix} 1-f(0) & f(0) & 0 & 0 \\ f(0) & 1-f(0) & 0 & 0 \\ 0 & 0 & 1-f(1) & f(1) \\ 0 & 0 & f(1) & 1-f(1) \end{pmatrix}$$

which is unitary since it is a permutation matrix i.e. there is exactly one non-zero entry namely 1 in each row and these non-zero entries belong to distinct columns. Hence each column is a unit vector and the different column vectors are orthogonal.

4. More generally, let $f : \{0,1\}^n \to \{0,1\}$ and $U_f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ with $U_f : |x, y\rangle \mapsto |x, y \oplus f(x)\rangle$. Now U_f has $n+1$ input qubits, the first n are input to f; it has also $n+1$ output qubits, the first n qubits are identical with the first n qubits of the input as in Figure 3. Check that for any integer n the operator U_f is a unitary transformation.

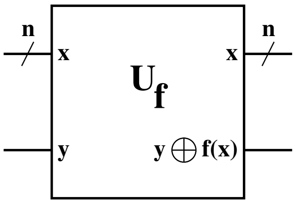

Figure 3: A Gate for Parallel Computation

Solution: Again this follows from in general from Section 4.4. We can generalise the proof in the previous question to have a direct proof as follows.

For each $x = x_1 \cdots x_n$ we have the two possible input qubits $x0$ and $x1$, which correspond to two adjacent rows of U_f. The action of U_f on the these two basis vectors is to either leave them unchanged or swap them. Hence, the matrix U_f has the following two-by-two submatrix

$$\begin{pmatrix} 1-f(x) & f(x) \\ f(x) & 1-f(x) \end{pmatrix}$$

in the $x0$ and $x1$ row and column positions. Therefore, U_f induces a permutation of basis vectors and is thus unitary.

5. Show that

$$H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle.$$

Solution: We have:

$$H^{\otimes n}|0\rangle^{\otimes n} = (H|0\rangle)^{\otimes n} = \frac{1}{\sqrt{2^n}}(|0\rangle + |1\rangle)^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x_1, x_2, \cdots, x_n \in \{0,1\}} |x_1 x_2 \cdots x_n\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle.$$

6. Show that

$$\begin{array}{rcl} H|x\rangle & = & \sum_{y \in \{0,1\}} (-1)^{xy} |y\rangle / \sqrt{2} \\ H^{\otimes n}|x\rangle & = & \sum_{y \in \{0,1\}^n} (-1)^{x.y} |y\rangle / \sqrt{2^n} \end{array}$$

where $x.y$ is the bitwise inner product of x and y modulo 2.

Solution: The first equality follows immediately by checking it for $x = 0$ and $x = 1$. As for the second, let $x = x_1 x_2 \cdots x_n$. Then by the first equality we can write:

$$H|x_i\rangle = \sum_{y_i \in \{0,1\}} (-1)^{x_i y_i} |y_i\rangle / \sqrt{2}.$$

Therefore,

$$H^{\otimes n}|x\rangle = \bigotimes_{i=1}^{n} H|x_i\rangle = \bigotimes_{i=1}^{n} \sum_{y_i \in \{0,1\}} (-1)^{x_i y_i} |y_i\rangle / \sqrt{2}$$

$$= \sum_{y \in \{0,1\}^n} \frac{1}{\sqrt{2^n}} (-1)^{\sum_{i=1}^{n} x_i y_i} |y\rangle = \sum_{y \in \{0,1\}^n} (-1)^{x.y} |y\rangle / \sqrt{2^n}.$$