Domain theory and Exact computation
Exercises 5- Solutions

1. This number satisfies $x = 1 + \frac{2}{x}$, i.e. $x^2 - x - 2 = 0$. The only solutions to this equation are 2 and -1. As the number expressed by the continued fraction is non-negative, the solution is 2. The associated normal product is

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \cdots$$

hence the intervals are $I_0 = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix}\right)[0,\infty] = [1,\infty]$, $I_1 = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix}\right)[0,\infty] = \left(\begin{smallmatrix} 3 & 2 \\ 1 & 2 \end{smallmatrix}\right)[0,\infty] = [1,3]$ and $I_2 = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix}\right)[0,\infty] = \left(\begin{smallmatrix} 3 & 2 \\ 1 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix}\right)[0,\infty] = \left(\begin{smallmatrix} 5 & 6 \\ 3 & 2 \end{smallmatrix}\right)[0,\infty] = [1.66\ldots,3]$.

2. $D_{-1}[0,\infty] = [0,1]$, $D_0[0,\infty] = [\frac{1}{3},3]$ and $D_1[0,\infty] = [1,\infty]$.

For contractivity note that $\frac{a-1}{a+1} = 1 - \frac{2}{a+1}$, hence

$$\rho(x,y) = \left|\frac{2}{x+1} - \frac{2}{y+1}\right|$$

for x and y both non-negative or both non-positive. So we calculate for $x, y \in [0,\infty]$

$$\begin{aligned}
\rho(D_{-1}(x), D_{-1}(y)) &= \rho(\frac{x}{x+2}, \frac{y}{y+2}) \\
&= \left|\frac{2}{\frac{x}{x+2}+1} - \frac{2}{\frac{y}{y+2}+1}\right| \\
&= \left|\frac{x+2}{x+1} - \frac{y+2}{y+1}\right| \\
&= \left|1 + \frac{1}{x+1} - (1 + \frac{1}{y+1})\right| \\
&= \left|\frac{1}{x+1} - \frac{1}{y+1}\right|
\end{aligned}$$

$$= \frac{1}{2}\rho(x,y)$$

The normal products D^{ω}_{-1}, D^{ω}_0 and D^{ω}_1 represent 0, 1 and ∞, respectively, fixed points of D_{-1}, D_0 and D_1.

3. To show that the right rule corresponds to composition from the right assume that $f : \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}^*$ where

$$f(x,y) = \frac{axy + cx + ey + g}{bxy + dx + fy + h}$$

and let $g : \mathbb{R}^* \to \mathbb{R}^*$ with $g(y) = \frac{my+p}{ny+q}$. We now compute

$$f(x,g(y)) = \frac{ax\frac{my+p}{ny+q} + cx + e\frac{my+p}{ny+q} + g}{bx\frac{my+p}{ny+q} + dx + f\frac{my+p}{ny+q} + h}$$

$$= \frac{(am+cn)xy + (ap+cq)x + (em+gn)y + ep + gq}{(bm+dn)xy + (bp+dq)x + (fm+hn)y + fp + hq}$$

$$= \frac{a'xy + c'x + e'y + g'}{b'xy + d'x + f'y + h'}$$

where $\begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} m & p \\ n & q \end{pmatrix}$ and $\begin{pmatrix} e' & g' \\ f' & h' \end{pmatrix} = \begin{pmatrix} e & g \\ f & h \end{pmatrix}\begin{pmatrix} m & p \\ n & q \end{pmatrix}$ as required. The left rule is similar.

4. From the unp for $\sqrt{2}$ on page 61, we get

$$\sqrt{2}/2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\cdots.$$

$$= \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\cdots$$

We thus evaluate the following truncated uext, which we call E, with the rules for absorption and emission, by first absorbing (as specified) the matrix $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ into the two tensors:

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}(\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}[0,\infty], \begin{pmatrix} 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 3 \end{pmatrix}(\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}[0,\infty],[0,\infty]))$$

$$= \begin{pmatrix} 0 & 3 & 0 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix}([0,\infty], \begin{pmatrix} 0 & 3 & 0 & 1 \\ 12 & 12 & 4 & 6 \end{pmatrix}([0,\infty],[0,\infty]))$$

The information contained in the second tensor is $[0,1/4]$ and thus we can emit the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}$ from it. Note that after this emission the second tensor becomes

$$\begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 3 & 0 & 1 \\ 12 & 12 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 3 & 0 & 1 \\ 12 & 12 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 12 & 0 & 4 & 2 \\ 0 & 3 & 0 & 1 \end{pmatrix},$$

(where matrix algebra has been performed up to scaling by a non-zero multiplicative factor) giving an unsigned tensor as expected. Absorbing the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}$ from right into the root tensor we get:

$$\begin{aligned} E &= \begin{pmatrix} 0 & 3 & 0 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix} ([0, \infty], \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix} [0, \infty]) \\ &= \begin{pmatrix} 3 & 12 & 1 & 4 \\ 4 & 19 & 2 & 9 \end{pmatrix} ([0, \infty], [0, \infty]), \end{aligned}$$

and the information in the new root tensor is $[\min(3/4, 12/19, 1/2, 4/9), \max(3/4, 12/19, 1/2, 4/9)] = [4/9, 3/4]$ which is the required interval.