Quantum Computing - Solutions 6

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1. Consider Grover's algorithm as in Section 7 of the lecture notes.

   (i) Show that
$$(2|\psi\rangle\langle\psi| - I)(\sum_x \alpha_x|x\rangle) = \sum_x(-\alpha_x + 2\langle\alpha\rangle)|x\rangle,$$
   where $\langle\alpha\rangle = \sum_x \alpha_x/N$.

   (ii) Explain why the operation $2|\psi\rangle\langle\psi| - I$ in Grover's algorithm is called *inversion about mean*.

Solution: (i) We have: $|\psi\rangle = \frac{1}{\sqrt{N}}\sum_y |y\rangle$ and hence $\langle\psi| = \frac{1}{\sqrt{N}}\sum_y\langle y|$. Therefore:

$$(2|\psi\rangle\langle\psi| - I)(\sum_x \alpha_x|x\rangle) = \frac{2}{N}((\sum_y |y\rangle)(\sum_y\langle y|) - I)(\sum_x \alpha_x|x\rangle)$$

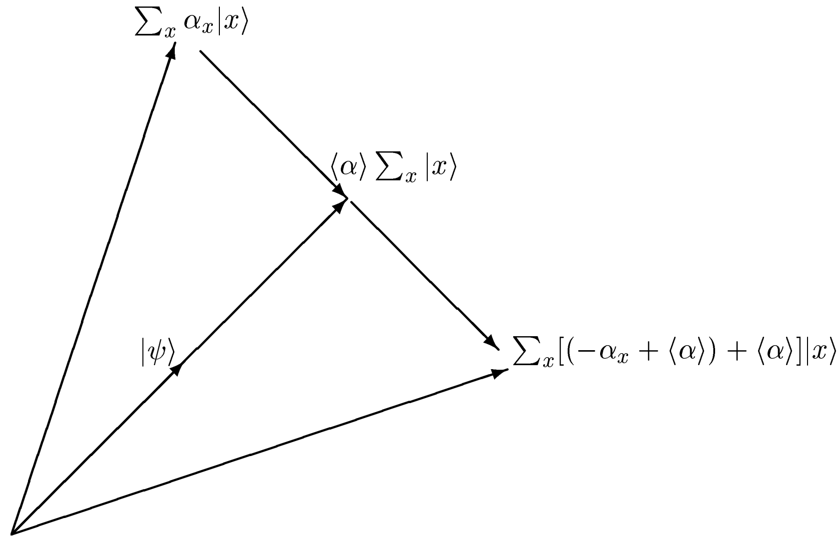
$$= \frac{2}{N}(\sum_y |y\rangle)\sum_x \alpha_x - \sum_x \alpha_x|x\rangle = \sum_x(-\alpha_x + 2\langle\alpha\rangle)|x\rangle.$$

(ii) From part (i), we have:

$$(2|\psi\rangle\langle\psi| - I)(\sum_x \alpha_x|x\rangle) = \sum_x(-\alpha_x + 2\langle\alpha\rangle)|x\rangle$$

$$= \sum_x[(-\alpha_x + \langle\alpha\rangle) + \langle\alpha\rangle]|x\rangle.$$

The two terms in the bracket is the difference between the mean and the state $\sum_x \alpha_x|x\rangle$; hence after adding this to the mean we have inverted the state $\sum_x \alpha_x|x\rangle$ about the mean. It is also easy by direct calculation to show that the two vectors $\sum_x(-\alpha_x + \langle\alpha\rangle)|x\rangle$ and $\langle\alpha\rangle\sum_x|x\rangle$ are orthogonal; hence the output is actually reflection about the direction $\sum_x|x\rangle$ (which is the same as the direction of $|\psi\rangle = \frac{1}{\sqrt{N}}\sum_y|y\rangle$).



2. Show that, in Grover's algorithm, the action of $2|\psi\rangle\langle\psi| - I$ in the $|\sigma\rangle$ and $|\tau\rangle$ plane is reflection about $|\psi\rangle$.

Solution: This follows from the solution to Problem 1 above. Here is a more geometric alternative proof. Let $|x\rangle$ be an arbitrary state. Consider $|y\rangle = |x\rangle - \langle\psi|x\rangle|\psi\rangle$. Then $\langle\psi|y\rangle = \langle\psi|x\rangle - \langle\psi|x\rangle\langle\psi|\psi\rangle = 0$. Hence $|y\rangle$ is orthogonal to $|\psi\rangle$. On the other hand, $\langle\psi|x\rangle|\psi\rangle$ is the projection of $|x\rangle$ in the state $|\psi\rangle$. We have:

$$(2|\psi\rangle\langle\psi| - I)|x\rangle = (2|\psi\rangle\langle\psi| - I)(\langle\psi|x\rangle|\psi\rangle + |y\rangle) = \langle\psi|x\rangle|\psi\rangle - |y\rangle.$$

Thus, the action of the operator is reflection about $|\psi\rangle$.

3. Show that if we change the computational basis so that $|\sigma\rangle$ and $|\tau\rangle$ are basis elements, then the matrix representation of the Grover's transform G will be;

$$G_{\sigma,\tau} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $\theta/2$ is the angle between $|\sigma\rangle$ and $|\psi\rangle$.

Solution: The action of the oracle O is reflection about $|\sigma\rangle$. By problem (3), the action of $2|\psi\rangle\langle\psi| - I$ in the $|\sigma\rangle$ and $|\tau\rangle$ plane is reflection about $|\psi\rangle$. The composition of two reflections is rotation by twice the angle between the axes of reflection. Hence, the action of $G = (2|\psi\rangle\langle\psi| - I)O$ is rotation by $2(\theta/2) = \theta$. But the matrix for rotation by θ is precisely

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

In fact any vector in the real plane of $|\sigma\rangle$ and $|\tau\rangle$ is of the form $\left(\begin{smallmatrix} \cos\gamma \\ \sin\gamma \end{smallmatrix}\right)$, where γ is the angle made from σ. A simple calculation with elementary trigonometry shows:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix} = \begin{pmatrix} \cos(\gamma+\theta) \\ \sin(\gamma+\theta) \end{pmatrix},$$

and hence the state vector has rotated by θ.

4. Coursework.

5. What is the matrix for the quantum Fourier transform F when $n = 1$ and $n = 2$? Show that F is unitary for any n.

Solution: In general for $N = 2^n$ we have: $(F_{kj}) = \frac{1}{\sqrt{N}} e^{2\pi ijk/N}$. For $n = 1$ we get: $F = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. For $n = 2$, we have:

$$F = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

The circuit in Figure 33 (page 87 of Lecture Notes) shows that F is unitary as it has is the composition of unitary matrics (Hadamard and controlled phase). Alternatively, we can show that the columns of the matrix (F_{kj}) form an orthonormal set of vectors.