Domain theory and Exact computation
Exercises 6- Solutions

1. (i) The boundary (in both cases) is the unit sphere $\{x \in \mathbb{R}^n \mid |x| = 1\}$.

(ii) Note that I_n is closed as it is the finite union of closed intervals (see page 23). Hence $C = \bigcap_n I_n$ is the intersection of closed sets and is closed. The boundary of C is C itself since any open set containing any point of C contains open intervals in the complement of C.

2. Let O be open then since O^c is closed we see that O is upward closed. Let $A \subseteq D$ be a directed subset with $\bigsqcup A \in O$, then $A \cap O \neq \emptyset$, since otherwise $A \subseteq O^c$ implies $\bigsqcup A \in O^c$ as O^c is closed. On the other hand if O satisfies the two stated properties then O^c is downward closed and closed under lubs of directed sets and is thus closed. Hence O is open.

3. (i) Suppose $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} x_n = b$ where $(x_n)_{n\in\geq 0}$ is a sequence in $\mathbb{R}^n$. Assume $a \neq b$, then each of the open balls $O(a, |a-b|/2)$ and $(b, |a-b|/2)$ contain all but a finite number of elements of the sequence, which is absurd. Let $A = \{\bot, \top\}$ be the two element cpo with $\bot \sqsubseteq \top$. Then the sequence $(x_n)_{n\geq 0}$ with $x_n = \top$ for all $n \geq 0$ has limits $\bot$ and $\top$.

(ii) Suppose C is closed and let $\lim_{n\to\infty} x_n = x$, with $x_n \in C$ for all $n \geq 0$. If $x \in C^c$ then, by definition of x as a limit, all but a finite number of the elements of the sequence must belong to the open set C^c, which is a contradiction.

(iii) Take the close set $\{x \in \mathbb{R} | \, x \geq 0\}$. We already know that $x \geq 0$ is not decidable.

4. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous with respect to the Euclidean topology and let $\lim_{n\to\infty} x_n = x$. Suppose $O \subseteq \mathbb{R}^n$ with $f(x) \in O$. By definition of continuity there exists an open set U with $x \in U$ such that $f(U) \subseteq O$. By definition of limit, there exists $N \geq o$ such that $x_n \in U$ for all $n \geq N$. Hence, $f(x_n) \in O$ for all $n \geq N$ and thus $\lim_{n\to\infty} f(x_n) = f(x)$.

Assume now that whenever $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} f(x_n) = f(x)$. Suppose f is not continuous at x. Then there exists open set $O \subseteq \mathbb{R}^n$ such that $f(O(x, 1/n)) \not\subseteq$ for all $n \in \mathbb{N}$. Hence, there exists $x_n \in O(x, 1/n)$ such that $f(x_n) \notin O$. But this means that $\lim_{n\to\infty} x_n = x$ but $f(x_n)$ does not converge to $f(x)$, a contradiction.

5. Let $|x| = 1$, then $\epsilon_D(x) = \mathsf{tt}$ but any open set U containing x will intersect D^c (as well as D) and thus $\epsilon_D(U) = \{\mathsf{tt}, \mathsf{ff}\}$. This shows that ϵ_D is not continuous at x.

On the other hand ϵ'_D is continuous since ${\epsilon'_D}^{-1}(\{\mathsf{tt}\}) = S^\circ$ and ${\epsilon'_D}^{-1}(\{\mathsf{ff}\}) = (S^c)^\circ$ are both open.