Domain theory and Continuous Data Types
Exercises 7

1. Show that a function $f : X \to Y$ of topological spaces is continuous iff the pre-image of every open set of Y is an open set of X.

2. Show that the interior of a closed set is a regular open set.

3. Define the m-ary union and intersection on the domain of geometric objects and show that they are continuous.

4. Draw the inner and outer convex hulls of the following three rectangles.

$$\begin{aligned} R_1 &= \{(-2,0),(-1,0),(-1,-1),(-2,-1)\} \\ R_2 &= \{(-1,3),(0,3),(0,2),(-1,2)\} \\ R_3 &= \{(1,1),(2,1),(2,0),(1,0)\}. \end{aligned}$$

5. Given any sequence $(x_n)_{n\geq 0}$ and any strictly increasing sequence of natural numbers $(n_i)_{i\geq 0}$, the sequence $(x_{n_i})_{i\geq 0}$ is called a *subsequence* of $(x_n)_{n\geq 0}$. Show that any sequence in a bounded closed subset of $\mathbb{R}^n$ has a convergent subsequence. (*Hint: The sequence, $(x_n)_{n\geq 0}$ say, is contained in a cube in $\mathbb{R}^n$. Let $n_0 = 0$. Divide this cube into 2^n equal sub-cubes. Then, at least one of these sub-cubes contains an infinite number of elements of $(x_n)_{n\geq 0}$; take $n_1 > n_0$ with x_{n_1} in such a sub-cube. Repeat recursively. If $n_i > n_{i-1} > \cdots > n_0$ have been selected, divide the current sub-cube into 2^m equal sub-cubes. Then, at least one of these sub-cubes contains an infinite number of elements of $(x_n)_{n\geq 0}$; take $n_{i+1} > n_i$ with $x_{n_{i+1}}$ in such a sub-cube. Finally show that $(x_{n_i})_{i\geq 0}$ converges.*)

6. Use Problem 5 to show that if C is a bounded closed subset of $\mathbb{R}^n$ with $C \subset \bigcup_{i\geq 0} O_i$ where $(O_i)_{i\geq 0}$ is an increasing sequence of open subsets of $\mathbb{R}^n$, then there exists $i \geq 0$ with $C \subset O_i$.

7. Consider the domain $\mathbf{S}([0,1]^n)$ of geometric objects in $[0,1]^n$ and the predicate $-R- : \mathbf{S}([0,1]^n) \times \mathbf{S}([0,1]^n) \to \{\mathtt{tt}, \mathtt{ff}\}_\perp$, defined by

$$(A,B)R(C,D) = \begin{cases} \mathtt{tt} & \text{if } B \cup C = [0,1]^n \\ \mathtt{ff} & \text{if } A \cap D \neq \emptyset \\ \perp & \text{otherwise} \end{cases}$$

What predicate does $-R-$ represent on the maximal elements? Show that $-R-$ is continuous as a predicate on the cpo $\mathbf{S}([0,1]^n) \times \mathbf{S}([0,1]^n)$.