Quantum Computing - Solutions 7

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1.

(i) Work out the matrix for the quantum Fourier transform F and the network which implements it for $n = 3$.

(ii) Show that the inverse quantum Fourier transform is given by:

$$F^{\dagger} : |j\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{-2\pi ijk/2^n} |k\rangle. \tag{1}$$

(iii) Work out the circuit for the inverse of the Fourier transform.

Solution: (i) $(F)_{kj} = \frac{1}{\sqrt{8}} e^{2\pi ikj/8}$. Let $a = e^{\pi i/4} = (1+i)/2$, then $a^2 = i$, $a^3 = ia = (-1+i)/2$, $a^4 = -1$, $a^5 = -a$, $a^6 = -i$, $a^7 = -ia$ and we have:

$$F = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a & i & ia & -1 & -a & -i & -ia \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & ia & -i & a & -1 & -ia & i & -a \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -a & i & -ia & -1 & a & -i & ia \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -ia & -i & -a & -1 & ia & i & a \end{pmatrix}$$

(ii) Since F is a symmetric matrix its adjoint is simply the complex conjugate of F. Hence $(F^{-1})_{kj} = (F^{\dagger})_{kj} = (F^*)_{jk} = (F^*)_{kj} = \frac{1}{\sqrt{2^n}} e^{-2\pi ijk/2^n}$.

(iii) Since $(AB)^{-1} = B^{-1}A^{-1}$, the circuit is the same as the circuit for F but reversed with all the gates replaced by their adjoints. We have $H^{\dagger} = H$ and $R_k^{\dagger} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & e^{-2\pi i/2^k} \end{smallmatrix}\right)$.

2.

(i) Find the eigenvalues and eigenvectors of the Pauli matrices, X, Y, Z. Locate the eigenvectors on the Bloch sphere.

(ii) Show that the eigenvalues of a unitary matrix have norm one.

(iii) Show that a change of basis does not alter the eigenvalues of a matrix. How do the eigenvectors alter with a change in the basis?

Solution:

(i)

- X has eigenvalues 1, with normalized eigenvector $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and -1, with normalized eigenvector $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.
- Y has eigenvalues 1, with normalized eigenvector $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, and -1, with normalized eigenvector $\frac{1}{\sqrt{2}}(i|0\rangle + |1\rangle)$.
- Z has eigenvalues 1, with normalized eigenvector $|0\rangle$, and -1, with normalized eigenvector $|1\rangle$.

(ii) Let U be a unitary matrix with eigenvalue a corresponding to normalized eigenvector v. Since U preserves unit vectors, we have $\|U(v)\| = \|v\| = 1$. Hence $1 = \|U(v)\| = \|av\| = |a|\|v\| = |a|$.

(iii) Let A have eigenvalue a with eigenvector v. Under a change of basis with matrix B, the transformed matrix BAB^{-1} will have eigenvalue a with eigenvector Bv since: $BAB^{-1}(Bv) = BAv = B(av) = aBv$.

3. Diagonalize the three Pauli matrices and the Hadamard matrix.

Solution: From Exercise 2 above, we immediately see that X, Y and Z have diagonal representation $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ respectively in the basis given by:

(i) $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

(ii) $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ and $\frac{1}{\sqrt{2}}(i|0\rangle + |1\rangle)$.

(iii) $|0\rangle$ and $|1\rangle$.