Domain theory and Exact computation
Exercises 7- Solutions

1. Suppose f is continuous and $O \subseteq Y$ is open. To show that $f^{-1}(O)$ is open let $x \in f^{-1}(O)$. Since $f(x) \in O$, by continuity of f at x there exists an open set U of X with $x \in U$ and $f(U) \subseteq O$, which implies $U \subseteq f^{-1}(O)$. Thus $f^{-1}(O)$ is open. Conversely, suppose the pre-image of every open set of Y is an open set of X. Let $x \in X$ and let O be an open subset of Y, with $f(x) \in O$. Then $f^{-1}(O)$ is open by assumption, and $x \in f^{-1}(O)$ with $f(f{-}1(O)) \subseteq O$.

2. Let C be a closed set. We want to prove that $\overline{C^\circ}^\circ = C^\circ$. Since $C^\circ \subseteq C$ we have $\overline{C^\circ} \subseteq \overline{C} = C$. Hence $\overline{C^\circ}^\circ \subseteq C^\circ$. To show the reverse relation, let $x \in C^\circ$. Then there exists an open ball $)(x,r) \subseteq C = \overline{C}$. But this means that $x \in \overline{C^\circ}^\circ$.

3. Coursework.

4. An done in the class.

5. Following the hint given, we only need to show that $(x_{n_i})_{i \geq 0}$ converges. We have a shrinking nested sequence of subcubes where the nth element of the sequence has edges of length $k/2^n$ where k is the length of the original cube. Hence the intersection is a singleton point x say. Any open set containing x will contain one of the sub-cubes of the sequence of the subcubes and thus all but a finite number of $(x_{n_i})_{i \geq 0}$. Hence $\lim_{i \to \infty} x_{n_i} = x$. Note that since C is closed we have $x \in C$.

6. *Use Problem 5 to show that if C is a bounded closed subset of $\mathbb{R}^n$ with $C \subset \bigcup_{i \geq 0} O_i$ where $(O_i)_{i \geq 0}$ is an increasing sequence of open subsets of $\mathbb{R}^n$, then there exists $i \geq 0$ with $C \subset O_i$.*

Solution: Suppose not. Thus for each $n \geq 0$ there exists $x_n \in C \setminus O_n$. Since O_n is increasing, we have $x_n \notin O_m$ for $n \geq m$. The sequence $(x_n)_{n \geq 0}$ has its elements in the closed bounded set C and thus has a convergent subsequence $(x_{n_i})_{i \geq 0}$ with $\lim_{i \to \infty} x_{n_i} = x \in C$ say. Then $x \in \bigcup_{i \geq 0} O_i$ implies that there exists $i \geq 0$ with $x \in O_i$. By the definition of limit, there exists $N \geq 0$ such that $n \geq N$ implies $x_n \in O_i$ but for $n = \max(i, N)$ this contradicts $x_n \notin O_i$.

7. Coursework.