Domain theory and Exact computation
Exercises 8 - Solutions

1, 2, 3. Routine checking.

4. Show that the function $[f \to g] : [D_1 \to E_1] \to [D_2 \to E_2]$ with $[f \to g](h) = g \circ h \circ f$, page 102 of the notes, is continuous.

Solution: We first check that, for cop's A, B and C, the composition map $- \circ - : (B \to C) \times (A \to B) \to (A \to C)$ with $(g, f) \mapsto g \circ f$ is continuous. All we need to check is that $- \circ -$ is continuous in each variable separately, which is an easy exercise (you should write out the details).

Now, monotonicity of $f \to g$ is easy to check. Assume $(h_i)_{i \in \omega}$ is an increasing sequence in $[D_1 \to E_1]$. Then, using the continuity of composition (twice) we get:

$$g \circ ((\bigsqcup_i h_i) \circ f) = g \circ (\bigsqcup h_i \circ f)$$

$$= g \circ \bigsqcup (h_i \circ f) = \bigsqcup_i f \circ (h_i \circ f).$$

5. Check that all the constructors listed on page 104 are actually functors.

Routine checking.

6. Let $(e, p) : D \to E$ be an embedding-projection pair between cpo's D and E. Check that e and p are strict, p is onto, and e is one-to-one and preserves lubs of arbitrary subsets.

Solution: 2. As $\bot \sqsubseteq p(\bot)$, monotonicity of e implies $e(\bot) \sqsubseteq e(p(\bot)) = \bot$, so e is strict. Then $p(\bot) = p(e(\bot)) \sqsubseteq \bot$, so p is strict, too. Too see that p is onto, note that $p(e(x)) = x$ for all $x \in d$. If $e(x) = e(y)$ then $x = pe(x) = pe(y) = y$ proving that e is one-to-one. Finally, assume that $A \subseteq D$ and the least upper bound $x = \bigsqcup A$ exists. By monotonicity,

$e(a) \sqsubseteq e(x)$ for all $a \in A$, thus $e(x)$ is an upper bound for $e(A)$. If y is another upper bound of $e(A)$, then for all $a \in A$ we have $e(a) \sqsubseteq y$, hence $a = pe(a) \sqsubseteq p(y)$, thus $p(y)$ is an upper bound for A, hence $x \sqsubseteq p(y)$. This implies $e(x) \sqsubseteq ep(y) \sqsubseteq y$. So $e(x) = \bigsqcup e(A)$. Note that this includes the case $A = \emptyset$, where $\bigsqcup A = \bot$.