Extensions of domain maps in differential and integral calculus

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Abstract

We introduce in the context of differential and integral calculus several key extensions of higher order maps from a dense subset of a topological space into a continuous Scott domain. These higher order maps include the classical derivative operator and the Riemann integration operator. Using a sequence of test functions, we prove that the subspace of real-valued continuously differentiable functions on a finite dimensional Euclidean space is dense in the space of Lipschitz maps equipped with the L-topology. This provides a new result in basic mathematical analysis, which characterises the L-topology in terms of the limsup of the sequence of derivatives of a sequence of C^1 maps that converges to a Lipschitz map. Using this result, it is also shown that the generalised (Clarke) gradient on Lipschitz maps is the extension of the derivative operator on C^1 maps. We show that the generalised Riemann integral (R-integral) of a real-valued continuous function on a compact metric space with respect to a Borel measure can be extended to the integral of interval-valued functions on the metric space with respect to valuations on the probabilistic power domain of the space of non-empty and compact sets of the metric space. We also prove that the Lebesgue integral operator on integrable functions is the extension of the R-integral operator on continuous functions. We finally illustrate an application of these results by deriving a simple proof of Green's theorem for interval-valued vector fields.

I. Introduction

The notion of injective spaces and densely injective spaces were introduced by Dana Scott in [1]. A T_0 topological space is said to be (densely) injective if every continuous map $f : X \to Z$ extends continuously to any space Y containing X as a (dense) subset. A celebrated result in domain theory is that a T_0 space is (densely) injective iff it is a continuous lattice (a bounded complete continuous domain) equipped with its Scott topology [2, p. 181-182]. This provides a topological characterisation of continuous lattices and bounded complete continuous dcop's, also referred to as bounded complete domains or as continuous Scott domains.

Simple extensions of first order maps have been used in interval analysis for over 50 years, specifically in the context of extending real-valued functions of a real variable to interval-valued functions with real intervals as input [3]. Scott's formulation, however, has allowed to consider extensions of real-valued functions over more abstract topological spaces and higher order functions. Domain extension of classical maps on locally compact Hausdorff spaces and regular Borel measures and integration on these spaces have been studied in [4], [5], [6], [7] at the foundation of mathematical analysis. Higher order functions, such as integration and supremum of functions, were also employed to enrich Real PCF with new operators in [8] and in lazy functional algorithms [9], [10]. In [11], a question that Scott raised in the context of a higher order function related to injective properties has been answered.

In this paper, we embark on a new road, which provides new and rather surprising connections between several fundamental operators in mathematical analysis. Recall first that the generalised subgradient was introduced by F. Clarke in the 1970's [12]. The so-called Clarke gradient of a real-valued Lipschitz map on a finite dimensional Euclidean space is a non-empty, convex, compact valued map which is continuous with respect to the upper topology, equivalently the Scott topology on the space of non-empty convex and compact subsets of the finite dimensional Euclidean space. It has become a main paradigm in non-smooth optimisation and control theory as well as convex analysis.

Using a domain-theoretic method, the so-called L-derivative of a Lipschitz map was developed in [13], [14] which on finite dimensional real Euclidean spaces was proved to be equivalent to the Clarke gradient [15] and was used in [16] in developing a language with a derivative operator. The L-topology on Lipschitz maps was introduced in [17] as the weakest refinement of the sup norm topology so that the Clarke gradient, equivalently the L-derivative, is continuous.

We will examine extensions of domain maps at the foundation of mathematical analysis in this paper with some rather surprising results. We show, in particular by using test functions, that the subset of continuous differentiable functions in the space of Lipschitz maps is dense with respect to the L-topology and then prove that the domain extension of the classical derivative operator on the space of continuously differentiable functions to the space of Lipschitz functions equipped with the L-topology is precisely Clarke's generalised subgradient. In the course of this proof, we obtain several new results in mathematical analysis. In particular, we show that the L-topology has a completely classical characterisation in terms of limsup of the sequence of derivatives of a sequence of C^1 functions converging in the sup norm to a Lipschitz map. Note that in basic mathematical analysis, one can say nothing about the limit of the sequence of derivatives without assuming that it converges in the sup norm. Here, we show that if the original sequence of functions converges in the L-topology then the limsup of the sequence of derivatives coincides with the Clarke gradient of the Lipschitz map.

Next we show using the densely injective property that the generalised Riemann integral operator, or the R-integral, on a compact metric space with respect to a Borel measure, extends as an interval-valued integral to the function space of interval-valued maps on the metric space and the normalised probabilistic power domain [18], [5] of its upper space, the space of non-empty compact subsets of X ordered by reverse inclusion. This provides, remarkably for the first time, a framework for integration in which both the function and the measure can be approximated by finitary objects, namely step functions and simple valuations respectively and we obtain an explicit formula how these finitary approximations are computed.

Then, we prove that we can obtain the Lebesgue integration of L^1 functions in $\mathbb{R}^n$ with respect to any locally finite measure (i.e, finite on compact sets) as the extension of the R-integration on compact sets. This shows the notable consequence that in order to compute the Lebesgue integral of an L^1 function one can use approximations of the L^1 function by continuous functions and R-integration instead of using measurable sets and their associated simple functions as in the classical theory of the Lebesgue integral.

Finally, we provide an application which uses results from the different sections of the paper to extend path integration over classical vector fields to Scott continuous convex and compact valued vector fields. We derive a simple proof of the interval version of Green's theorem, a main result in [14].

All in all, our results show that domain theory provides a certain unifying framework for basic differential and integral calculus.

A. Notation and Terminology

We assume the reader is familiar with elements of domain theory [19], [2] as well as differentiable and integral calculus. We write $\mathbf{I}\mathbb{R} = \{[a, b] \mid a \leq b \in \mathbb{R}\} \cup \{\mathbb{R}\}$ for the interval domain, i.e. the set of compact, nonempty real intervals together with $\mathbb{R}$, ordered by reverse inclusion. We write a non-bottom element $v \in \mathbf{I}\mathbb{R}$ as $v = [v^-, v^+]$. As usual, we identify any real number $x \in \mathbb{R}$ with the singleton $\{x\} \subset \mathbb{R}$ so that we identify the set of maximal elements of $\mathbf{I}\mathbb{R}$ as $\mathbb{R}$.

For an open subset $U \subset \mathbb{R}^n$, we let $(U \to \mathbf{I}\mathbb{R})$ denote the domain of all Scott continuous functions of type $U \to \mathbf{I}\mathbb{R}$. A function $f \in (U \to \mathbf{I}\mathbb{R})$ is given by a pair of respectively lower and upper semi-continuous functions $f^-, f^+ : U \to \mathbb{R}$ with $f(x) = [f^-(x), f^+(x)]$. Given an open subset $a \subset X$ of a topological space X and an element $b \in D$ of a continuous Scott domain with bottom $\bot$, the *single step function* $b\chi_a : X \to D$ is defined as $(b\chi_a)(x) = b$ if $x \in a$ and $\bot$ otherwise. Single-step functions are continuous with respect to the Scott topology. Any bounded finite set of single-step functions has a least upper bound, called a *step function*, in the space of all Scott continuous functions of type $X \to D$. We denote the continuous Scott domain of the nonempty, compact and convex subsets of $\mathbb{R}^n$, taken together with $\mathbb{R}^n$ as the bottom element and ordered by reverse inclusion, by $\mathbf{C}(\mathbb{R}^n)$. We denote as $(U \to \mathbf{C}\mathbb{R}^n)$ the domain of all Scott continuous functions of type $U \to \mathbf{C}\mathbb{R}^n$, ordered pointwise by the partial order in $\mathbf{C}\mathbb{R}^n$. The way-below relation in both $\mathbf{I}\mathbb{R}$ and $\mathbf{C}(\mathbb{R}^n)$ is given by $A \ll B$ iff B is contained in the interior of A.

B. Injective spaces

Suppose Z is a continuous Scott domain. We recall the precise statement of the densely injective property:

Proposition I.1. *[2, p. 181-182] If X is a dense subset of Y then, any continuous map $f : X \to Z$ has a maximal continuous extension $f^* : Y \to Z$ given by*

$$f^*(y) := \sup\{\inf f(O \cap X) : O \textit{ is open, } y \in O\}.$$

If $g : Y \to Z$ is any continuous extension of f, then $g \sqsubseteq f^*$.

Extensions of maps as such have a simple compositional property given in the following.

Corollary I.2. *Suppose X_1 is a dense subset of Y and $f_1 : X_1 \to Z_1$ is any continuous map into the bounded complete domain Z_1. Assume also that X_2 is a dense subset of Z_1 with $Im(f_1) \subset X_2$ and $f_2 : X_2 \to Z_2$ is any continuous map into the bounded complete domain*

Z_2. Then $f_2^* \circ f_1^* : Y \to Z_2$ is a continuous extension of $f_2 \circ f_1$ with $f_2^* \circ f_1^* \sqsubseteq (f_2 \circ f_1)^*$.

In Section V, we will present an example with higher order maps which shows that in general we do not have $f_2^* \circ f_1^* = (f_2 \circ f_1)^*$ in Corollary I.2.

C. Generalised gradient operator

We recall the notion of the generalized gradient (also called subgradient) of a real-valued Lipschitz function $f : U \subset \mathbb{R}^n \to \mathbb{R}$, where U is an open set, as introduced by Clarke and presented in [12, section 2.6]. The Clarke gradient $\partial f(x)$ of f at $x \in U$ is a non-empty convex compact subset of $\mathbb{R}^n$ such that for all $v \in \mathbb{R}^n$:

$$\sup((\partial f(x))) \cdot v = \limsup_{y \to x \ t \to 0^+} \frac{f(y+tv) - f(y)}{t}, \quad (1)$$

where $A \cdot v = \{x \cdot v : x \in A\}$ for any $A \subseteq \mathbb{R}^n$. For example, $f : \mathbb{R} \to \mathbb{R} : x \mapsto |x|$ has $\partial f(x) = -1$ for $x < 0$, $\partial f(x) = 1$ for $x > 0$, and $\partial f(0) = [-1, 1]$. This definition can be extended to a function of type $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$; see the Appendix.

II. Extension of the Derivative Operator

Let $U \subset \mathbb{R}^n$ be an open subset, and let $\mathrm{Lip}(U)$ denote the set of Lipschitz maps of type $U \subset \mathbb{R}^n \to \mathbb{R}$ equipped with the sup norm $\| \,.\, \|_\infty$. For any function $f : U \to \mathbb{R}$, let $Df(x) := f'(x)$ be the derivative of f at x when it exists. In particular, we have a map

$$D : (U \to_1 \mathbb{R}) \to (U \to_0 \mathbb{R}^n),$$

where $(U \to_1 \mathbb{R}) = C^1(U)$ is the set of continuously differentiable functions equipped with the C^1 norm, and $(U \to_0 \mathbb{R}^n) = C^0(U)$ is the space of continuous functions $U \to \mathbb{R}^n$ equipped with the sup norm.

Recall that the L-topology is defined as the weakest refinement of the sup norm topology such that the Clarke gradient $\partial : \mathrm{Lip}(U) \to \mathbf{C}(\mathbb{R}^n)$ is continuous; it admits a complete metric [17].

In this section, we will show that $\mathrm{Lip}(U) \cap C^1(U)$, the subset of continuously differentiable functions in $\mathrm{Lip}(U)$, is dense with respect to the L-topology. Note that if U relatively compact, i.e., has a compact closure, then any map in $\mathrm{Lip}(U)$ extends by continuity to the closure of U and in this case we can use the closure $\overline{U}$ of U and we have: $C^1(\overline{U}) \subset \mathrm{Lip}(\overline{U})$ will be a dense subset.

We will then use the canonical embedding

$$s : (U \to_0 \mathbb{R}) \to (U \to \mathbf{C}(\mathbb{R}^n))$$

with $s(f)(x) = \{f(x)\}$ to consider D as a map $(U \to_1 \mathbb{R}) \to (U \to \mathbf{C}(\mathbb{R}^n))$ and prove that the Clarke generalised gradient operator ∂ is the extension of D, i.e., $\partial = D^*$.

We note the property below.

Lemma II.1. *Suppose $f_k : U \to \mathbb{R}$ is a sequence of C^1 functions such that $f_k \to f$ in the sup norm as $k \to \infty$ where f is a Lipschitz map. Then for all $x \in U$ and $v \in \mathbb{R}^n$, we have*

$$\limsup_{y \to x, k \to \infty} f_k'(y) \cdot v \geq \limsup_{y \to x, t \to 0^+} \frac{f(y+tv) - f(y)}{t}$$

Proof: Let

$$a := \limsup_{y \to x, t \to 0^+} \frac{f(y+tv) - f(y)}{t},$$

which is a real number, in fact bounded by the Lipschitz constant of f. Then, by the definition of $\limsup_{y \to x, t \to 0^+} \frac{f(y+tv)-f(y)}{t}$, for all positive integers k there exists y_k and t_k with $|x - y_k| < 1/k$, $0 < t_k < 1/k$ such that

$$\frac{f(y_k + t_k v) - f(y_k)}{t_k} > a - \frac{1}{k}$$

Fix y_k and t_k as above. Since $f_k \to f$ in the sup norm, there exists n_k such that for $\ell \geq n_k$ we have

$$\frac{f_\ell(y_k + t_k v) - f_\ell(y_k)}{t_k} > a - \frac{1}{k}$$

For two points $a, b \in \mathbb{R}^n$ let $\mathrm{L}(a, b)$ be the one dimensional open line segment from a to b. Without loss of generality we can assume n_k is a strictly increasing sequence of positive integers. By the higher dimensional mean value theorem [20, p. 355] applied to f_{n_k} in the direction v, there exists $z_k \in \mathrm{L}(y_k, y_k + t_k v)$, such that

$$f'_{n_k}(z_k) \cdot v = \frac{f_{n_k}(y_k + t_k v) - f_{n_k}(y_k)}{t_k}$$

i.e.,

$$f'_{n_k}(z_k) \cdot v > a - \frac{1}{k}$$

Thus, $\limsup_{k \to \infty} f'_{n_k}(z_k) \cdot v \geq a$. Since $\lim_{k \to \infty} z_k = x$ as $k \to \infty$, we conclude that

$$\limsup_{y \to x, k \to \infty} f_k'(y) \cdot v \geq a$$

■

Since

$$\limsup_{y \to x, w \to v, k \to \infty} f_k'(y) \cdot w \geq \limsup_{y \to x, k \to \infty} f_k'(y) \cdot v$$

we also obtain:

Corollary II.2.

$$\limsup_{y \to x, w \to v, k \to \infty} f_k'(y) \cdot w \geq \limsup_{y \to x, t \to 0^+} \frac{f(y+tv) - f(y)}{t}$$

We also have:

Proposition II.3. *If $f : U \to \mathbb{R}$ is a Lipschitz map then*

$$\begin{array}{l}\limsup_{y\to x, w\to v, t\to 0^+} \frac{f(y+tw)-f(y)}{t}\\ = \limsup_{y\to x, t\to 0^+} \frac{f(y+tv)-f(y)}{t}\end{array}$$

Proof: Let $c > 0$ be a Lipschitz constant for f and $\epsilon > 0$ be given. Then there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|y-x| < \delta_1$ and $0 < t < \delta_2$ implies:

$$\frac{f(y+tv)-f(y)}{t} < \frac{\epsilon}{2} + \limsup_{y\to x, t\to 0^+} \frac{f(y+tv)-f(y)}{t}$$

Hence for $|v-w| < \frac{\epsilon}{2c}$ we have:

$$\begin{array}{l}\frac{f(y+tw)-f(y)}{t}\\ = \frac{f(y+tw)-f(y+tv)+f(y+tv)-f(y)}{t}\\ \leq \frac{tc|w-v|}{t} + \frac{f(y+tv)-f(y)}{t}\\ \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} + \limsup_{y\to x, t\to 0^+} \frac{f(y+tv)-f(y)}{t}\end{array}$$

and thus

$$\begin{array}{l}\limsup_{y\to x, w\to v, t\to 0^+} \frac{f(y+tw)-f(y)}{t}\\ \leq \limsup_{y\to x, t\to 0^+} \frac{f(y+tv)-f(y)}{t}\end{array}$$

The other inequality is rather obvious. ∎

Recall that the generalised gradient operator is a map of type $\partial : \mathrm{Lip}(U) \to (U \to \mathbf{C}(\mathbb{R}^n)$. Also recall the definition of L-topology (or weak topology) on the space of Lipschitz maps as the weakest refinement of the C^0 topology such that $\partial : \mathrm{Lip}(U) \to (U \to \mathbf{C}(\mathbb{R}^n)$ is continuous with respect to the Scott topology on $(U \to \mathbf{C}(\mathbb{R}^n))$ [17]. Since $\partial f = f'$ for $f \in C^1(U)$, it follows that the restriction of the L-topology on $C^1(U)$ is precisely the C^1 norm topology:

Proposition II.4. *The relative subspace L-topology induced on the subset $C^1(U) \cap \mathit{Lip}(U)$ coincides with the C^1 norm topology.*

We now obtain one of our main results, which gives a characterisation of the L-topology by completely classical notions in mathematical analysis.

Theorem II.5. *A sequence $f_k \in C^1(U)$ converges to $f \in \mathit{Lip}(U)$ in the L-topology as $n \to \infty$ if and only if $f_k \to f$ in the sup norm topology as $k \to \infty$ and for all $x \in U$ and all $v \in \mathbb{R}^n$, we have:*

$$\limsup_{y\to x, w\to v, k\to\infty} f'_k(y)\cdot w \leq \limsup_{y\to x, t\to 0^+} \frac{f(y+tv)-f(y)}{t}$$

Proof: Suppose f_k converges to $f \in \mathrm{Lip}(U)$ in the L-topology. Since the L-topology is a refinement of the C^0 topology, it follows that $f_k \to f$ in the sup norm topology. To show the inequality, assume $x \in U$ and $v \in \mathbb{R}^n$ are given. Let $C \ll \partial f(x)$ in $\mathbf{C}(\mathbb{R}^n)$ and let $O \subset U$ be any open set containing x with $O \ll \partial f^{-1}(\Uparrow C)$ in the lattice of open sets of $(U \to \mathbf{C}(\mathbb{R}^n))$. Then, $C\chi_O \ll \partial f$ in $(U \to \mathbf{C}(\mathbb{R}^n))$ [2, Proposition II-4.20(iv)]. Note that step functions made up of such single-step functions of the form $C\chi_O$, where $O \subset U$ is a relatively compact set and $C \in \mathbf{C}(\mathbb{R}^n)$ provide a basis of the Scott topology on $(U \to \mathbf{C}(\mathbb{R}^n))$. Since, by the definition of L-topology, $\partial^{-1}(\Uparrow(C\chi_O))$ is open with $f \in \partial^{-1}(\Uparrow(C\chi_O))$, there exists N such that for all $k \geq N$, we have $C\chi_O \ll \partial f_k$. But $\partial f_k = f'_k$ and thus $C \ll f'_k(y)$ for all $k \geq N$ and $y \in O$, which implies $C \cdot w \ll f'_k(y)\cdot w$ and thus $f'_k(y)\cdot w < \sup(C\cdot w)$ for all $y \in O$, $w \in R^n$ and $k \geq N$. Taking $\limsup$ on both sides of the last inequality, we obtain:

$$\limsup_{w\to v, y\to x, k\to\infty} f'_k(y)\cdot w \leq \limsup_{w\to v} \sup(C\cdot w) = \sup(C\cdot v)$$

Since $C \ll \partial f(x)$ is arbitrary, it follows by the definition of $\partial f(x)$ that

$$\limsup_{w\to v, y\to x, k\to\infty} f'_k(y)\cdot w \leq \limsup_{y\to x, t\to 0^+} \frac{f(y+tv)-f(y)}{t}$$

Suppose now that $f_k \to f$ in the sup norm topology and for all $x \in U$ and all $v \in \mathbb{R}^n$, we have:

$$\limsup_{w\to v, y\to x, k\to\infty} f'_k(y)\cdot w \leq \limsup_{y\to x, t\to 0^+} \frac{f(y+tv)-f(y)}{t}$$

Let $C\chi_O$ be a single-step function with $C\chi_O \ll \partial f$ in $(U \to \mathbf{C}(\mathbb{R}^n))$. We will show that there exists N such that $k \geq N$ implies $C\chi_O \ll \partial f_k = f'_k$, which will show the convergence of f_k to f in the L-topology. Let $S \subset \mathbb{R}^n$ be the unit sphere equipped with the subspace Euclidean topology. Recall that $C\chi_O \ll \partial f$ is equivalent to $O \ll (\partial f)^{-1}(\Uparrow C)$ [2]. Thus, for all $x \in \overline{O}$ we have $C \ll \partial f(x)$. Take $C_x \in \mathbf{C}(\mathbb{R}^n)$ with $C \ll C_x \ll \partial f(x)$ for each $x \in \overline{O}$. By the Scott continuity of ∂f at x, there exists an open neighbourhood O_x of x such that $C_x \ll \partial f(z)$ for all $z \in O_x$. Then the collection O_x for $x \in \overline{O}$ covers the compact set $\overline{O}$. Suppose O_{x_i} with $1 \leq i \leq l$ is a finite cover. Put

$$C' = \bigcap\{C_{x_i} : 1 \leq i \leq l\}.$$

Then $C \ll C' \ll \partial f(z)$ for all $z \in \overline{O}$. By compactness of S, there exists $\delta > 0$ such that $\sup(C'\cdot v) < \sup(C\cdot v) - \delta$ for all $v \in S$. Since the support map of C, namely, $v \mapsto \sup(C\cdot v) : S \to \mathbb{R}$ is continuous [21], it is uniformly continuous on the compact set S and thus there exists $\theta > 0$ such that $|v-w| < \theta$ implies

$$|\sup(C\cdot v) - \sup(C\cdot w)| < \delta/2.$$

We have:

$$\begin{aligned}
&\limsup_{w\to v, y\to x, k\to\infty} f_k'(y)\cdot w \\
&\leq \limsup_{y\to x, t\to 0^+} \tfrac{f(y+tv)-f(y)}{t} \\
&= \sup(\partial f(x)\cdot v) \\
&< \sup(C'\cdot v) < \sup(C\cdot v) - \delta
\end{aligned}$$

Therefore, for each $x \in \overline{O}$ and $v \in S$ there exists an open neighbourhood O_x of x in $\overline{O}$, an open neighbourhood $U_v = \{w \in S : |w - v| < r_v\}$ of v in S, for $0 < r_v < \theta$, and integer $N_{x,v}$ such that $f_k'(y)\cdot w < \sup(C\cdot v) - \delta$ for $y \in O_x$, $w \in U_v$ and $k \geq N_{x,v}$. By compactness of $\overline{O}$ and that of S, there exists $x_i \in \overline{O}$ for $i = 1, \ldots, M_1$ and $v_j \in S$ for $j = 1, \ldots, M_2$, say, such that for each pair $y \in \overline{O}$ and $w \in S$, there exist i and j such that we have $(y, w) \in O_{x_i} \times U_{v_j}$ and thus $f_k'(y)\cdot w < \sup(C\cdot v_i) - \delta$ for all $k \geq N = \max\{N_{x_i,v_j} : 1 \leq i \leq M_1, 1 \leq j \leq M_2\}$. Since $r_{v_i} < \theta$ we have $|\sup(C\cdot w) - \sup(C\cdot v_i)| < \delta/2$ and thus:

$$\begin{aligned}
&f_k'(y)\cdot w < \sup(C\cdot v_i) - \delta < \sup(C\cdot w) + \delta/2 - \delta \\
&= \sup(C\cdot w) - \delta/2
\end{aligned}$$

It follows that for $k \geq N$, we have $C \ll f_k'(y)$ for all $y \in \overline{O}$. In other words $C\chi_O \ll f_k'$ for $k \geq N$, which shows that $f_k \to f$ in the L-topology.

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From Lemma II.1, we also obtain:

Corollary II.6. *A sequence $f_k \in C^1(U)$ converges to $f \in Lip(U)$ in the L-topology, as $k \to \infty$, iff $f_k \to f$ in the sup norm topology and for all $v \in \mathbb{R}^n$ and $x \in U$ we have:*

$$\limsup_{w\to v, y\to x, k\to\infty} f_k'(y)\cdot w = \limsup_{y\to x, t\to 0^+} \frac{f(y+tv)-f(y)}{t}$$

Recall that, given any metric space (X, d), the collection $\mathrm{Lip}(X, d)$ of bounded real-valued Lipschitz functions on X is equipped with its *Lipschitz norm* $\|\cdot\|_{Lip}$ defined as

$$\|f\|_{Lip} = \|f\| + \|f\|_d \tag{2}$$

where $\|f\| = \sup\{|f(x)| : x \in X\}$ is the sup norm and

$$\|f\|_d = \sup\{|f(x) - f(y)|/d(x, y) : x, y \in X, x \neq y\}.$$

If (X, d) is complete then so is the Lipschitz norm [22]. We now present a simple example of a sequence of C^1 functions that converge in the sup norm and in the L-topology, but not in the Lipschitz norm, to a Lipschitz map.

Example II.7. Consider the sequence of C^1 and Lipschitz functions $f_k : [-1, 1] \to \mathbb{R}$, for $k \geq 1$, with

$$f_k(x) = \begin{cases} |x| & \text{if} \quad |x| \geq 1/k \\ \frac{kx^2}{2} + \frac{1}{2k} & \text{if} \quad |x| < 1/k \end{cases}$$

and the Lipschitz map $f : [-1, 1] \to \mathbb{R}$ with $f(x) = |x|$. Clearly, $\lim_{k\to\infty} f_k = f$ in the sup norm. It is easily checked that $\|f_k - f\|_d \geq 1$ for all $k \geq 1$ and thus f_k does not tend to f as $k \to \infty$ in the Lipschitz norm topology. However,

$$\limsup_{w\to v, y\to x, k\to\infty} f_k'(y)w = v = \limsup_{y\to x, t\to 0^+} \frac{f(y+tv)-f(y)}{t}$$

Therefore, by Corollary II.6, $f_k \to f$ in the L-topology. This therefore gives a simple application of our new results in basic mathematical analysis.

In order to prove the next theorem, we need to construct a sequence of test functions which we will now describe. Let $B : \mathbb{R} \to \mathbb{R}$ be the C^∞ function defined by

$$B(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Let $T : \mathbb{R}^n \to \mathbb{R}$ be the bump function defined by $T(x) = \alpha B(1 - |x|^2)$ where $\alpha^{-1} = \int_{\mathbb{R}^n} B(1 - |x|^2)\, dx$ so that $\int_{\mathbb{R}^n} T(x)\, dx = 1$. For any positive integer k, we define the C^∞ function $T_k : \mathbb{R}^n \to \mathbb{R}$ with $T_k(x) = k^n T(kx)$. Then for all k, the map T_k is positive every where and we have $\int_{\mathbb{R}^n} T_k(x)\, dx = 1$. These sequences are usually used in treating C^∞ functions on the one hand and distributions on the other hand. We will show that they can also be used in relation to Lipschitz maps.

Theorem II.8. *For any map $f : U \to \mathbb{R}$ with Lipschitz constant c, there is a sequence f_k of C^∞ functions with Lipschitz constant c such that $\lim_{k\to\infty} f_k = f$ in the L-topology.*

Proof: We will construct $f_k := f * T_k$, the convolution of f and the map T_k as constructed above. Define a sequence of C^∞ functions $f_k : U \to \mathbb{R}$ with

$$f_k(x) = \int_U f(y) T_k(x - y)\, dy$$

for $x \in U$. Now let $w = x - y$ so that $y = x - w$. Then $w \in U - U$, where for subsets $A, B \subset \mathbb{R}^n$, the *Minkowski* sum and difference are defined by $A \pm B = \{a \pm b : a \in A, b \in B\}$. Since U is open so is $U - U$ and as $0 \in U - U$ it follows that there exists an open ball of radius $\delta > 0$ centred at 0 that is contained in $U - U$. Thus, by the above change of variable in the integral we obtain:

$$\begin{aligned}
&f_k(x) \\
&= \int_{U-U} f(x - w) T_k(w)\, dw \\
&= k^n \int_{U-U} f(x - w) T(kw)\, dw
\end{aligned}$$

Then, for $k > 1/\delta$:

$$\begin{aligned}
&f_k(x) \\
&= \int_{O_{1/k}(0)} f(x - w) T_k(w)\, dw \\
&= \int_{O_1(0)} f(x - \tfrac{z}{k}) T(z)\, dz
\end{aligned} \tag{3}$$

where we have changed the variable again by putting $z = kw$.

We have:

$$\begin{aligned}&|f_k(x_1) - f_k(x_2)|\\ &\leq \int_{\mathbb{R}^n} |f(x_1 - y) - f(x_2 - y)| T_k(y)\, dy\\ &\leq c|x_1 - x_2| \int_U T_k(y)\, dy\\ &= c|x_1 - x_2|\end{aligned}$$

Therefore, f_k has Lipschitz constant c and we have:

$$\begin{aligned}&|f_k(x) - f(x)|\\ &\leq \int_{O_1(0)} |f(x - \tfrac{z}{k}) - f(x)| T(z)\, dz\\ &\leq \tfrac{c}{k} \int_{O_1(0)} T(z)\, dz = c/k\end{aligned} \tag{4}$$

showing that $\lim_{k\to\infty} f_k = f$ in the sup norm. By Theorem II.5, it remains to show that

$$\limsup_{w\to v, y\to x, k\to\infty} f'_k(y)\cdot w \leq a := \limsup_{y\to x, t\to 0^+} \frac{f(y + tv) - f(y)}{t}$$

Let $\epsilon > 0$ and $v \in \mathbb{R}^n$ be given. Then, by Proposition II.3 there exist $\delta_1, \delta_2 > 0, \delta_3 > 0$ such that for $|x - y| < \delta_1$, $0 < t < \delta_2$ and $|v - w| < \delta_3$, we have:

$$\frac{f(y + tw) - f(y)}{t} < a + \epsilon \tag{5}$$

Now for $k \geq 1/\delta_1$ we have $\frac{|z|}{k} \leq \delta_1$ when $|z| \leq 1$. Thus using Equation 3 to compute $f(y)$ and $f(y+tw)$ and using Equation 5, we obtain for $k \geq 1/\delta_1$ and $0 < t < \delta_2$:

$$\begin{aligned}&\tfrac{f_k(y+tw) - f_k(y)}{t}\\ &= \int_{O_1(0)} \left(\tfrac{f(y+tv-\frac{z}{k}) - f(y-\frac{z}{k})}{t}\right) T(z)\, dz\\ &\leq (a + \epsilon) \int_{O_1(0)} T(z)\, dz\\ &= a + \epsilon\end{aligned}$$

Hence, for $k \geq 1/\delta_1$, $|x - y| < \delta_1$ and $|v - w| < \delta_3$, we have:

$$f'_k(y) \cdot w \leq a + \epsilon$$

and thus

$$\limsup_{w\to v, y\to x, k\to\infty} f'_k(y) \cdot w \leq a + \epsilon,$$

which implies

$$\limsup_{w\to v, y\to x, k\to\infty} f'_k(y) \cdot w \leq a$$

since $\epsilon > 0$ was arbitrary. ■

The following main result characterises the L-topology on Lipschitz maps in terms of the density of the subspace of C^1 and C^∞ maps.

Corollary II.9. *The subspace $C^\infty(U) \cap Lip(U)$, and thus $C^1(U) \cap Lip(U)$, is dense in Lip(U) with respect to the L-topology.*

Now we are able to prove our final main result in this section. Note that the differential operator $D : C^1(U) \cap \mathrm{Lip}(U) \to C^0(U)$ with $D(f) = f'$ can be regarded as having type $D : C^1(U) \cap \mathrm{Lip}(U) \to (U \to \mathbf{C}(\mathbb{R}^n))$ since $C^0(U)$ can be identified as a subset of the maximal elements of $(U \to \mathbf{C}(\mathbb{R}^n))$. As we have seen $C^1(U) \cap \mathrm{Lip}(U)$ is, by Corollary II.9, dense in Lip(U), and moreover, since the restriction of the L-topology on $C^1(U) \cap \mathrm{Lip}(U)$ is the C^1 norm topology, D is continuous on $C^1(U) \cap \mathrm{Lip}(U)$. Therefore, its extension $D^* : \mathrm{Lip}(U) \to (U \to \mathbf{C}(\mathbb{R}^n))$ is well-defined and continuous and we have $D^*(f) = f'$ for $f \in \mathrm{Lip}(U) \cap C^1(U)$.

Theorem II.10. *The extension of the differential operator $D : C^1(U) \cap Lip(U) \to (U \to \mathbf{C}(\mathbb{R}^n))$ is the generalised gradient operator, i.e., $D^* = \partial$ with $D^*(f) = \partial f$.*

Proof: First note that the L-topology on Lip(U), being the meet of the sup norm topology and the Scott topology, is itself second countable. Thus, in the definition of D^* we can use a countable set of open sets. In fact, for each $f \in \mathrm{Lip}(U)$, we can assume we have a shrinking sequence O_n of open subsets $O_0 \supset O_1 \supset O_2 \supset \cdots$ with $f \in O_i$ for all $i \geq 0$ that form a local basis for the L-topology at f. Let $f \in \mathrm{Lip}(U)$ and its countable set of open subsets as above be given. Then, putting $C := C^1(U) \cap \mathrm{Lip}(U)$, we have: $D^*(f) : U \to \mathbf{C}(\mathbb{R}^n)$ given by

$$D^* f = \sup_{k\geq 0} \inf_{g\in O_k \cap C} g'$$

Next note that $D^*(f)(x) \in \mathbf{C}(\mathbb{R}^n)$ for any given $x \in U$ and thus $D^*(f)(x)$, as a non-empty convex compact subset, is completely determined by its support function $S_{D^*(f)(x)} : \mathbb{R}^n \to \mathbb{R}$ with $S_{D^*(f)(x)}(v) = \sup(v \cdot D^*(f)(x))$, for $v \in \mathbb{R}^n$. Let $x \in U$ and $v \in \mathbb{R}^n$ be given. We will show that

$$\sup(v \cdot D^* f(x)) = \sup(v \cdot \partial f(x))$$

from which the result follows.

Let $a := \sup(v \cdot D^* f(x))$ and let $\epsilon > 0$ be given. Since the support function is continuous and $D^*(f)$ is Scott continuous it follows that the composition function of type $U \times \mathbb{R}^n \to \mathbb{R}$ with

$$(x, w) \mapsto \sup(w \cdot D^* f(x))$$

is upper continuous. Thus, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for $|x - y| < \delta_1$ and $|v - w| < \delta_2$:

$$\sup(w \cdot D^* f(y)) < a + \epsilon$$

i.e.,

$$\sup(w \cdot \sup_{k\geq 0} \inf_{g\in O_n \cap C} g'(y)) < a + \epsilon$$

Thus, there exists N such that $k \geq N$ implies

$$\sup(w \cdot \inf_{g\in O_k \cap C} g'(y)) < a + \epsilon \tag{6}$$

Now let $g_k \in O_k \cap C$. Then, from Relation 6, it follows that for $k \geq N$ and $|x - y < \delta_1$ and $|v - w| < \delta_2$, we have:

$$w \cdot g'_k(y) < a + \epsilon$$

and thus

$$\limsup_{w \to v, y \to x, n \to \infty} w \cdot g'_k(y) \leq a + \epsilon$$

But $g_k \to f$ as $k \to \infty$ in the L-topology and, from Corollary II.6, we obtain:

$$\begin{array}{l} \limsup_{w \to v, y \to x, k \to \infty} w \cdot g'_k(y) \\ = \limsup_{y \to x, t \to 0^+} \frac{f(y+tv)-f(y)}{t} \\ = \sup(v \cdot \partial f(x)) \end{array}$$

where the latter equality follows from the definition of the generalised gradient of f. We conclude that $\sup(v \cdot \partial f(x)) \leq a + \epsilon$ for all $\epsilon > 0$, i.e., $\sup(v \cdot \partial f(x)) \leq a$.

In order to show the reverse inequality, let $\epsilon > 0$ be given. We will obtain a sequence $g_k \in O_k \cap C$ such that

$$a - \epsilon < \limsup_{w \to v, y \to x, n \to \infty} w \cdot g'_k(y)$$

By the definition of a and continuity of the support function, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - y| < \delta_1$ and $|v - w| < \delta_2$ implies

$$a - \epsilon < \sup(v \cdot \sup_{k \geq 0} \inf_{g \in O_k \cap C} g'(y))$$

This means that for all k there exists $g_k \in O_k \cap C$ with $a - \epsilon < w \cdot g'_k(y)$ for $|x - y| < \delta_1$ and $|v - w| < \delta_2$. Hence,

$$a - \epsilon \leq \limsup_{w \to v, y \to x, k \to \infty} g'_k(y),$$

which, by Corollary II.6, implies: $\sup(v \cdot \partial f(x)) \geq a$.

∎

III. Integral of interval valued functions

In this section, we will show that the generalised Riemann integral of a real-valued continuous function on $[0,1]^n$ with respect to a Borel measure μ on $[0,1]^n$ can be extended to the integral of interval-valued functions on $[0,1]^n$ with respect to values on the probabilistic power domain of $\mathbf{C}([0,1]^n)$. We will see in the next section that working with compact subsets of $\mathbb{R}^n$ is sufficient to present an account of L^1, i.e., Lebesgue integrable, functions on $\mathbb{R}^n$.

Recall from [23], [18], [4], [24] that a (sub-probability) valuation on a topological space Y is a map $\nu : \Omega(Y) \to [0,1]$, where $\Omega(Y)$ is the lattice of open subsets of Y, with, (i) $\nu(\emptyset) = 0$, (ii) $\nu(O_1) \leq \nu(O_2)$ if $O_1 \subset O_2$, (iii) $\nu(O_1) + \nu(O_2) = \nu(O_1 \cup O_2) + \nu(O_1 \cap O_2)$, such that ν is continuous with respect to the Scott topologies on $\Omega(Y)$ and on $\mathbb{R}$ ordered with the usual ordering of real numbers. A simple valuation is of the form $\nu = \sum_{i=1}^n r_i \delta_{y_i}$ where $y_i \in Y$ for $1 \leq i \leq n$ and $r_i > 0$ with

$$\nu(O) = \sum \{r_i : y_i \in O\}.$$

The probabilistic power domain $\mathbf{P}(Y)$ of Y is the set of valuations of Y, i.e., of type $\Omega(Y) \to \mathbb{R}$, ordered pointwise. If D is a (countably based continuous dcpo, then so is $\mathbf{P}(D)$ with a basis of simple valuations. In particular, $\mathbf{P}(\mathbf{C}([0,1]^n))$ is a countably based continuous dcpo with a countable basis of simple valuations. The same is true for the normalised probabilistic power subdomain $\mathbf{P}^1(\mathbf{C}([0,1]^n)) \subset \mathbf{P}(\mathbf{C}([0,1]^n))$, consisting of normalised valuations, i.e., $\nu \in \mathbf{P}(\mathbf{C}([0,1]^n))$ with $\nu([0,1]^n) = 1$. Recall that the partial order on simple valuations in $\mathbf{P}^1(Y)$, or the so-called splitting lemma, takes a simple form [5]. For two simple valuations

$$\mu_1 = \sum_{b \in B} r_b \delta_b, \qquad \mu_2 = \sum_{c \in C} s_c \delta_c$$

in $P^1(Y)$, we have: $\mu_1 \sqsubseteq \mu_2$ iff, for all $b \in B$ and all $c \in C$, there exists a nonnegative number $t_{b,c}$ such that

$$\sum_{c \in C} t_{b,c} = r_b \qquad \sum_{b \in B} t_{b,c} = s_c$$

and $t_{b,c} \neq 0$ implies $b \sqsubseteq c$.

The generalised Riemann integral, as the name suggests, has properties similar to the classical Riemann integral. In particular, a real-valued function f on $[0,1]^n$ will have a generalised Riemann integral $\int f d\mu \in \mathbb{R}$ if and only if f is continuous almost everywhere with respect to μ. Furthermore, if the generalised Riemann integral does exist then it is equal to the Lebesgue integral of f with respect to μ. In this case, if $\mu = \sup_{i \geq 0} \nu_i$ where μ_i is a simple valuation for $i \geq 0$, then $\int f \, d\mu = \sup_{i \geq 0} S^\ell(f, \mu_i) = \inf_{i \geq 0} S^u(f, \mu_i)$, where the generalised lower and upper sums of any bounded map $g : [0,1]^n \to \mathbb{R}$ f with respect to a simple valuation $\nu = \sum_{i=1}^k r_i \delta_{C_i} \in \mathbf{P}^1(\mathbf{C}([0,1]^n))$ are defined by:

$$S^\ell(g, \nu) = \sum_{i=1}^k r_i \inf g[C_i], \; S^u(f, \nu) = \sum_{i=1}^k r_i \sup g[C_i].$$

For a continuous map $f : [0,1]^n \to \mathbb{R}$, the values of $\inf f[X]$ and $\sup f[X]$ are attained on the compact set $[0,1]^n$.

The generalised Riemann integral provides a map $\int_R : ([0,1]^n \to \mathbb{R}) \times \mathbf{M}([0,1]^n) \to \mathbb{R}$, where $([0,1]^n \to \mathbb{R})$ is the space of real-valued continuous function on $[0,1]^n$ with the sup norm and $\mathbf{M}([0,1]^n)$ is the space of finite Borel measures on $[0,1]^n$ with the weak topology, i.e., the weakest topology such that for all real valued continuous functions $f \in C([0,1]^n)$ we have: $\lim_{n \to \infty} \mu_n = \mu$ implies $\lim_{n \to \infty} \int_X f \, d\mu_n = \int_X f \, d\mu$. The map $\int$:

$([0,1]^n \to \mathbb{R}) \times \mathbf{M}([0,1]^n) \to \mathbb{R}$, where $C([0,1]^n)$ is equipped with the sup norm topology and $\mathbf{M}([0,1]^n)$ with the weak topology, is separately continuous in its two arguments and it can easily be checked that it is in fact a continuous map. For the extension result, we write $\int(f,\mu) := \int f\, d\mu$.

Proposition III.1. *The function space* $([0,1]^n \to \mathbb{R})$ *is dense in* $([0,1]^n \to \mathbf{I}\mathbb{R})$ *with respect to the Scott topology.*

Proof: We will use the interior of convex subset of $[0,1]^n$ with rational vertices as the topological base for $[0,1]^n$. It is sufficient to show that for any step function $f = \sup_{i\in I} a_i \chi_{O_i}$, where I is a finite indexing set and $\Uparrow f \neq \emptyset$, we have $([0,1]^n \to \mathbb{R}) \cap \Uparrow f \neq \emptyset$. (Note that as usual we identify a function $f \in ([0,1]^n \to \mathbb{R})$ with the maximal element $g \in ([0,1]^n \to \mathbf{I}\mathbb{R})$ where $g(x) = \{f(x)\}$ for all $x \in X$.) The open subsets O_i with $i \in I$ partition the domain of f into a finite number of crescents (i.e., intersections of open and closed sets) C_j $(j \in J)$ in which the value of f is a constant b_j, say. Each crescent C_j is a simple polyhedra with rational vertices. Recall $f \ll h$ iff $O_i \ll h^{-1}(\Uparrow a_i)$ for each $i \in I$. Thus, by replacing f, if necessary, with a step function h such that $f \ll h$ and $\Uparrow h \neq \emptyset$ we can assume that if $\bigcap_{t\in T} \partial C_t \neq \emptyset$, for any $T \subset J$, then $(\bigcap_{t\in T} b_t)^\circ \neq \emptyset$. Assume now that $n = 2$ as the extension to $n > 2$ is straightforwrd. We triangulate each crescent C_j for $j \in J$ and for each node x of the triangulation we choose a value $v(x) \in \{\bigcap b_t : x \in \partial C_t\}^\circ$. Now we define the piecewise linear map $k : [0,1]^n \to \mathbb{R}$ that takes value $v(x)$ on each node of the triangulation and is linear on each crescent C_j for $j \in J$. Then, $k : [0,1] \to \mathbb{R}$ is continuous and satisfies $f \ll k$ as required. ∎

The map $e : \mathbf{M}([0,1]^n) \to \mathbf{P}^1(\mathbf{C}([0,1]^n))$ with $e(\mu)(A) = \mu(s^{-1}(A))$ where $s : [0,1]^n \to \mathbf{C}([0,1]^n)$ with $s(x) = \{x\}$ is an embedding onto the set of maximal elements of $\mathbf{P}^1(\mathbf{C}([0,1]^n)$ [5], [25]. It follows that $\mathbf{M}([0,1]^n)$ is dense in $\mathbf{P}^1(\mathbf{C}([0,1]^n))$.

Thus, from the extension theorem we know that we have a map $\int^* : ([0,1]^n \to \mathbf{I}\mathbb{R}) \times \mathbf{P}^1(\mathbf{C}([0,1]^n)) \to \mathbf{I}\mathbb{R}$ which extends $\int$. As usual, we write $\int^* f\, d\nu := \int^*(f,\nu)$. In this section, we will find an explicit expression for this extension, which extends the generalised Riemann integral to interval-valued functions on the one hand and continuous valuations on $\mathbf{P}^1(\mathbf{C}([0,1]^n))$ on the other. Since by the general properties of the extended map $\int^*$ is continuous, it is sufficient to determine its value on pairs (f,ν) where $f \in ([0,1]^n \to \mathbf{I}\mathbb{R})$ is a step function and $\nu \in \mathbf{P}^1(\mathbf{C}([0,1]^n))$ is a simple valuation. Note that we can write $f = [f^-, f^+]$ with $f^-, f^+ : [0,1]^n \to \mathbb{R}$ where f^- is lower semi-continuous and f^+ is upper semi-continuous.

We thus define the *interval-valued* integral

$$\int : ([0,1]^n \to \mathbf{I}\mathbb{R}) \times \mathbf{P}^1(\mathbf{C}([0,1]^n)) \to \mathbf{I}\mathbb{R} \qquad (7)$$

by defining it on pairs (f,ν) where $f \in ([0,1]^n \to \mathbf{I}\mathbb{R})$ is a step function and $\nu \in \mathbf{P}^1(\mathbf{C}([0,1]^n))$ is a simple valuation as

$$\int(f,\nu) = \left[S^\ell(f^-,\nu), S^u(f^+,\nu)\right]$$

and check the continuity condition with respect to these basis elements and then extend the definition by continuity to $([0,1]^n \to \mathbf{I}\mathbb{R}) \times \mathbf{P}^1(\mathbf{C}([0,1]^n))$.

The continuity with respect to step functions has a short proof using Dini-Cartan's lemma:

Proposition III.2. *If* $f_i \in ([0,1]^n \to \mathbf{I}\mathbb{R})$ *for* $i \geq 0$ *is an increasing sequence of step functions with* $\sup_i f_i = f$, *where* f *is a step function, then for any simple valuation* ν*:*

$$\sup_i \int(f_i,\nu) = \int(f,\nu).$$

Proof: Let $C \in \mathbf{C}([0,1]^n)$. By Dini-Cartan's lemma for lower semi-continuous functions [26, Lemma 2.2.9], we have: $\sup_i \inf f_i^-[C] = \inf \sup_i f_i^-[C] = \inf f^-[C]$. Thus, $S^\ell(f^-,\nu) = \sup_i S^\ell(f_i^-,\nu)$. Similarly, $S^u(f^+,\nu) = \inf_i S^u(f_i^+,\nu)$. ∎

The next result however has a much longer proof.

Proposition III.3. *If* $\nu_i \in \mathbf{P}^1(\mathbf{C}([0,1]^n))$ *is for* $i \geq 0$ *is an increasing sequence of simple valuations with* $\sup_i \nu_i = \nu$, *where* ν *is a simple valuation, then for any step function* f*:*

$$\sup_i \int(f,\nu_i) = \int(f,\nu).$$

Proof: Note that for simple valuations $\mu_1, \mu_2 \in \mathbf{P}^1(\mathbf{C}([0,1]^n))$ with $\mu_1 \sqsubseteq \mu_2$, we have $S^\ell(f^-,\mu_1) \leq S^\ell(f^-,\mu_2)$: In fact if $\mu_1 = \sum_{a\in A} r_a \delta_a$ and $\mu_2 = \sum_{b\in B} r_b \delta_b$, then

$$\begin{aligned}
&\textstyle\sum_{b\in B} r_b \inf f^-[b] = \sum_{b\in B} \sum_{a\in A, a\supset b} t_{ba} \inf f^-[b] \\
&\geq \textstyle\sum_{b\in B} \sum_{a\in A, a\supset b} t_{ba} \inf f^-[a] \\
&= \textstyle\sum_{a\in A} \sum_{b\in B, a\supset b} t_{ba} \inf f^-[a] \\
&= \textstyle\sum_{a\in A} r_a \inf f^-[a]
\end{aligned}$$

Thus, $S^\ell(f^-,\nu) \geq \sup_i S^\ell(f^-,\nu_i)$.

To show the reverse inequality, let $f = [f^-, f^+]$, $\nu = \sum_{j=1}^n r_j \delta_{C_j}$, where $C_j \in \mathbf{C}([0,1]^n)$, $r_j > 0$ for $1 \leq j \leq n$, and let $\nu_i = \sum_{j=1}^{j_i} r_{ij} \delta_{C_{ij}}$, with $C_{ij} \in \mathbf{C}([0,1]^n)$, $r_{ij} > 0$ for $i \geq 0$ and $1 \leq j \leq j_i$.

Let $\epsilon > 0$ be given with $\epsilon < 1$. We will show that there exists N such that $S^\ell(f^-,\nu) < S^\ell(f^-,\nu_N) + \epsilon$ from which the result follows as the dual relation for $S^u(f^+,\nu)$ would be similar. Since f^- is bounded and for any non-empty set A and real number c we have $\inf(f^- + c)[A] = \inf f^-[A] + c$, without loss of generality we can assume that $f^- > 0$. Let $M > 1$ be such that $f^-(x) \leq M$ for all $x \in X$ for which f^- is defined. Consider the poset P consisting of the finite number of elements C_j for $1 \leq j \leq n$ with the partial order of reverse inclusion induced from $\mathbf{C}([0,1]^n)$. For each $j = 1,\ldots,n$, we have $C_j = \bigcap_{m\geq 1}(C_j)_{1/m}$ where, for any compact set C, we define the open subset $C_\delta = \{x \in X : \exists y \in C.\, d(x,y) < \delta\}$. Thus, there exists an open neighbourhood O_j, say, of C_j such that $C_t \subseteq O_j$ implies $C_t \subseteq C_j$.

Next, note that f^-, being a lower semi-continuous function, attains its infimum on any compact subset [27]. Thus, assume $\inf f^-[C_j] = f^-(x_j)$ for some $x_j \in C_j$ for $1 \leq i \leq n$, and $\inf f^-[C_{ij}] = f^-(x_{ij})$ for some $x_{ij} \in C_{ij}$ for $i \geq 0$ and $1 \leq j \leq j_i$. Then for all $x \in C_j$ we have $f^-(x) \geq f^-(x_j)$. Thus, by the lower semi-continuity of f^- at x there exists an open set O_x containing x with $f^-(y) > f^-(x_j) - \epsilon$ for $y \in O_x$. The collection of open sets O_x has a finite sub-cover of C_j, say by O_{x_t} with $1 \leq t \leq K$. Put $V_j := \bigcup_{t=1}^K O_{x_t}$. Then $C_j \subset V_j$ and we have $f^-(y) > f(x_j) - \epsilon$ for $y \in V_j$. Let $W_j := \Box(O_j \cap V_j)$.

Since $\nu = \sup_i \nu_i$ it follows that there exists N_j such that $i \geq N_j$ implies $\nu(W_j) < \nu_i(W_j) + \frac{\epsilon}{Mk}$, where $k > 0$ is a sufficiently large number which would be determined later. Take $N = \max_{1\leq j\leq n} N_j$. Then

$$\nu(W_j) < \nu_i(W_j) + \frac{\epsilon}{Mk}, \tag{8}$$

for all $i \geq N$ and $1 \leq j \leq n$.

Now, for each $j = 1,\ldots,n$, let

$$W_j^+ = \bigcup\{W_k : W_k \subseteq W_j, W_k \neq W_j\}.$$

Recall that any continuous valuation on a dcpo extends uniquely to a measure [28] and in particular it extends to crescents, i.e., intersection of an open and a closed set. Then, we have:

$$\begin{aligned} \nu(W_j \setminus W_j^+) &= \nu(W_j) - \nu(W_j^+) \\ \nu_N(W_j \setminus W_j^+) &= \nu_N(W_j) - \nu_N(W_j^+) \end{aligned}$$

Since $\nu_N \sqsubseteq \nu$ we have: $-\nu(B) \leq -\nu_N(B)$ for any open set B. Thus, from Inequality 8, we obtain:

$$\nu(W_j \setminus W_j^+) < \nu_N(W_j \setminus W_j^+) + \frac{\epsilon}{Mk} \tag{9}$$

We claim that for any non-empty subset $A \subseteq \{1,\ldots,n\}$ we have:

$$\nu(\bigcup_{j\in A} W_j) \leq \nu_N(\bigcup_{j\in A} W_j) + \frac{2^{|A|}\epsilon}{Mk} \tag{10}$$

where $|A|$ is the number of elements in A. First note that for any valuation μ and open sets B_t for $t = 1,\ldots,p$ we have [6]:

$$\begin{aligned} &\mu(\textstyle\bigcup_{1\leq t\leq p} B_t) = \sum_{t=1}^p \mu(B_t) - \sum_{t_1<t_2} \mu(B_{t_1} \cap B_{t_2}) \\ &+ \textstyle\sum_{t_1<t_2<t_3} \mu(B_{t_1} \cap B_{t_2} \cap B_{t_3}) - \ldots + \ldots \\ &(-1)^{p+1}\mu(\textstyle\bigcap_{1\leq t\leq p} B_t) \end{aligned}$$

Thus, applying this rule to $\bigcup_{j\in A} W_j$, we get:

$$\nu(\bigcup_{j\in A} W_j) = \sum_{j\in A} \nu(W_j) - \sum_{j_1\neq j_2} \nu(W_{j_1} \cap W_{j_2}) + \ldots, \tag{11}$$

and a similar equation for $\nu_N(\bigcup_{j\in A} W_j)$.

Now note that by construction and since $\mathbf{C}([0,1]^n)$ is bounded complete we have for any set of distinct indices $1 \leq j_q \leq n$ with $q = 1,\ldots \ell$:

$$\bigcap_{1\leq q\leq \ell} W_{j_q} = W_{j_0}$$

where $j_0 \in \{1,\ldots,n\}$, is given by: $\bigcap_{1\leq q\leq \ell} C_{j_q} = C_{j_0}$. Note that since $\nu_N \sqsubseteq \nu$, for any open set B, we always have

$$-\nu(B) \leq -\nu_N(B). \tag{12}$$

Thus, using Relation (8) for the positive terms in Equation 11 and its analogue for ν_N while using Relation 12 for the negative terms in Equation 11 and its analogue for ν_N, we have:

$$\nu(\bigcup_{j\in A} W_j) < \nu_N(\bigcup_{j\in A} W_j) + \frac{m\epsilon}{Mk} \leq \nu_N(\bigcup_{j\in A} W_j) + \frac{2^{|A|}\epsilon}{Mk}$$

where m is the total number of values of valuations with a positive sign in Equation 11.

We now compute (recalling that $\forall x.\, f^-(x) \leq M$):

$$\begin{aligned} &S^\ell(f^-,\nu) \\ &= \textstyle\sum_{i=1}^n r_j \inf f^-[C_j] = \sum_{j=1}^n \nu(W_j \setminus W_j^+) f^-(x_j) \\ &\leq \textstyle\sum_{j=1}^n (\nu_N(W_j \setminus W_j^+) + \frac{\epsilon}{Mk}) f^-(x_j) \\ &= \textstyle\frac{n\epsilon}{k} + \sum_{j=1}^n \sum_t \{r_{Nt} f^-(x_j) : C_{Nt} \in W_j \setminus W_j^+\} \end{aligned}$$

Thus, (recalling that $\sum_{t=1}^{j_N} r_{Nt} = 1$),

$$\begin{aligned} &S^\ell(f^-,\nu) - \textstyle\frac{n\epsilon}{k} \\ &\leq \textstyle\sum_{j=1}^n \sum\{r_{Nt} f^-(x_j) : C_{Nt} \in W_j \setminus W_j^+\} \\ &= \textstyle\sum_{j=1}^n \sum\{r_{Nt} f^-(x_j) : C_j \subseteq C_{Nt} \in W_j \setminus W_j^+\} \\ &+ \textstyle\sum_{j=1}^n \sum\{r_{Nt} f^-(x_j) : C_j \not\subseteq C_{Nt} \in W_j \setminus W_j^+\} \\ &\leq \textstyle\sum_{j=1}^n \sum\{r_{Nt}(f^-(x_{Nt}) + \frac{\epsilon}{k}) : C_j \subseteq C_{Nt} \in W_j \setminus W_j^+\} \\ &+ \textstyle\sum_{j=1}^n \sum\{r_{Nt} f^-(x_j) : C_j \not\subseteq C_{Nt} \in W_j \setminus W_j^+\} \\ &\leq \textstyle\frac{n\epsilon}{k} + \sum_{j=1}^n \sum\{r_{Nt} f^-(x_{Nt}) : C_j \subseteq C_{Nt} \in W_j \setminus W_j^+\} \\ &+ M \textstyle\sum_{j=1}^n \sum\{r_{Nt} : C_j \not\subseteq C_{Nt} \in W_j \setminus W_j^+\} \end{aligned}$$

Now by the splitting lemma applied to the term $r_{Nt}\delta_{C_{Nt}}$ of the valuation ν_N in the relation $\nu_N \sqsubseteq \nu$, it follows that

$$\begin{aligned} &\textstyle\sum\{r_{Nt} : C_j \not\subseteq C_{Nt} \in W_j \setminus W_j^+\} \\ &\leq \nu(W_j^+) - \nu_N(W_j^+) \leq \textstyle\frac{2^n\epsilon}{Mk}, \end{aligned}$$

by Relation 10. Therefore:

$S^\ell(f^-,\nu) - \frac{n\epsilon}{k} \leq \frac{n\epsilon}{k} + \frac{n2^n\epsilon}{k}$
$+\sum_{j=1}^n \sum\{r_{Nt}f^-(x_{Nt}) : C_j \subseteq C_{Nt} \in W_j \setminus W_j^+\}$

We conclude that,

$S^\ell(f^-,\nu) - \frac{2n\epsilon}{k} - \frac{n2^n\epsilon}{k}$
$\leq \sum_{j=1}^n \sum\{r_{Nt}f^-(x_{Nt}) : C_j \subseteq C_{Nt} \in W_j \setminus W_j^+\}$
$\leq \sum_{t=1}^{j_N} r_{Nt}f^-(x_{Nt})$
$= S^\ell(f^-,\nu_N)$

Putting $k := 2n(1+2^{n-1})$, the proof is complete.

■

Extending the interval-valued integral by continuity, we have established that the interval-valued integral 7 is a continuous map and in particular we obtain an explicit way of computing it when the function and the valuation (measure) are given in terms of sups of step functions and simple valuations respectively:

Corollary III.4. *If $f = \sup_{i\geq 0} f_i$ and $\nu = \sup_{i\geq 0}\nu_i$ are sups of increasing sequences of step functions and simple valuations respectively then*

$$\int f\,d\nu = \sup_{i\geq 0}[S^\ell(f_i,\nu_i), S^u(f_i,\nu_i)]$$

Theorem III.5. *The extension of the integral is the interval-valued integral:*

$$\int^* = \int$$

Proof: Recall that the $\int^* : ([0,1]^n \to \mathbb{R}) \times \mathbf{P}^1(\mathbf{C}([0,1]^n)) \to \mathbf{I}\mathbb{R}$ is the maximal extension of the generalised Riemann integral and we have just seen that the interval-valued integral $\int : ([0,1]^n \to \mathbb{R}) \times \mathbf{P}^1(\mathbf{C}([0,1]^n)) \to \mathbf{I}\mathbb{R}$ is continuous and is an extension of the generalised Riemann integral on $[0,1]^n$. Thus, all we need to check is that for each pair $(f,\nu) \in \mathbf{P}^1(\mathbf{C}([0,1]^n))$, where ν is a simple valuation, we have:

$$\int^*(f,\nu) \sqsubseteq \int(f,\nu).$$

For $i \geq 0$, let $f_i \ll f$ and $\nu_i \ll \nu$, where $f_i \ll f_{i+1}$ and $\nu_i \ll \nu_{i+1}$, be sequences of step functions and simple valuations respectively, with $f = \sup_i f_i$ and $\nu = \sup_i \nu_i$. Let

$$O_i = (\Uparrow f_i) \times (\Uparrow \nu_i).$$

Then, O_i for $i \geq 0$, define a local base of open sets at (f,v), where $(f_{i+1},\nu_{i+1}) \in O_i$, and thus:

$$\int^*(f,v) = \sup_i \inf \int O_i \cap (([0,1]^n \to \mathbb{R}) \times M^1([0,1]^n))$$

Let $\nu_i = \sum_{j=1}^{j_i} r_{ij}\delta_{C_{ij}}$ and $\nu = \sum_{j=1}^n r_j\delta_{C_j}$. Assume the lower semi-continuous function f_i attains its minimum on the compact set C_{ij} at $x_{ij} \in C_{ij}$ for $i \geq 0$ and $j = 1,\ldots j_i$, and f^- attains its minimum on the compact set C_j at x_j for $1 \leq j \leq n$. Let

$$\mu_i = \sum_{j=1}^{j_i} r_{ij}\delta_{x_{ij}} \in M^1([0,1]^n).$$

and

$$\mu = \sum_{j=1}^n r_j\delta_{x_j} \in M^1([0,1]^n).$$

Then $\nu_i \sqsubseteq \mu_i$ and thus $(f_{i+1},\mu_{i+1}) \in O_i$ for each $i \geq 0$ and $\nu \sqsubseteq \mu$. By construction, we also have:

$$S^\ell(f_i^-,\nu_i) = S^\ell(f_i^-,\mu_i)$$

On the other hand, since f and f_i are step functions, f^- and f_i^- for $i \geq 0$ are continuous except for a finite number of points, and are thus R -integrable. Since f_i^- is an increasing sequence of lower semi-continuous functions converging on a compact set $[0,1]^n$ to a lower semi-continuous function f, it follows that the convergence is uniform, and, by [4, Theorem 6.8], the sequence of R-integrals of f_i with respect to any measure converges to that of f. Since $\nu = \sup_i \nu_i$, and the Scott topology of $\mathbf{P}^1(\mathbf{C}([0,1]^n))$ restricted to the subspace $M^1([0,1]^n)$ coincides with the weak topology [6], we have $\mu = \lim_{i\to\infty}\mu_i$ in the weak topology. Thus, by the continuity of the interval-valued integral we obtain:

$S^\ell(f^-,\nu) = \sup_i S^\ell(f_i^-,\nu_i)$
$= \sup_i S^\ell(f_i,\mu_i) = \sup_i \int f_i\,d\mu_i = \int f\,d\mu$

Since for each $i \geq 0$,

$$S^\ell(f_{i+1},\nu_{i+1}) \in \int O_i \cap (([0,1]^n \to \mathbb{R})),$$

we conclude that

$$S^\ell(f^-,\nu) \in \sup_i \inf \int O_i \cap (([0,1]^n \to \mathbb{R}) \times M^1([0,1]^n))$$

Similarly,

$$S^u(f^+,\nu) \in \sup_i \inf \int O_i \cap (([0,1]^n \to \mathbb{R}) \times M^1([0,1]^n))$$

and we conclude that $\int(f,\nu) \sqsupseteq \int^*(f,\nu)$.

■

IV. Lebesgue Integration via R-integration

In this section we show that the Lebesgue integral of L^1 functions on $\mathbb{R}^n$ with respect to any locally finite Borel measure, is the extension of the R-integral of continuous functions on compact subsets of $\mathbb{R}^n$. In addition, we extend the Lebesgue integral to locally finite valuations in $\mathbf{P}(\mathbf{C}(\mathbb{R}^n))$ and to closed balls of the space $L^1(\mu)$, the collection of Lebesgue integrable functions on $\mathbb{R}^n$ with respect to a locally finite Borel measure μ, i.e., finite on compact subsets of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is a vector space, is makes more sense to use $\mathbf{C}(\mathbb{R}^n)$ than the much larger space $\mathbf{C}(\mathbb{R}^n)$. To fix the ideas, in this section we will use $R\int$ to indicate the R-integral and $\int$ to indicate the Lebesgue integral.

Recall that $C_c^0(\mathbb{R}^n)$, the collection of real-valued continuous functions on $\mathbb{R}^n$ with compact support, is dense in $L^1(\mathbb{R}^n,\mu)$ [29, p. 68]. Note that locally finite measures on $\mathbb{R}^n$ can be obtained as the supremum of simple valuations on $\mathbf{C}(\mathbb{R}^n)$, or $\mathbf{U}(\mathbb{R}^n)$ [30]. Recall also that for any separable Banach space Y, the set $\mathbf{B}(Y)$ of closed balls of Y ordered by reverse inclusion is a countably based continuous domain [7]. Thus, $C_c^0(\mathbb{R}^n)$ is also dense in $\mathbf{B}(L^1(\mathbb{R}^n,\mu))$. It follows that the R-integral $R\int : C_c^0(\mathbb{R}^n) \to \mathbf{I}\mathbb{R}$ has a maximal extension

$$\left(R\int\right)^* : \mathbf{B}(L^1(\mathbb{R}^n,\mu)) \to \mathbf{I}\mathbb{R}$$

For convenience, we write the image of $f \in L^1(\mathbb{R}^n,\mu)$ under $(R\int)^*$ as $R\int^* f\,d\mu$. In this extended abstract, we will restrict ourselves to characterise the value of the extended integral on the maximal elements and show that the classical Lebesgue integral of an L^1 map with respect to a locally finite measure is obtained by the above extension:

Theorem IV.1. *If $f \in L^1(\mathbb{R}^n,\mu)$, where μ is a locally finite measure, then $R\int^* f\,d\mu = \int f\,d\mu$.*

Proof:

Let $f \in L^1(\mathbb{R}^n,\mu)$. The open balls $O_n := O_{1/n}(f)$ with centre f and radius $1/n$ in $L^1(\mathbb{R}^n,\mu)$ provide a countable local base of the Banach space for integers $n \geq 1$. Therefore,

$$\begin{aligned}&R\textstyle\int^* f\,d\mu\\ &= [a^-,a^+] := \sup_n \inf\{R\textstyle\int g\,d\mu : g \in O_n \cap C_c(\mathbb{R}^n)\}\end{aligned}$$

Recall that the R-integral, when it exists, coincides with the Lebesgue integral. Thus, by the density of $C_c(\mathbb{R}^n)$, there exists, for each $n \geq 1$, some $f_n \in C_c(\mathbb{R}^n)$ with support on some compact set X_n, say, such that

$$\begin{aligned}&|R\textstyle\int_{X_n} f_n\,d\mu - \int f\,d\mu| = |\int_{X_n} f_n\,d\mu - \int_{\mathbb{R}^n} f\,d\mu|\\ &\leq \textstyle\int_{\mathbb{R}^n} |f_n - f|\,d\mu < 1/n\end{aligned}$$

Now, $\inf\{R\int g\,d\mu : g \in O_n \cap C_c(\mathbb{R}^n)$ is a compact real interval which contains $R\int_{X_n} f_n\,d\mu$, for each $n \geq 1$. Since $\lim_{n\to\infty} R\int_{X_n} f_n\,d\mu = \int f\,d\mu$ it follows that

$$\int f\,d\mu \in \sup_n \inf\{R\int g\,d\mu : g \in O_n \cap C_c(\mathbb{R}^n)\} = [a^-,a^+]$$

It remains to show that $a^- = a^+ = \int f\,d\mu$. Let $\epsilon > 0$ be given. Then, for each $n \geq 0$, by the definition of infimum, there exists $g_n \in O_n \cap C_c(\mathbb{R}^n)$ with X_n as compact support such that $R\int_{X_n} g_n < a^- + \epsilon$. Since, as for f_n above, we have $\lim_{n\to\infty} R\int_{X_n} g_n\,d\mu = \int f\,d\mu$, it follows that $\int_{\mathbb{R}^n} f\,d\mu \leq a^- + \epsilon$. Since $\epsilon > 0$ is arbitrary, we obtain $\int_{\mathbb{R}^n} f\,d\mu \leq a^-$. Similarly, $\int_{\mathbb{R}^n} f\,d\mu \geq a^+$ and we conclude that $R\int^* f\,d\mu = \int_{\mathbb{R}^n} f\,d\mu$.

■

V. Extension of Green's Theorem

We now combine the results of sections I-C and III.5 regarding the extension of the classical derivative operator and the generalised Riemann integration to deduce a simpler alternative proof of the interval version of the Green's theorem that was obtained using interval valued integration in [14].

Let $U \subset \mathbb{R}^n$ be a relatively compact open set and $p : [0,1] \to U$ a continuous piecewise C^1 path in U from a given point $p(0) = a$ to a point $p(1) = b$. If $f : U \to \mathbb{R}$ is a C^1 map then the path integral $\int_0^1 f'(p(t) \cdot p'(t)\,dt = f(p(1)) - f(p(0)) = f(a) - f(b)$ is independent of the path p. In particular if $a = b$, then the path integral is always zero independent of the closed path p.

We now fix a piecewise C^1 map $p : [0,1] \to U$ and define the operator

$$D_p : C^1(U) \cap \mathrm{Lip}(U) \to ([0,1] \to \mathbf{I}\mathbb{R})$$

by $D_p(g) = \lambda t.\, g'(p(t)) \cdot p'(t)$, i.e., $D_p(g)$ gives the derivative of the composition $g \circ p$. Note that when $p'(t)$ is undefined as a real number, we put $p'(t)$ to be the interval given by the limit of the left and right derivatives of p at t. Thus, the path integral of the derivative of f is given by $\int_p f = \int_{[0,1]} D_p f$. In other words, the path integration of the derivative of a function with respect to path p is a map of type

$$\int_{[0,1]} \circ D_p : C^1(U) \cap \mathrm{Lip}(U) \to \mathbb{R}$$

which is the composition of D_p and the integral with respect to the Lebesgue measure on $[0,1]$. Now the maximal extension of this composition with type

$$\left(\int_{[0,1]} \circ D_p\right)^* : \mathrm{Lip}(U) \to \mathbf{I}\mathbb{R}$$

is still given by $\int_p f = f(p(1)) - f(p(0))$, since for any sequence $f_n \in C^1(U) \cap \mathrm{Lip}(U)$ with $\lim_n f_n = f$ in the L-topology, we have $\int_p f_n = f_n(p(1)) - f_n(p(0))$ and $\lim_{n\to\infty} f_n(p(1)) - f_n(p(0)) = f(p(1) - f(p(0))$.

However, the composition $(\int_{[0,1]})^* \circ (D_p)^*$ of the two extensions is not equal to $(\int_{[0,1]} \circ D_p)^*$. To see this, first note that we have:

$$(D_p)^* : \mathrm{Lip}(U) \to ([0,1] \to \mathbf{I}\mathbb{R})$$

with $(D_p)^*(f) = \lambda t.\,((\partial f)(p(t)) \cdot p'(t)$. Now consider the interval valued integral with respect to the Lebesgue measure of functions of type $([0,1] \to \mathbb{R})$, which as we have shown in Theorem III.5 is the extension $\int^*$ of the classical Riemann integral (equivalent to the R-integral): $\int : ([0,1] \to \mathbb{R}) \to \mathbb{R}$. Thus, we immediately obtain the interval version of Green's theorem, which was a main result in [14].

Theorem V.1. *The composition*

$$\left(\int_{[0,1]}\right)^* \circ (D_p)^* : \mathit{Lip}(U) \to \mathbf{I}\mathbb{R}$$

is given by

$$\begin{aligned}&(\textstyle\int_{[0,1]})^* \circ (D_p)^*(f)\\ &= \left[\int_0^1 ((\partial f)(p(t)) \cdot p'(t))^-, \int_0^1 ((\partial f)(p(t)) \cdot p'(t))^+\right]\end{aligned}$$

and satisfies:

$$f(p(1)) - f(p(0)) \in \left(\int_{[0,1]}\right)^* \circ (D_p)^*(f)$$

Clearly, the above composition is in general interval valued rather than real valued. For example, let $n = 1$, $U = (-1,2)$ and $p(t) = t$ and consider the Lipschitz function $f : (-1,2) \to \mathbb{R}$ with $\partial(f)(t) = [0,1]$ for all $t \in (-1,2)$ as in [15, Lemma 7.8]. Then, $f(p(1)) - f(p(0)) = f(1) - f(0)$ but since $p'(t) = 1$ and $\partial f(t) = [0,1]$ for all $t \in [0,1]$, we obtain $(\int_{[0,1]})^* \circ (D_p)^*(f) = [0,1]$.

We have thus constructed an example in which the extension of a composition of two higher order maps is not equal to the composition of the extension of the two maps.

VI. Conclusion and Further Work

We have used Scott's celebrated extension theorem for densely injective spaces to show that the Clarke gradient operator is the domain extension of the classical derivative operator and that Lebesgue integration as well as interval-valued integration are domain extensions of the generalised Riemann integral. These rather surprising results indicate that domain theory gives a unifying framework for basic differential and integral calculus, whose further implications would need to be addressed in future work.

We can extend all our results in Section II to functions of type $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, where $\mathbb{R}^m$ is equipped with the usual Euclidean norm. The results of Section III can be used to obtain effectively the R-integral of an effectively given continuous function on a compact metric space in the effectively given function space $([0,1]^n \to \mathbf{I}\mathbb{R})$ with respect to an effectively given finite measure in the effectively given probabilistic power domain $\mathbf{P}^1(\mathbf{C}([0,1]^n))$. This will provide an effective structure for continuous functions with compact support in $\mathbb{R}^n$ and for locally finite measures in $\mathbb{R}^n$. Using the results of Section IV, we can then obtain an effective domain-theoretic framework for computability of the Lebesgue integral which would be an alternative to those in Type Two Theory [31] and to the approach used in [32] that employs interval valued measurable functions.

There are several immediate questions for future work; we give three examples. Can the results in Section II be extended to complex Lipschitz maps using the L-topology induced from the complex L-derivative [33] on them? Can we obtain the chain rule for Clarke's gradient using the extension theorem? Can we extend Lebesgue integration to integration of interval-valued functions with respect to simple valuations?

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