AN EXTENSION RESULT FOR CONTINUOUS VALUATIONS

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Abstract

We show, by a simple and direct proof, that if a bounded valuation on a monotone convergence space is the supremum of a directed family of simple valuations then it has a unique extension to a Borel measure. In particular this holds for any directed complete partial order (dcpo) with the Scott topology. It follows that every bounded and continuous valuation on a continuous dcpo can be extended uniquely to a Borel measure. The last result also holds for σ-finite valuations, but fails for dcpo's in general.

1 Introduction

The motivation of the present work goes back to Horn and Tarski [11] who gave conditions for extending a measure μ defined on a subalgebra A of a given Boolean algebra L to the whole algebra. Pettis [21] continued this work for the case L is a lattice and μ is a valuation. The term valuation is borrowed from Birkhoff [4] to designate a nonnegative real function μ, defined on a lattice L, such that it is modular, monotone and $\mu(0) = 0$. Pettis gave the necessary conditions for extending a valuation to a measure on the σ-algebra generated by L. He showed applications of this result in the proof of the Riesz-Markoff theorem and the existence of the Haar measure.

In this paper a valuation is a real valued function defined on the lattice of open sets of a given topological space that is modular, monotone and such that $\mu(\emptyset) = 0$ (see below); a continuous valuation is one which preserves directed suprema. For a directed complete partial orders (dcpo), the problem of extending a continuous valuation to a Borel measure reappeared in the context of probabilistic nondeterminism [23]. In [23] a proof was given for ω-algebraic dcpo's but it contained a gap. Norberg established the result for

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σ-finite valuations on ω-continuous dcpo's and gave applications in random set theory and a proof of the Daniell-Kolmogorov theorem for continuous lattices.

Lawson [16] showed that a continuous valuation defined on a distributive continuous lattice L has a unique extension to a regular Borel measure on L (on the Borel σ-algebra of its Lawson topology). It follows that every continuous valuation on a locally quasicompact sober topological space extends uniquely to a measure, provided that the space is second countable or the patch topology is compact. Norberg and Vervaat gave an extension theorem for modular capacities defined on quasicompact sets [20, Theorem 3.7]. For coherent spaces such capacities and continuous valuations are equivalent notions ([20, Theorem 5.4] and [26, Chapter 3]). As an immediate consequence it follows that a valuation on a coherent space extends uniquely to a Borel measure (on the σ-algebra generated by the patch topology). This generalises Lawson's latter result. These results imply the extension result for ω-continuous dcpo's and coherent continuous dcpo's.

Finally Jones and Plotkin ([14, Theorem 8.2] and [13, Chapter 5]), following the approach in [23], claimed the result for continuous dcpo's, without presenting a correct proof.

Continuous valuations have in recent years played a key role in the domain theoretic approach to classical measure theory which has led to a new generalisation of the Riemann integral with applications in computation in fractal geometry [7]. In this paper we give a short and direct proof, based on the original idea in [23]. We show that if a σ-finite valuation on a monotone convergence space (see below), is the supremum of a directed family of simple valuations then it has a unique extension to a Borel measure. As a corollary we have that every bounded and continuous valuation on a continuous dcpo can be extended uniquely to a Borel measure. We also show that the extension fails in general for a continuous valuation on a dcpo.

2 Basic definitions and auxiliary results

We recall some basic definitions, our main references for domain theory are [8, 12, 1]. A topological space is a pair (X, τ) where X is a set and τ is a topology for X. We denote by $\mathcal{B}(X)$ the Borel $\sigma-$algebra of (X, τ). A Borel measure μ is finite if $\mu(X) < \infty$ and σ-**finite** if there exists a countable family $\{C_i\}_{i \in \mathbb{N}}$ of sets in $\mathcal{B}(X)$ such that $X = \bigcup_{i \in \mathbb{N}} C_i$ and $\mu(C_i) < \infty$ for all $i \in \mathbb{N}$. We say that a measure is normalised if $\mu(X) = 1$. Observe that every finite measure with $\mu(X) > 0$ can be normalised. For the measurable space $(X, \mathcal{B}(X))$ we denote by $\mathbf{M}X$ the set of all positive Borel measures bounded by 1 and by $\mathbf{M}^1X$ the set of all probability measures.

Let $(P, \sqsubseteq)$ be a **partially ordered set** (poset). For $A \subseteq P$ we define $\downarrow A = \{x \in P \mid \exists a \in A.x \sqsubseteq a\}$. We often abbreviate $\downarrow\{a\}$ by $\downarrow a$. In a similar way we define $\uparrow A$ and $\uparrow a$. The set A is **lower** if $A = \downarrow A$ and **upper** if $A = \uparrow A$. A nonempty subset $A \subseteq P$ is said to be **directed** if for all $x, y \in A$ there exists $z \in A$ such that $x, y \sqsubseteq z$. A nonempty subset $T \subseteq P$ is called a **chain** if for all $x, y \in T$ we have $x \sqsubseteq y$ or $y \sqsubseteq x$. A **directed complete partial order** is a poset P in which every directed subset A has a least upper bound (lub) denoted by $\bigsqcup^{\uparrow} A$. The **Scott topology** of a poset $(P, \sqsubseteq)$ is defined as follows: $Q \subseteq P$ is open if Q is upper and for all directed subsets $A \subseteq P$ such that $\bigsqcup^{\uparrow} A$ exists, we have $\bigsqcup^{\uparrow} A \in Q$ implies that $Q \cap A \neq \emptyset$. Note that $\downarrow a$ is closed for all $a \in P$. The Scott topology will be denoted by ΩP. A dcpo with the Scott topology is a T_0 space.

Conversely, any T_0 topological space (X, τ) can be partially ordered via the **specialisation order**. Let $cl(B)$ denote the topological **closure** of any $B \subseteq X$. The specialisation order is defined by $x \sqsubseteq y$ if and only if $x \in cl(\{y\})$. We define a **monotone convergence** space [8, Chapter II,3.9] as a triple $(X, \tau, \sqsubseteq)$ satisfying: (X, τ) is a T_0 space with $\sqsubseteq$ the specialisation order, $(X, \sqsubseteq)$ is a dcpo and τ is coarser than the Scott topology. It is easy to see that in a monotone convergence space every directed set A converges to its supremum, i.e. for all open sets O such that $\bigsqcup^{\uparrow} A \in O$ there exists $x \in A$ such that for all $y \in A$ if $x \sqsubseteq y$, then $y \in A$. A dcpo $(D, \sqsubseteq)$ with the Scott topology is a monotone convergence space and the specialisation order coincides with $\sqsubseteq$.

Assume for the rest of this section that (X, τ) is any topological space. A closed subset C of X is called **irreducible**, if for any closed subsets C_1, C_2 of X with $C \subseteq C_1 \cup C_2$, we have $C \subseteq C_1$ or $C \subseteq C_2$. We say X is **sober** if all closed irreducible subsets of X are the closure of a unique point. A sober space is always a monotone convergence space (hence T_0), but there exist monotone convergence spaces that are not sober (see the last section of the paper). A subset Q of X is called **quasicompact** if it has the Heine-Borel property, i.e. every open cover of Q has a finite subcover. X is compact if it is quasicompact and Hausdorff. X is said to be **locally quasicompact** if for all $y \in X$ and $U_1 \in \tau$ with $y \in U_1$, there exists a quasicompact subset Q and an open set U_2 such that $y \in U_2$ and $U_2 \subseteq Q \subseteq U_1$. X is **coherent** if it is sober, locally quasicompact and the finite intersection of upper quasicompact sets is quasicompact.

Let x, y be elements of a dcpo D. We say x approximates y or x is **way below** y, denoted by $x \ll y$, if whenever $y \sqsubseteq \bigsqcup^{\uparrow} A$ and A is a directed set, then there exists $a \in A$ with $x \sqsubseteq a$. We have that $x \ll y$ implies that $x \sqsubseteq y$. For $C \subseteq D$ we define $\twoheaduparrow C = \{x \in D \mid \exists c \in C. c \ll x\}$. We abbreviate $\twoheaduparrow\{c\}$ by $\twoheaduparrow c$. In a similar way we define $\twoheaddownarrow C$ and $\twoheaddownarrow c$. A subset $B \subseteq D$ is called a **basis** if for all $x \in D$ the set $B_x = B \cap \twoheaddownarrow x$ is directed and $x = \bigsqcup^{\uparrow} B_x$. D is a **continuous dcpo** if it has a basis and an ω-continuous dcpo if it has a countable basis. In a continuous dcpo D for all $C \subseteq D$ the set $\twoheaduparrow C$ is open. A continuous dcpo with the Scott topology is always sober, locally quasicompact but not necessarily coherent.

Given a topological space (X, τ), a **valuation** [4, 16, 14] ν is a map $\nu : \tau \to [0, \infty]$ which satisfies:

- $\nu(\emptyset) = 0$ (strictness)
- $Q_1 \subseteq Q_2 \Rightarrow \nu(Q_1) \leq \nu(Q_2)$ (monotonicity)
- $\nu(Q_1 \cup Q_2) + \nu(Q_1 \cap Q_2) = \nu(Q_1) + \nu(Q_2)$ (modularity).

A valuation ν is said to be continuous [16, 14] if for any $D \subseteq \tau$, which is directed with respect to $\subseteq$, we have $\nu(\bigcup D) = \sup_{Q \in D} \nu(Q)$. For any $a \in X$ we define the point valuation based at a as the function $\delta_a : \tau \to [0, \infty)$ such that

$$\delta_a(Q) = \begin{cases} 1 & \text{if } a \in Q \\ 0 & \text{otherwise.} \end{cases}$$

A **simple valuation** ν is any finite linear combination $\sum_{i=1}^{n} r_i \delta_{a_i}$ of point valuations with $r_i \in \mathbb{R}^+ \backslash \{0\}$ for $i = 1, \ldots, n$. The set $\{a_1, a_2, \ldots, a_n\}$ is called the support of ν and is denoted by $|\nu|$. Simple valuations are always continuous. A simple valuation can be extended to a measure on the whole power set of X and its extension to a measure on $\mathcal{B}(X)$ is unique. For simplicity we will use the same name for a simple valuation and its extension.

The **probabilistic power domain** $\mathbf{P}X$ of a topological space (X, τ) is defined as the set of continuous valuations on X bounded by 1 with the following order: $\mu \sqsubseteq \nu$ if and only if $\mu(Q) \leq \nu(Q)$ for all $Q \in \tau$. $(\mathbf{P}X, \sqsubseteq)$ is a dcpo having the constant zero valuation as bottom element. For any directed subset $\{\nu_i\}_{i \in I}$ of $\mathbf{P}X$, the lub $\nu = \bigsqcup^{\uparrow}_{i \in I} \nu_i$ is given by $\nu(Q) = \sup_{i \in I} \nu_i(Q)$ for all $Q \in \tau$. We denote by $\mathbf{P}_s X$ the subset of simple valuations of $\mathbf{P}X$. The normalised probabilistic power domain of X is defined as $\mathbf{P}^1 X = \{\mu \in \mathbf{P}X \mid \mu(X) = 1\}$. If D is a dcpo with bottom element $\perp$ then $\mathbf{P}^1 D$ has $\delta_\perp$ as bottom element.

Let J be a directed set and L be a set. A **net** is a function $j \mapsto x_j : J \to L$. If L is a poset and $j \sqsubseteq k$ implies that $x_j \sqsubseteq x_k$ for all $j, k \in J$ then we say that the net is **monotone**. Nets will also be denoted as $\langle x_j \rangle_{j \in J}$. Observe that the image of a monotone net is a directed subset of L. Conversely for any directed $D \subseteq L$ the inclusion map from D to L is a monotone net. We will use monotone nets and directed subsets interchangeably. Let $\langle x_j \rangle_{j \in J}$ be a net ranging over the real numbers. Recall that $\lim_{j \in J} x_j = l \in \mathbb{R}$ if and only if for all $\varepsilon > 0$ there exists $k \in J$ such that $|x_j - l| < \varepsilon$ if $k \sqsubseteq j$.

Let X be a set. A collection $\mathcal{S}$ of subsets of X is called a (Boolean) **semialgebra of subsets** of X if: (i) $\emptyset, X \in \mathcal{S}$; (ii) $\mathcal{S}$ is closed under finite intersections; (iii) if $A \in \mathcal{S}$ then its complement A^c, can be expressed as a finite disjoint union of elements of $\mathcal{S}$. If $\mathcal{S}$ is a semialgebra of subsets of X, then the set of finite disjoint unions of elements of $\mathcal{S}$ is the minimum algebra which contains $\mathcal{S}$.

Lemma 2.1 *(Splitting lemma) Let $(D, \tau, \sqsubseteq)$ be a monotone convergence space and $\nu_1 = \sum_{b \in |\nu_1|} r_b \delta_b$, $\nu_2 = \sum_{c \in |\nu_2|} s_c \delta_c \in \mathbf{P}^1 D$. Then $\nu_1 \sqsubseteq \nu_2$ if and only if there exists a function $f : |\nu_1| \times |\nu_2| \to [0,1]$ such that for all $b \in |\nu_1|$ and $c \in |\nu_2|$*

(1) $\sum_{c \in |\nu_2|} f(b,c) = r_b$.

(2) $\sum_{b \in |\nu_1|} f(b,c) = s_c$.

(3) $f(b,c) > 0$ *implies* $b \sqsubseteq c$.

Proof This result is known for dcpo's with the Scott topology [6, Proposition 3.1]. So we only need to note that $|\nu_1| \cup |\nu_2|$ with the inherited order is a finite dcpo. The Scott topology in this space is the subspace topology induced by τ. Therefore, there exists f satisfying the above conditions. Conversely, if such f exists, then $\nu_1 \sqsubseteq \nu_2$ with respect to ΩD. Since $\tau \subseteq \Omega D$ the conclusion follows. ∎

A general version of the above result exists for Polish spaces [25, Theorem 11] [15, Theorem 1-(ii)]. A subset T of a topological space (X, τ) is called a **crescent** if there are open sets U and V such that $V \subseteq U$ and $T = U \backslash V$. We denote the set of all crescents of (X, τ) by $Cres(\tau)$. The next Proposition is a reformulation of Lemma 1 of [23] for monotone convergence spaces.

Proposition 2.2 *Let $(D, \tau, \sqsubseteq)$ be a monotone convergence space.*

(1) $\tau \subseteq Cres(\tau)$.

(2) $T \in Cres(\tau)$ if and only if there exist an open set A and a closed set B such that $T = A \cap B$.

(3) If $T \in Cres(\tau)$ then

- *If $x, y \in T$ and $x \sqsubseteq z \sqsubseteq y$ then $z \in T$.*
- *If $S \subseteq T$ is a directed subset then $\bigsqcup^{\uparrow} S \in T$.*
- *If $S \subseteq D$ is a directed subset and $\bigsqcup^{\uparrow} S \in T$ then there exists $d_0 \in S$ such that $d_0 \in T$.*

(4) $Cres(\tau)$ is a semialgebra.

(5) Let $\mu : Cres(\tau) \to [0, 1]$ be a function. Then μ has a unique extension to a probability measure on the Borel σ-algebra of (D, τ) if and only if μ is finitely additive, σ-subadditive and $\mu(D) = 1$.

Proof (1) and (2) are trivial, (4) follows from (2). As a consequence we have that the σ-algebra generated by $Cres(\tau)$ is exactly the Borel σ-algebra of (D, τ).

(3) Assume that $T = U \backslash V$ with $U, V \in \tau$ and $V \subseteq U$. Recall $\tau \subseteq \Omega D$.

- Suppose that $x, y \in T$ and $x \sqsubseteq z \sqsubseteq y$. We have $z \in U$ since $x \in U$ and U is upper. For the same reason if $z \in V$ then we would have $y \in V$ which is false. Therefore $z \notin V$ and we conclude that $z \in T$.
- Suppose that $S \subseteq T$ is a directed set. $\bigsqcup^{\uparrow} S \in U$ since U is upper. If $\bigsqcup^{\uparrow} S \in V$ then there exists $d \in S$ such that $d \in V$. This contradicts $S \subseteq T$. Therefore $\bigsqcup^{\uparrow} S \notin V$ and $\bigsqcup^{\uparrow} S \in T$.
- Since $\bigsqcup^{\uparrow} S \in U$ and U is open then there exists $d_0 \in S$ with $d_0 \in U$. If there exists $d \in S \cap V$ then $\bigsqcup^{\uparrow} S \in V$, since V is upper, contradicting $\bigsqcup^{\uparrow} S \in T$. Therefore $S \subseteq V^c$ and $d_0 \in T$.

(5) Let $\mu : \mathcal{S} \to [0, \infty]$ be a function defined on a semialgebra $\mathcal{S}$ and such that $\mu(\emptyset) = 0$. Then μ has a unique extension to a measure on the algebra $\mathcal{A}$ generated by $\mathcal{S}$ if μ is finitely additive and σ-subadditive. By Carathéodory's extension theorem a σ-finite measure on an algebra

$\mathcal{A}$ has a unique extension to a measure on the σ-algebra generated by $\mathcal{A}$ [22, p. 295-298]. ∎

Lemma 2.3 *(Smiley, Horn, Tarski) A normalised valuation ν on a topological space (X, τ) extends uniquely to a finitely additive measure $\overline{\nu}$ on the algebra of sets generated by the topology.*

Proof A normalised valuation is a partial measure in the sense of Horn and Tarski [11, Theorem 1.9]. Therefore it can be extended uniquely to a finitely additive measure on the Boolean algebra generated by the lattice of open sets (Theorem 1.22). See also [24]. ∎

Note that the algebra of sets generated by τ consists of finite disjoint unions of crescents. For any $T \in Cres(\tau)$ with $T = U \backslash V$ and $V \subseteq U$ we have $\overline{\nu}(U \backslash V) = \nu(U) - \nu(V)$.

Lemma 2.4 *Let D be a continuous dcpo. Then D is ω-continuous if and only if ΩD is second countable.*

Proof (cf. [19, Proposition 3.1]) For the 'only if' part suppose that D is an ω-continuous dcpo with countable basis B. Then it is easy to see that the set $\{\Uparrow b \mid b \in B\}$ is a countable basis for the topology.

For the 'if' part suppose that S is a countable basis for the topology. Consider the set $P = \{(u, v) \mid u, v \in S \text{ and } \exists w \in D. u \subseteq\uparrow w \subseteq v\}$. Using a choice function we can construct a set $B = \{w_p \in D \mid p \in P\}$ such that for all $w_{(u,v)} \in B$ we have $u \subseteq\uparrow w_{(u,v)} \subseteq v$. Since $P \subseteq S \times S$ then B is a countable set.

We will show that B has the interpolation property [8, Chapter I,1.15], i.e. if $x, z \in D$ and $x \ll z$ then there exists $w \in B$ such that $x \ll w \ll z$. Suppose that $x, z \in D$ with $x \ll z$. Since D is a continuous dcpo then there exists $y \in D$ such that $x \ll y \ll z$. Therefore $y \in \Uparrow x$ and $z \in \Uparrow y$. It follows that there exist two basic open sets $v, u \in S$ such that $y \in v \subseteq \Uparrow x$ and $z \in u \subseteq \Uparrow y$. Hence $u \subseteq \Uparrow y \subseteq\uparrow y \subseteq v$. As a consequence we have that $(u, v) \in P$. Therefore there exists $w_{(u,v)} \in B$ such that $z \in u \subseteq\uparrow w_{(u,v)} \subseteq v \subseteq \Uparrow x$. We conclude that $x \ll w_{(u,v)} \ll z$ with $w_{(u,v)} \in B$.

Since D is a continuous dcpo, checking that B is a basis is an easy consequence of the interpolation property. ∎

A valuation ν on a topological space (X, τ) is called **countably continuous** if for all ω-chains $\{Q_i\}_{i\in\mathbb{N}}$ in τ we have $\nu(\bigcup_{i\in\mathbb{N}} Q_i) = sup_{i\in\mathbb{N}}\nu(Q_i)$.

Lemma 2.5 *Let D be an ω-continuous dcpo. A valuation ν on D is continuous if and only if it is countably continuous.*

Proof The 'only if' part is immediate. For the 'if' part let $B = \{b_n \mid n \in \mathbb{N}\}$ be a countable basis for the dcpo. Suppose that we have a directed union $Q = \bigcup^{\uparrow}_{k\in K} Q_k$ where $Q_k \in \Omega D$ for all $k \in K$. We must show that $\nu(Q) = \sup_{k\in K} \nu(Q_k)$.

Since ν is a valuation, by monotonicity we have $\sup_{k\in K} \nu(Q_k) \leq \nu(Q)$. Let $c_i = \{b_1, b_2, \ldots, b_i\}$ for all $i \in \mathbb{N}$. Since B is a basis and Q is an open set, for all $x \in Q$ there exists $m \in \mathbb{N}$ such that $b_m \ll x$ and $b_m \in Q$. Hence $x \in \Uparrow b_m \subseteq \Uparrow(Q \cap c_m)$. Therefore $Q = \bigcup_{i\in\mathbb{N}} \Uparrow(Q \cap c_i)$ and $\{\Uparrow(Q \cap c_i)\}_{i\in\mathbb{N}}$ is an ω-chain in ΩD. Since we are assuming that ν is a countably continuous valuation we have $\nu(Q) = \sup_{i\in\mathbb{N}} \nu(\Uparrow(Q\cap c_i))$. Observe that for all $i \in \mathbb{N}$ and $b \in Q \cap c_i$ there exists a $k \in K$ such that $b \in Q_k$. Since the union $\bigcup^{\uparrow}_{k\in K} Q_k$ is directed we can find $k_i \in K$ such that $Q \cap c_i \subseteq Q_{k_i}$. Therefore for all $i \in \mathbb{N}$ we have $\Uparrow(Q \cap c_i) \subseteq Q_{k_i}$. Hence $\nu(\Uparrow(Q \cap c_i)) \leq \nu(Q_{k_i}) \leq \nu(Q)$ for all $i \in \mathbb{N}$. We conclude that $\sup_{k\in K} \nu(Q_k) = \nu(Q)$. ∎

3 A generalised Splitting Lemma

Let $(D, \tau, \sqsubseteq)$ be a monotone convergence space and $\{\nu_n\}_{n\in\mathbb{N}}$ be an ω-chain of normalised simple valuations on D. Consider a sequence of functions $f_i : |\nu_i| \times |\nu_{i+1}| \to [0, 1]$ such that f_i satisfies the conditions of Lemma 2.1 for ν_i and ν_{i+1}. Let $S = \prod_{i\in\mathbb{N}} |\nu_i|$ endowed with the product topology of the discrete spaces $|\nu_i|$. For any A, B subsets of $\mathbb{N}$ with $A \subseteq B$, let $\pi_{B,A} : \prod_{i\in B} |\nu_i| \to \prod_{i\in A} |\nu_i|$ denote the natural projection of $\prod_{i\in B} |\nu_i|$ onto $\prod_{i\in A} |\nu_i|$. For any $E \subseteq S$ and $n \in \mathbb{N}$ define $E\restriction_n = \pi_{\mathbb{N},\{0,1,\ldots,n\}}(E)$, i.e. the projection of the elements of E into their first $n+1$ coordinates. For a given $s = (x_0, x_1, \ldots, x_n) \in S^n$ we define

$$p_n(\{s\}) = \nu_0(\{x_0\}) \prod_{0\leq i<n} \frac{f_i(x_i, x_{i+1})}{\nu_i(\{x_i\})}.$$

Each factor $f_i(x, y)/\nu_i(\{x\})$ gives the conditional probability that $y \in \nu_{i+1}$ given $x \in \nu_i$. According to this interpretation, $p_n(\{s\})$ is the probability of the orbit s. Since the set $S\restriction_n$ is finite, the function p_n extends naturally to all subsets of $S\restriction_n$ and the following lemma holds.

Lemma 3.1 *For all $k, n \in \mathbb{N}$ with $k \leq n$,*

- $\nu_k = p_n \circ \pi^{-1}_{\{0,\ldots,n\},\{k\}}$
- $p_n = p_{n+1} \circ \pi^{-1}_{\{0,\ldots,n,n+1\},\{0,\ldots,n\}}$.

In particular $p_n(S\lceil_n) = \nu_n(|\nu_n|) = 1$, and therefore p_n is a probability measure on $S\lceil_n$ for all $n \in \mathbb{N}$. We now show that the functions p_n extend to a probability measure on the whole product space S, which is a generalisation of a standard result as in [3, Example 1.2]. Let A denote any finite subset of $\mathbb{N}$ and $m = max(A)$. We define a probability on $\prod_{i\in A}|\nu_i|$ by $P_A = p_m \circ \pi^{-1}_{\{0,1,\ldots,m\},A}$. If B is any other finite subset of $\mathbb{N}$ such that $A \subseteq B$, then $P_A = P_B \circ \pi^{-1}_{B,A}$ by Lemma 3.1. Therefore $\{P_F \mid F \text{ finite}, F \subseteq \mathbb{N}\}$ is a consistent family of finite-dimensional probability distributions and by the Daniell-Kolmogorov theorem (as stated in [5, Theorem 12.1.2]), there exists a probability measure P on the Borel σ-algebra $\mathcal{B}(S)$ satisfying $P_F = P\circ\pi^{-1}_{\mathbb{N},F}$ for all finite $F \subseteq \mathbb{N}$. Suppose $j, k \in \mathbb{N}$ and $A_i \subseteq |\nu_{j+i}|$ for $i = 0, \ldots, k$. Define the cylinder $\langle j; A_0, \ldots, A_k\rangle$ as

$$\begin{aligned}\langle j; A_0, \ldots, A_k\rangle &= \pi^{-1}_{\mathbb{N},\{j,\ldots,j+k\}}(A_0 \times \cdots \times A_k) \\ &= \{(x_m) \in S \mid x_{j+i} \in A_i \text{ for } i = 0, \ldots, k\}.\end{aligned}$$

Then,

- $P(\langle n; A\rangle) = \nu_n(A)$
- $P(\langle n; \{x\}, \{y\}\rangle) = f_n(x, y)$
- $P(\langle 0; A_0, \ldots, A_n\rangle) = p_n(A_0 \times \cdots \times A_n)$.

Now for all $E \subseteq S$ and $n \in \mathbb{N}$ define

$$\begin{aligned}[n, E] &= \pi^{-1}_{\mathbb{N},\{0,\ldots,n\}}(\pi_{\mathbb{N},\{0,\ldots,n\}}(E)) \\ &= \{(x_i) \in S \mid (x_0, x_1, \ldots, x_n) \in E\lceil_n\}.\end{aligned}$$

Note that $[n, E] \supseteq [n+1, E] \supseteq E$ for all $n \in \mathbb{N}$. Let C be the set of chains in S, i.e. $C = \{(x_k) \in S \mid x_k \sqsubseteq x_{k+1} \text{ for all } k \in \mathbb{N}\}$. For every $E \subseteq S$, the set $[n, E]$ is closed, and hence $C = \bigcap_{n\in\mathbb{N}}[n, C]$ is closed. Since $f_i(x, y) > 0$ only if $x \sqsubseteq y$, it follows that $P(C) = 1$.

Proposition 3.2 *Let $\{\nu_n\}_{n\in\mathbb{N}}$ be an ω-chain in $\mathbf{P}^1_s D$. Let (A_n) be a sequence of pairwise disjoint sets such that $A_n \subseteq |\nu_n|$. Suppose that for some fixed $\alpha > 0$, $\nu_n(A_n) \geq \alpha$, for all $n \in \mathbb{N}$. Then there exists $(x_k) \in C$ such that $x_n \in A_n$ for infinitely many $n \in \mathbb{N}$.*

Proof Let $C(A_n) = C \cap \langle n; A_n \rangle$, i.e. the set of chains $(x_k) \in S$ such that $x_n \in A_n$. Since C is closed, it is Borel measurable, therefore

$$\begin{aligned} P(C(A_n)) &= P(C \cap \langle n; A_n \rangle) \\ &= P(\langle n; A_n \rangle) \qquad \text{since } P(C) = 1 \\ &= \nu_n(A_n) \\ &\geq \alpha. \end{aligned}$$

Now observe that the subset of chains $(x_k) \in C$ such that $x_n \in A_n$ for infinitely many $n \in \mathbb{N}$, is exactly $\bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} C(A_n) = \limsup_n C(A_n)$. Therefore $P(\limsup_n C(A_n)) \geq \alpha > 0$. Since P is a measure we conclude that $\limsup_n C(A_n) \neq \emptyset$. This last result also follows from extended versions of the Borel-Cantelli Lemma for nonindependent events (see [17, Lemma 3.5] or [18]). ∎

4 The extension results

We first prove the extension theorem in the case of normalised simple valuations.

Theorem 4.1 *Let $(D, \tau, \sqsubseteq)$ be a monotone convergence space and $\langle \mu_j \rangle_{j \in J}$ be a directed subset of $\mathbf{P}^1_s D$ with $\mu = \sup_{j \in J} \mu_j$. Then μ has a unique extension to a measure on the Borel σ-algebra of (D, τ).*

Proof Since μ is the supremum of a directed family of continuous valuations, it is easy to check that μ is a continuous valuation and $\mu(D) = 1$. As a consequence of Lemma 2.3, μ extends uniquely to a finitely additive set function $\overline{\mu}$ defined on $Cres(\tau)$. In order to apply part (5) of Proposition 2.2 we only need to check that $\overline{\mu}$ is σ-subadditive on $Cres(\tau)$.

Suppose that $T \in Cres(\tau)$ is the union of a countable disjoint sequence (T_i) in $Cres(\tau)$. We show that $\overline{\mu}(T) \leq \sum_{i=1}^{\infty} \overline{\mu}(T_i)$. In order to derive a contradiction assume $\lim_m \sum_{i=1}^{m} \overline{\mu}(T_i) = \sum_{i=1}^{\infty} \overline{\mu}(T_i) < \overline{\mu}(T)$. This happens if and only if there exists some $\alpha > 0$ such that for all $m \in \mathbb{N}$ we have $\sum_{i=1}^{m} \overline{\mu}(T_i) \leq \overline{\mu}(T) - 2\alpha$. Therefore $2\alpha \leq \lim_j \mu_j(\bigcup_{i>m} T_i)$. This means that if we fix $m \in \mathbb{N}$ there exists $n_m \in J$ such that $n_m \sqsubseteq j$ implies that $\mu_j(\bigcup_{i>m} T_i) \geq \alpha$.

Let $N_0 \in J$ be such that $\mu_{N_0}(T) \geq \alpha$. Observe that for all $j \in J$, the set $|\mu_j| \cap T$ is finite. In particular since $|\mu_{N_0}|$ is finite, there exists $p_1 \in \mathbb{N}$ such that $|\mu_{N_0}| \cap T \subseteq \bigcup_{m \leq p_1} T_m$. Since we are assuming $\lim_{j \in J} \mu_j(\bigcup_{m > p_1} T_m) \geq$

2α, there exists $N_1 \in J$ such that $\mu_{N_1}(\bigcup_{m>p_1} T_m) \geq \alpha$. Because $\langle \mu_j \rangle_{j\in J}$ is directed we can also assume $\mu_{N_0} \sqsubseteq \mu_{N_1}$. In general, since $|\mu_{N_j}| \cap T \subseteq \bigcup_{m \leq p_{j+1}} T_m$ is finite we can repeat the argument and find $N_{j+1} \in J$ and $p_{j+2} \in \mathbb{N}$ such that

- $\mu_{N_j} \sqsubseteq \mu_{N_{j+1}}$
- $|\mu_{N_{j+1}}| \cap T \subseteq \bigcup_{m \leq p_{j+2}} T_m$
- $A_{j+1} = |\mu_{N_{j+1}}| \cap \bigcup_{p_{j+1} < m \leq p_{j+2}} T_m$ with $\mu_{N_{j+1}}(A_{j+1}) \geq \alpha$.

Note that the sets A_j are pairwise disjoint for all $j \in \mathbb{N}$. Then $\{\mu_{N_n}\}_{n\in\mathbb{N}}$ is an ω-chain of normalised simple valuations with $\mu_{N_n}(A_n) \geq \alpha$. Applying Proposition 3.2 we know that there exists an ω-chain $\{x_k\}_{k\in\mathbb{N}}$ in D such that $x_n \in A_n$ for infinitely many $n \in \mathbb{N}$. Since we can remove from this chain all the terms x_k such that $x_k \notin A_k$ we can assume, without loss of generality, that each x_k belongs to one and only one A_n. Since T is a crescent, by part (3) of Proposition 2.2 it follows that $\bigsqcup_{k\in\mathbb{N}} x_k \in T$. Hence there exists i_0 such that $\bigsqcup_{k\in\mathbb{N}} x_k \in T_{i_0}$. Again by part (3) of Proposition 2.2 an infinite number of x_k will be in T_{i_0} and this is a contradiction since each x_k was chosen in a different T_i and the T_i are pairwise disjoint.

We conclude that $\overline{\mu}$ is σ-subadditive in $Cres(\tau)$. By part (5) of Proposition 2.2 we conclude that μ has a unique extension to a measure on the σ-algebra generated by τ. ∎

Corollary 4.2 *Let $(D, \tau, \sqsubseteq)$ be a monotone convergence space and $\langle \mu_j \rangle_{j\in J}$ a directed subset of $\mathbf{P}_s D$ with $\mu = \sup_{j\in J} \mu_j$. If $\mu(D) < \infty$ then μ has a unique extension to a measure on the Borel σ-algebra of (D, τ).*

Proof We will show that this case reduces to the normalised case. First observe that if $\mu(D) = 0$ then μ is identically zero therefore the constant zero measure is the only possible extension of μ. If $\mu(D) > 0$, let $\mu' = \mu/\mu(D)$ and $\mu'_j = \mu_j/\mu(D)$. Then $\mu'(D) = 1$ and $\mu' = \sup_{j\in J} \mu'_j$ where $\langle \mu'_j \rangle_{j\in J}$ is a directed set of simple valuations. It is clear that μ will have a unique extension to a measure if and only if μ' does.

If D has a bottom element $\perp$ then for all $j \in J$, let

$$\nu_j = \mu'_j + (1 - \mu'_j(D))\, \delta_\perp.$$

Then ν_j is a simple valuation, $\nu_j(D) = \nu_j(\uparrow \perp) = 1$ and $\nu_j(Q) = \mu'_j(Q)$ for all $Q \in \tau \backslash \{D\}$. Therefore $\langle \nu_j \rangle_{j\in J}$ is a directed subset of $\mathbf{P}^1_s D$ with $\mu' = \sup_{j\in J} \nu_j$ and the conclusion follows from Theorem 4.1.

If D does not have a bottom element we can consider the lifting $D_\perp = D \cup \{\perp\}$ where $\perp$ is the least element of $D_\perp$. Then we have $\tau_\perp = \tau \cup \{D_\perp\}$. Since $\mu'(D) = 1$, it is clear that we can extend μ' to $D_\perp$ by letting $\mu'(D_\perp) = 1$. The definition given above for the ν_j will do as an extension for the μ'_j. Now we can apply Theorem 4.1 and find a unique extension of μ' to a measure $\overline{\mu}'$ on $\mathcal{B}(D_\perp)$. The restriction of this measure to $\mathcal{B}(D)$ will give us the desired measure since $\overline{\mu}'(\perp) = \overline{\mu}'(D_\perp \backslash D) = \mu'(D_\perp) - \mu'(D) = 0$. ∎

Corollary 4.3 *If D is a continuous dcpo then every bounded valuation has a unique extension to a measure on the Borel σ-algebra of $(D, \Omega D)$.*

Proof If D is a continuous dcpo then the set of continuous valuations on D is also a continuous dcpo with a basis of simple valuations [14, 13]. Hence every bounded valuation is the supremum of a directed set of simple valuations and the conclusion follows from the last corollary. ∎

This shows that the extension results considered in this paper are not directly implied by the extension theorem of Norberg and Vervaat. A continuous dcpo need not be coherent and the latter theorem is given for spaces where the intersection of upper quasicompact sets is quasicompact. On the other hand, we do not know if a continuous valuation on a coherent space is always the lub of a directed set of simple valuations. Hence, our results might not apply in that setting. Similar remarks apply to Lawson's result for locally quasicompact second countable sober spaces.

5 The σ-finite case

Next we present the extension result for σ-finite valuations.

Theorem 5.1 *$(D, \tau, \sqsubseteq)$ be a monotone convergence space. Let $\langle \mu_j \rangle_{j \in J}$ be a directed set of simple valuations on D with $\mu = \sup_{j \in J} \mu_j$. If $D = \bigcup_{i \in \mathbb{N}} Q_i$ with $Q_i \in \tau$ and $\mu(Q_i) < \infty$ then μ has a unique extension to a measure on the Borel σ-algebra of (D, τ).*

Remark 5.2 *This theorem is only more general in case D is not quasi-compact. Otherwise, under the assumptions of the theorem, $\{Q_i\}_{i \in \mathbb{N}}$ would have a finite subcover, implying $\mu(D) < \infty$ and Theorem 4.1 would apply. In particular, this is true if $D = \uparrow F$ for some finite set $F \subseteq D$.*

Proof of Theorem 5.1 Without loss of generality we can assume $Q_i \subseteq Q_{i+1}$ for all $i \in \mathbb{N}$. As in the proof of Theorem 4.1 we will first extend μ

to the semialgebra $Cres(\tau)$ and show that this extension is finitely additive and σ-subadditive. The idea is to extend μ to a finite measure on Q_i and then take the supremum of such measures.

Let ν be a continuous valuation on D and $Q \in \tau$. The restriction of ν to Q defined as $\nu|_Q(A) = \nu(A \cap Q)$ for all $A \in \tau$ is also a continuous valuation on D. If $\nu = \sum_{b \in |\nu|} r_b \delta_b$ is a simple valuation then $\nu|_Q = \sum_{b \in |\nu| \cap Q} r_b \delta_b$ is also a simple valuation. It is straightforward to check that $\langle \mu_j|_Q \rangle_{j \in J}$ is a directed set of simple valuations on D and $\mu|_Q = \sup_{j \in J} \mu_j|_Q$. Notice that for all $i \in \mathbb{N}$, $\mu|_{Q_i}(D) = \mu(Q_i) < \infty$; hence by Corollary 4.2 it has a unique extension η_i to a Borel measure on (D, τ). Let $T = U \backslash V \in Cres(\tau)$ with $U, V \in \tau$ and $V \subseteq U$. We define the extension of μ to $Cres(\tau)$ by $\overline{\mu}(T) = \sup_{i \in \mathbb{N}} \eta_i(T)$. The sequence $(\eta_i(T))$ is increasing since for all $i \in \mathbb{N}$ we have

$$
\begin{aligned}
\eta_i(T) &= \eta_i(U) - \eta_i(V) \\
&= \mu|_{Q_i}(U) - \mu|_{Q_i}(V) \\
&= \lim_{j \in J} \mu_j|_{Q_i}(U) - \lim_{j \in J} \mu_j|_{Q_i}(V) \\
&= \lim_{j \in J} \mu_j(T \cap Q_i).
\end{aligned}
$$

But $Q_i \subseteq Q_{i+1}$ implies that $\mu_j(T \cap Q_i) \leq \mu_j(T \cap Q_{i+1})$ for all $j \in J$ and hence $\eta_i(T) = \lim_{j \in J} \mu_j(T \cap Q_i) \leq \lim_{j \in J} \mu_j(T \cap Q_{i+1}) = \eta_{i+1}(T)$. So we can write $\overline{\mu}(T) = \bigsqcup^{\uparrow}_{i \in \mathbb{N}} \eta_i(T)$. Next, $\overline{\mu}$ is an extension of μ since for all $A \in \tau$ we have

$$
\begin{aligned}
\overline{\mu}(A) &= \textstyle\bigsqcup^{\uparrow}_{i \in \mathbb{N}} \eta_i(A) \\
&= \textstyle\bigsqcup^{\uparrow}_{i \in \mathbb{N}} \mu|_{Q_i}(A) && \text{since } A \in \tau \\
&= \textstyle\bigsqcup^{\uparrow}_{i \in \mathbb{N}} \mu(A \cap Q_i) \\
&= \mu(\textstyle\bigcup_{i \in \mathbb{N}} A \cap Q_i) && \text{since } \mu \text{ is continuous} \\
&= \mu(A \cap \textstyle\bigcup_{i \in \mathbb{N}} Q_i) \\
&= \mu(A).
\end{aligned}
$$

We can easily check that $\overline{\mu}$ is finitely additive and σ-subadditive, therefore μ has a unique extension to a Borel measure. ∎

We can now give a direct proof of a result by Norberg.

Corollary 5.3 *[19, Theorem 3.9] Let D be a continuous dcpo with a second countable Scott topology. A function $\mu : \Omega D \to [0, \infty]$ that is finite on the set $\{U \in \Omega D \mid U \ll D\}$ has a unique extension to a Borel measure if and only if it is a countably continuous valuation.*

Proof The 'only if' part is immediate since the restriction of a Borel measure to the open sets gives a countably continuous valuation.

For the 'if' part notice that since D is a continuous dcpo with second countable Scott topology then, by Lemma 2.4, D is an ω-continuous dcpo. In an ω-continuous dcpo countably continuous valuations are continuous by Lemma 2.5. Hence μ is a continuous valuation. Moreover, on a continuous dcpo the set of continuous valuations is a continuous dcpo with a basis of simple valuations [14, 13]. Therefore μ is the supremum of a directed set of simple valuations. Let $B = \{b_n \mid n \in \mathbb{N}\}$ be a countable basis for D. We have $\Uparrow b_n \ll D$ for all $n \in \mathbb{N}$. Hence $\mu(\Uparrow b_n) < \infty$ for all $n \in \mathbb{N}$ and $D = \bigcup_{n \in \mathbb{N}} \Uparrow b_n$. The conclusion follows from Theorem 5.1. ∎

Suppose, more generally, that μ is a continuous valuation on a monotone convergence space $(D, \tau, \sqsubseteq)$ and $\mu(D) = \infty$. If D has a bottom element $\perp$ then the set $\{\perp\}$ is closed. Therefore $D' = D \backslash \{\perp\} \in \tau$. Hence $\mu(D') < \infty$ or $\mu(D') = \infty$. In the first case it is straightforward to check that $\mu\,|_{D'} = \sup_{j \in J} \mu_j\,|_{D'}$; then by Corollary 4.2 $\mu\,|_{D'}$can be extended uniquely to a Borel measure on D'. In order to extend this measure to the whole space D we only need to assign some weight to $\{\perp\}$. Since a measure is finitely additive and we are assuming $\mu(D') < \infty$ the only possible choice is to have $\overline{\mu}(\{\perp\}) = \infty$. So if $\mu(D') < \infty$ the extension is unique. If $\mu(D') = \infty$ and $\mu\,|_{D'}$ has a unique extension to a Borel measure on D', $\overline{\mu}(\{\perp\})$ can be assigned any nonnegative extended real value and this function will still be a measure; therefore the extension to D is not unique in this case.

6 A counterexample for a dcpo

The question remains if the extension result holds in more general settings. We will give an example of a bounded continuous valuation on a dcpo which cannot be extended to a Borel measure. The following dcpo is defined in [12, Chapter II, 1.9], as an example of a dcpo with a non-sober Scott topology. Consider $X = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ with $(j, k) \sqsubseteq (m, n)$ if and only if either $j = m$ and $k \leq n$ or $n = \infty$ and $k \leq m$. It is easy to check that $(X, \sqsubseteq)$ is a dcpo and that every nonempty open set contains all but a finite number of points (m, ∞).

Define the function $\nu : \Omega X \to [0, \infty]$ by

$$\nu(Q) = \begin{cases} 1 & \text{if } Q \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Then ν is a modular function since the intersection of any pair of nonempty open sets in X is again nonempty. Strictness, monotonicity and continuity are easily verified, therefore ν is a bounded continuous valuation. But ν cannot be extended to a Borel measure. In fact $Q_n = (\bigcup_{j=0}^{n} \downarrow (j, \infty))^c$ is a decreasing sequence of open sets with $\bigcap_{n \in \mathbb{N}} Q_n = \emptyset$ but $\lim_n \nu(Q_n) = 1$. By Theorem 4.1 ν also gives an example of a continuous valuation on a dcpo that is not the supremum of a directed family of simple valuations.

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LIST OF SYMBOLS

$\mathcal{A}$	Script capital A
$\mathcal{B}$	Script capital B
$\mathbb{N}$	Open face capital N
$\mathbb{R}$	Open face capital R
$\mathcal{S}$	Script capital S
ε	lowercase epsilon
$\in$	belongs to
ν	lowercase nu
v	lowercase v
ω	lowercase omega
w	lowercase w
τ	tau
$\restriction$	restriction symbol
$\ll$	way below relationship
$\blacksquare$	end of proof symbol
$\perp$	bottom element symbol
$\Uparrow$	double up arrow
$\Downarrow$	double down arrow
$\subseteq$	subset of
$\sqsubseteq$	dominated by
$\bigsqcup^{\uparrow}$	Directed supremum (note that arrow is superscripted)