I-Categories as a framework for solving domain equations

Abbas Edalat Michael B. Smyth

Department of Computing
Imperial College of Science, Technology and Medicine
180 Queen's Gate
London SW7 2BZ
England.

Abstract

An abstract notion of category of information systems or I-category is introduced as a generalisation of Scott's well-known category of information systems. As in the theory of partial orders, I-categories can be complete or ω-algebraic, and it is shown that ω-algebraic I-categories can be obtained from a certain completion of countable I-categories. The proposed axioms for a complete I-category introduce a global partial order on the morphisms of the category, making them a cpo. An initial algebra theorem for a class of functors continuous on the cpo of morphisms is proved, thus giving canonical solution of domain equations; an effective version of these results for ω-algebraic I-categories is also provided. Some basic examples of I-categories representing the categories of sets, Boolean algebras, Scott domains and continuous Scott domains are constructed.

1 Introduction

A distinctive feature of information systems representing Scott domains, as expressed in [Sco82, LW84], is that the collection of all information systems itself has an "information ordering". This has the consequence that domain equations of the form

$$D \cong F(D)$$

for the recursive specification of types, can be handled by ordinary (cpo) fixed point techniques. The same feature had been noticed already in the context of the presentation of domains for sequentiality by "concrete data structures" in [BC81]. We shall use "information systems" loosely to cover all such presentations of domains, as well as, indeed, many systems of presentation of spaces and structures that are not domains at all.

The ordering of information systems is, typically, defined as follows. Let I, J be information systems, with token sets L, L'. Then $I \trianglelefteq J$ iff $L \subseteq L'$ and the operations and relations of L are the restrictions to L of those of L'. If we further ask that the tokens of information systems all be drawn from a common pool of tokens and that the very *same* token, Δ, appear as the distinguished member in all information systems that we consider (in a given context), we find (typically) that the collection of information systems under $\trianglelefteq$ is a cpo (even, for example, an ω-algebraic cpo) with the trivial information system with token set $\{\Delta\}$ as its least element; see Section 5.1. Some versions, of course, dispense with Δ, and have the empty system as the least one. The result is an approach to domain equations which is appreciably simpler than other standard approaches in terms of inverse limits, universal objects, or O-categories. But it must be pointed out that the simple approach, in terms of the $\trianglelefteq$ -ordering of information systems, does not give us everything that the more elaborate category-theoretic methods yield: Specifically, we do not get an initial algebra theorem, characterising the intended solutions of the above domain equation.

Our enquiry really starts with the question: What needs to be added to the information system method, to get something fully equivalent to the other standard methods for solving a domain equation?

The answer, we suggest, is that a global information ordering of morphisms, $\trianglelefteq^m$, is needed, in addition to (or, including) that for objects. Concretely, we can define:

$$(f : I_1 \to I_2) \trianglelefteq^m (g : J_1 \to J_2) \text{ if } I_n \trianglelefteq J_n \quad (n = 1, 2) \text{ and } f \subseteq g$$

(note that f, g are sets of pairs of tokens). For the general theory, however, we do not work concretely in this way: rather, we abstract the required properties of the global ordering of objects and morphisms, arriving at an *axiomatic* notion of "category of information systems", or "I-category". All the required theory of domain equations, at least as far as the (effective) initial algebra theorem, can be developed at the abstract level, and is then available to be applied routinely to various concrete situations, such as Scott domains, stable domains, Stone spaces, and even metric spaces. Various constructions, such as the functor category of two categories of information systems, can also be handled.

We summarise the remainder of the paper. Section 2 introduces I-categories as elaborated partial orders, with complete I-categories corresponding to cpo's, and an appropriate notion of ω-algebraic I-category. Section 3 presents the general initial algebra theorem for complete I-categories. In Section 4, it is shown that the material of the preceding sections can be straightforwardly effectivized. Finally, in Section 5, some basic examples are considered: Scott domains; the continuous generalisation of these domains; and in a little more detail, Boolean algebras regarded as information systems for Stone spaces.

2 Basic Definitions and Axioms

In this section we give the basic definitions of a category of information systems, which we simply call an *I*-category, and the related notions. Our notations are fairly standard. For a category P, we denote its class of objects by Obj_P and its class of morphisms by Mor_P, and we often delete the subscript P. The mappings dom and $\mathrm{cod} : \mathrm{Mor} \to \mathrm{Obj}$ denote the domain and the codomain mappings associated with P, and $\mathrm{Id} : \mathrm{Obj} \to \mathrm{Mor}$ denotes the mapping which takes an object to the identity morphism of that object so that the identity morphism on A is denoted by $\mathrm{Id}(A)$. We denote the morphism f with domain A and codomain B by $f : A \to B$. For $f, g \in \mathrm{Mor}$ with $\mathrm{cod}(f) = \mathrm{dom}(g)$, we denote the composition of f and g by $f;g$ (instead of the more common notation $g \circ f$), which should be read as f followed by g. Given a functor $F : P_1 \to P_2$ between two categories P_1 and P_2 we write $F_o : \mathrm{Obj}_{P_1} \to \mathrm{Obj}_{P_2}$ and $F_m : \mathrm{Mor}_{P_1} \to \mathrm{Mor}_{P_2}$ for the induced mappings on objects and morphisms respectively. The identity endofunctor is denoted by ID. We say P is a *countable category* if Mor_P is countable. $\mathcal{P}_f$ denotes the *finite* power set constructor, i.e. for a set A, $\mathcal{P}_f(A)$ is the set of all finite subsets of A. Given a partial order $(A, \sqsubseteq)$ and $a, b \in A$, we write $a \uparrow b$ if a and b are bounded above i.e. if there exists $c \in A$ with $a \sqsubseteq c$ and $b \sqsubseteq c$. We denote the class of compact elements of an algebraic cpo A by $\mathcal{K}_A$.

2.1 I-categories

An *I-category* is, intuitively speaking, an ordered category with a distinguished class of morphisms, called *inclusion morphisms*, which induces a partial order on the class of morphisms as well as on the class of objects. Here is the precise definition.

Definition 2.1 An *I-category* is a four tuple $(P, \mathrm{Inc}, \sqsubseteq, \Delta)$ where:

- P is a category with ordered hom-sets,
- $\mathrm{Inc} \subseteq \mathrm{Mor}$ is the subclass of *inclusion morphisms* of P such that in each hom-set, $\mathrm{hom}(A, B)$, there is at most one inclusion morphism which we denote by $\mathrm{in}(A, B)$ or $A \rightarrowtail B$,
- $\sqsubseteq^{A,B}$ is the partial order on $\mathrm{hom}(A, B)$, for all $A, B \in \mathrm{Obj}$,
- $\Delta \in \mathrm{Obj}$ is a distinguished object,

which satisfy the following two axioms:

Axiom 1 (i) *The class of objects Obj and the inclusion morphisms Inc form a partial order represented as a category.*

(ii) *$in(\Delta, A)$ exists, for all $A \in Obj$ and $in(\Delta, A) \sqsubseteq f$ for all morphisms $f \in \mathrm{hom}(\Delta, A)$.*

(iii) *$f; in(A, B) \sqsubseteq g; in(A, B) \Rightarrow f \sqsubseteq g$, for all $f, g \in Mor, in(A, B) \in Inc$, such that the compositions are defined.*

Axiom 2 *Composition of morphisms is monotone with respect to the partial order on hom-sets, i.e.*

$$f_1 \sqsubseteq f_2 \;\&\; g_1 \sqsubseteq g_2 \;\Rightarrow\; f_1; g_1 \sqsubseteq f_2; g_2$$

whenever the compositions are defined.

Remark 2.2 (i) *Axiom 2 says that an I-category is an* order-enriched *category.*

(ii) *The initial algebra theorem (Section 3) holds without Axiom 1(iii) which can be dispensed with in a more general setting. However in the present paper we choose to retain this axiom which implies that inclusion morphisms are monomorphisms and that the subcategory of strict morphisms of an ω-algebraic I-category is itself an ω-algebraic I-category (Proposition 3.3).*

For convenience, we often denote an I-category by its carrier and write P for $(P, \mathrm{Inc}, \sqsubseteq, \Delta)$. We now define partial orders $\trianglelefteq$ on Obj_P and $\trianglelefteq^m$ on Mor_P of an I-category $(P, \mathrm{Inc}, \sqsubseteq, \Delta)$ as follows:

- $A \trianglelefteq B$ if $\mathrm{in}(A, B)$ exists;
- $f \trianglelefteq^m g$ if
  - (i) $\mathrm{dom}(f) \trianglelefteq \mathrm{dom}(g)$,
  - (ii) $\mathrm{cod}(f) \trianglelefteq \mathrm{cod}(g)$,
  - (iii) $f; \mathrm{in}(\mathrm{cod}(f), \mathrm{cod}(g)) \sqsubseteq \mathrm{in}(\mathrm{dom}(f), \mathrm{dom}(g)); g$.

Note that $f \trianglelefteq^m g$ iff the diagram

$$\begin{tikzcd}
\mathrm{dom}(g) \arrow[r, "g"] & \mathrm{cod}(g) \\
\mathrm{dom}(f) \arrow[u] \arrow[r, "f"] \arrow[ur, phantom, "\sqsupseteq"] & \mathrm{cod}(f) \arrow[u]
\end{tikzcd}$$

weakly commutes. It is easily checked that the relations $\trianglelefteq$ and $\trianglelefteq^m$ are in fact partial orders. Note that if in Axiom 1 we require to have just a pre-order, we obtain a *pre-I-category* , in which $\trianglelefteq$ and $\trianglelefteq^m$ will be just pre-orders.

Proposition 2.3 (i) *$Id(\Delta)$ is the least element of $(Mor, \trianglelefteq^m)$.*

(ii) *Inclusion morphisms are monomorphisms.*

(iii) *Composition of morphisms is monotonic with respect to* $\trianglelefteq^m$.

Proof (i) Let $(f : A \to B) \in \mathrm{Mor}$ be given. By Axiom 1(ii), we have

$$\begin{aligned} \mathrm{in}(\Delta, B) &\sqsubseteq \mathrm{in}(\Delta, A); f \quad \Rightarrow \\ \mathrm{Id}(\Delta); \mathrm{in}(\Delta, B) &\sqsubseteq \mathrm{in}(\Delta, A); f \quad \Rightarrow \\ \mathrm{Id}(\Delta) &\trianglelefteq^m f. \end{aligned}$$

(ii) Suppose $f; \mathrm{in}(A, B) = g; \mathrm{in}(A, B)$. By Axiom 1(iii), we get $f \sqsubseteq g$ and $g \sqsubseteq f$, i.e. $f = g$.

(iii) Suppose we have

$$(f_1 : A_1 \to B_1) \trianglelefteq^m (f_2 : A_2 \to B_2) \qquad \text{and} \qquad (g_1 : B_1 \to C_1) \trianglelefteq^m (g_2 : B_2 \to C_2).$$

Then by monotonicity of composition of morphisms with respect to $\sqsubseteq$ we have:

$$f_1; g_1; \mathrm{in}(C_1, C_2) \sqsubseteq f_1; \mathrm{in}(B_1, B_2); g_2 \sqsubseteq \mathrm{in}(A_1, A_2); f_2; g_2$$

i.e. $f_1; g_1 \trianglelefteq^m f_2; g_2$. $\square$

Example 2.4 Partial Orders. Any partial order with least element, $(Q, \sqsubseteq)$, considered as a category in the standard way, gives rise to an I-category $(Q, \mathrm{Inc}, =, \bot)$, where $\mathrm{Inc} = \mathrm{Mor}$, "$=$" is the discrete partial order on hom-sets and $\Delta = \bot$. Similarly, pre-orders are examples of pre-I-categories.

Example 2.5 Sets. Consider the category of sets which we denote by **Sets**. It can be easily checked that $(\mathbf{Sets}, \mathrm{Inc}, =, \emptyset)$ is a large I-category, where Inc is just the set inclusions, "$=$" is the discrete partial order on hom-sets and $\emptyset$ is the empty set. Here, we have $(f : A \to B) \trianglelefteq^m (g : C \to D)$ iff $A \subseteq B, C \subseteq D$ and $f = g\restriction_A$.

2.2 Complete I-categories

In the same way that we define a cpo as a partial order having lubs of increasing chains, we can define a complete I-category as an I-category with some completion properties:

Definition 2.6 A *complete I-category* $(P, \mathrm{Inc}, \sqsubseteq, \Delta)$ is an I-category which satisfies the following three axioms:

Axiom 3 *$(Mor, \trianglelefteq^m)$ is a cpo.*

Axiom 4 *$(Inc, \trianglelefteq^m)$ is a subcpo of $(Mor, \trianglelefteq^m)$.*

Axiom 5 *Composition of morphisms is a continuous operation with respect to $\trianglelefteq^m$, i.e. $\bigsqcup_i (f_i; g_i) = (\bigsqcup_i f_i); (\bigsqcup_i g_i)$ whenever $\langle f_i \rangle_{i \geq 0}$ and $\langle g_i \rangle_{i \geq 0}$ are increasing chains in $(Mor, \trianglelefteq^m)$ with $cod(f_i) = dom(f_i)$, for all $i \geq 0$.*

Proposition 2.7 *In a complete I-category, $(Obj, \trianglelefteq)$ is a cpo and the mappings $Id : (Obj, \trianglelefteq) \to (Mor, \trianglelefteq^m)$, and dom, cod : $(Mor, \trianglelefteq^m) \to (Obj, \trianglelefteq)$ are continuous.*

Proof Let $\langle A_i \rangle_{i \geq 0}$ be an increasing chain of objects. Then $\langle \mathrm{Id}(A_i) \rangle_{i \geq 0}$ is an increasing chain of inclusion morphisms and, hence, by Axiom 4 has a lub $\mathrm{in}(A, B)$, say. On the other hand, by Axiom 5, the increasing chain $\langle \mathrm{Id}(A_i); \mathrm{Id}(A_i) \rangle_{i \geq 0}$ has lub $\mathrm{in}(A, B); \mathrm{in}(A, B)$. It follows that $A = B$ and therefore, by Axiom 1(i), we have

$$\bigsqcup \mathrm{Id}(A_i) \;=\; \mathrm{in}(A, A) \;=\; \mathrm{Id}(A).$$

We now claim that $A = \bigsqcup A_i$. In fact, since $\mathrm{Id}(A_i) \trianglelefteq^m \mathrm{Id}(A)$, we have $A_i \trianglelefteq A$ for all $i \geq 0$. If $C \in \mathrm{Obj}$ is another upper bound for $\langle A_i \rangle_{i \geq 0}$ we immediately obtain $\mathrm{Id}(A_i) \trianglelefteq^m \mathrm{Id}(C)$ for all $i \geq 0$ and hence

$$\mathrm{Id}(A) = \bigsqcup \mathrm{Id}(A_i) \trianglelefteq^m \mathrm{Id}(C)$$

i.e. $A \trianglelefteq C$. This proves that $(\mathrm{Obj}, \trianglelefteq)$ is a cpo and Id is continuous. To establish the continuity of dom, let $\langle (f_i : A_i \to B_i) \rangle_{i \geq 0}$ be an increasing chain of morphisms. By Axiom 5, the increasing chain of morphisms $\langle \mathrm{Id}(A_i); f_i \rangle_{i \geq 0}$ has lub

$$(\bigsqcup \mathrm{Id}(A_i)); \bigsqcup f_i = \mathrm{Id}(\bigsqcup A_i); \bigsqcup f_i.$$

Hence, we obtain $\mathrm{dom}(\bigsqcup f_i) = \bigsqcup A_i = \bigsqcup \mathrm{dom}(f_i)$. The continuity of cod is proved in a similar way. □

Example 2.8 Complete Partial Orders. Any cpo, considered as a category, is a complete I-category. See Example 2.4. It is straightforward to check the axioms.

Example 2.9 Sets. Observe that $(\mathbf{Sets}, \mathrm{Inc}, \subseteq, \emptyset)$ of Example 2.5 is a complete large I-category. An increasing chain of objects $A_0 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ has lub $\bigcup_i A_i$; whereas an increasing chain of morphisms $\langle f_i : A_i \to B_i \rangle_{i \geq 0}$ has lub $\bigcup_i f_i : (\bigcup_i A_i) \to (\bigcup_i B_i)$, where $(\bigcup_i f_i)(x) = f_j(x)$ if $x \in A_j$. It is routine to verify the axioms.

2.3 ω-algebraic I-categories

An *ω-algebraic I-category* is a complete I-category in which $(\mathrm{Mor}, \trianglelefteq^m)$ is ω-algebraic and compact morphisms and objects are related in a desirable way as follows.

Definition 2.10 An *ω-algebraic I-category* is a complete I-category which satisfies the following two axioms:

Axiom 6 *$(Mor, \trianglelefteq^m)$ is ω-algebraic.*

Axiom 7 (i) *$f \in (Mor, \trianglelefteq^m)$ is compact $\Rightarrow$ $cod(f) \in (Obj, \trianglelefteq)$ is compact.*

(ii) *$A, B \in (Obj, \trianglelefteq)$ are compact with $A \trianglelefteq B$ $\Rightarrow$ $in(A, B) \in (Mor, \trianglelefteq^m)$ is compact.*

(iii) *The composition of compact morphisms is compact.*

See Proposition 2.11 below about Axiom 7(i) and the remarks after Proposition 2.17 about Axiom 7(ii) and Axiom 7(iii) .

Proposition 2.11 *In an ω-algebraic I-category, $(Obj, \trianglelefteq)$ is an ω-algebraic cpo and compact morphisms have compact domains.*

Proof First note that the set of compact objects is at most countable since, by Axiom 7(ii), $\mathrm{Id}(A)$ is compact whenever A is compact and the set of compact morphisms is at most countable. Next, let $A \in \mathrm{Obj}$ be given. Then there exists an increasing chain of compact morphisms $\langle (g_i : C_i \to D_i) \rangle_{i \geq 0}$ with $\mathrm{Id}(A) = \bigsqcup g_i$. Hence $A = \bigsqcup D_i$ where $D_i = \mathrm{cod}(g_i)$ is compact for all $i \geq 0$ by Axiom 7(i). This proves that the cpo $(\mathrm{Obj}, \trianglelefteq)$ is ω-algebraic. Now let $(f : A \to B)$ be a compact morphism of an ω-algebraic I-category. Then there exists an increasing chain of compact objects $\langle A_i \rangle_{i \geq 0}$ with $A = \bigsqcup A_i$. Define $f_i = \mathrm{in}(A_i, A); f$. Then

$$\bigsqcup f_i = (\bigsqcup \mathrm{in}(A_i, A); f = \mathrm{in}(\bigsqcup A_i, A); f = \mathrm{in}(A, A); f = f$$

using the continuity of dom. Since f is compact, $f_i = f$ for some $i \geq 0$, which implies $A = A_i$ and hence A is compact. $\Box$

Example 2.12 ω-ALGEBRAIC CPO'S. Any ω-algebraic cpo, considered as a category, is an ω-algebraic I-category. See Example 2.4. The compact objects are precisely the compact points and the compact morphisms are precisely the morphisms between the compact points. It is routine to check all the axioms.

Example 2.13 SETS WITH ELEMENTS FROM A COUNTABLE ALPHABET. Consider the category of sets with elements from a given countable alphabet. Denoting this category by **Set-ISys**, we can easily check that $(\textbf{Set-ISys}, \mathrm{Inc}, =, \emptyset)$ is an ω-algebraic I-category. In fact, given a morphism $f : A \to B$ we can construct an increasing chain of morphisms between finite sets with lub f as follows. Choose finite sets $A_i, B_i, i \geq 0$ with $A = \bigcup_i A_i, B = \bigcup_i B_i$ and for each $i \geq 0$ let n_i be the least integer with $n_i \geq i$ and $f(A_i) \subseteq B_{n_i}$. Define $f_i : A_i \to B_{n_i}$ by $f_i = f \restriction_{A_i}$. Then we have $f = \bigcup_i f_i$ and $B = \bigcup_i B_{n_i}$.

Finally in this section, we note that the product $P_1 \times P_2$ of two I-categories P_1 and P_2 is an I-category, with $\Delta_{P_1 \times P_2} = (\Delta_{P_1}, \Delta_{P_2})$ and the partial order on hom-sets and inclusion morphisms defined co-ordinatewise in the obvious way. Furthermore $P_1 \times P_2$ is complete (ω-algebraic) if P_1 and P_2 are complete (ω-algebraic).

2.4 Completion of an I-category

Recall that the ideal completion of a countable poset, $(L, \sqsubseteq)$, gives rise to an ω-algebraic cpo, $(\overline{L}, \subseteq)$, of directed ideals of L ordered by inclusion. In the same way we can take the "ideal completion" of a countable I-category to obtain an ω-algebraic I-category as follows.

Definition 2.14 For a countable I-category $P = (P, \mathrm{Inc}, \sqsubseteq, \Delta)$, its *I-completion*, $\overline{P} = (\overline{P}, \overline{\mathrm{Inc}}, \subseteq, \overline{\Delta})$ is given by:

- $\mathrm{Obj}_{\overline{P}} = \overline{\mathrm{Obj}_P}$;
- $\mathrm{Mor}_{\overline{P}} = \overline{\mathrm{Mor}_P}$,

  $$\begin{array}{ll} \mathrm{dom}(\mathcal{F}) = \{\mathrm{dom}(f) \mid f \in \mathcal{F}\}, & \mathrm{cod}(\mathcal{F}) = \downarrow\{\mathrm{cod}(f) \mid f \in \mathcal{F}\}, \\ \mathrm{Id}(\mathcal{A}) = \downarrow\{\mathrm{Id}(A) \mid A \in \mathcal{A}\}, & \mathcal{F};\mathcal{G} = \downarrow\{f;g \mid f \in \mathcal{F}, g \in \mathcal{G}\}; \end{array}$$

  (assuming in the latter equation that $\mathrm{cod}(\mathcal{F}) = \mathrm{dom}(\mathcal{G})$, and denoting the downward closure of S by $\downarrow S$)
- $\mathrm{in}(\mathcal{A}, \mathcal{B})$ exists iff $\mathcal{A} \subseteq \mathcal{B}$, and when it exists we have:

  $$\mathrm{in}(\mathcal{A}, \mathcal{B}) \;=\; \downarrow\{\mathrm{in}(A, B) \mid A \in \mathcal{A}, B \in \mathcal{B}, A \trianglelefteq B\};$$
- the inclusion $\subseteq$ as the partial order on hom-sets;
- $\overline{\Delta} = \{\Delta\}$.

In order to show that $\overline{P}$ is a well defined category, it is convenient to use the following simple lemma whose proof we omit.

Lemma 2.15 *Suppose P, Q are posets with P directed and $h : P \to Q$ a monotonic surjection. Then Q is also directed and if S, T are finite subsets of P, Q respectively, there exists $a \in P$ such that a is an upperbound of S in P and $h(a)$ is an upper bound of T in Q.*

We will now show:

Proposition 2.16 *$\overline{P}$ is a well-defined category.*

Proof First we need to show that $\mathrm{dom}(\mathcal{F})$, $\mathrm{cod}(\mathcal{F})$, $\mathrm{Id}(\mathcal{A})$, $\mathcal{F};\mathcal{G}$, $\mathrm{in}(\mathcal{A},\mathcal{B})$ as defined in the definition of $\overline{P}$ are in fact directed ideals of Obj_P or Mor_P. We will only verify this for $\mathcal{F};\mathcal{G}$. Assuming $\mathrm{cod}(\mathcal{F}) = \mathrm{dom}(\mathcal{G}) = \mathcal{A}$, say, let $f_1;g_1$ and $f_2;g_2$ be in $\mathcal{F};\mathcal{G}$, i.e. $f_1, f_2 \in \mathcal{F}$ and $g_1, g_2 \in \mathcal{G}$ with $\mathrm{cod}(f_1) = \mathrm{dom}(g_1)$ and $\mathrm{cod}(f_2) = \mathrm{dom}(g_2)$. Since $\mathcal{F}$ is directed, there exists $f_3 \in \mathcal{F}$ above f_1, f_2. Applying Lemma 2.15 to $\mathcal{G}, \mathcal{A}$ and the mapping dom, we can find $g \in \mathcal{G}$ above g_1, g_2 with $\mathrm{cod}(f_3) \trianglelefteq \mathrm{dom}(g)$. Let $f = f_3;\mathrm{in}(\mathrm{cod}(f_3), \mathrm{dom}(g))$. By Proposition 2.3(iii), f is above f_1, f_2. Hence $f;g$ is above $f_1;g_1$ and $f_2;g_2$. It remains to show that $f \in \mathcal{F}$. Applying Lemma 2.15 to $\mathcal{F}, \mathcal{A}$ and the mapping cod, we can find $f_4 \in \mathcal{F}$ above f_3 with $\mathrm{dom}(g) \trianglelefteq \mathrm{cod}(f_4)$. Hence, by Proposition 2.3(iii) again, $f = f_3;\mathrm{in}(\mathrm{cod}(f_3), \mathrm{dom}(g)) \trianglelefteq f_4$ and therefore $f \in \mathcal{F}$.

Next we need to verify that $\overline{P}$ satisfies the basic axioms of a category. It is in fact routine to check that $\mathrm{Id}(\mathcal{A})$ is indeed the identity morphism on $\mathcal{A}$, that $\mathcal{F};\mathcal{G} \in \mathrm{hom}(\mathrm{dom}(\mathcal{F}), \mathrm{cod}(\mathcal{G}))$ and that the composition of morphisms is associative. $\Box$

Finally, we establish the expected result:

Proposition 2.17 *$\overline{P}$ is an ω-algebraic I-category.*

Proof The proof is divided into three parts.

(i) $\overline{P}$ is an I-category: We will only verify Axiom 1(iii) here. Suppose $\mathcal{F};\mathrm{in}(\mathcal{A},\mathcal{B}) \subseteq \mathcal{G};\mathrm{in}(\mathcal{A},\mathcal{B})$. We want to show that $\mathcal{F} \subseteq \mathcal{G}$. Let $(f : C \to A) \in \mathcal{F}$. Then $A \in \mathcal{A}$ and $f = f;\mathrm{in}(A, A) \in \mathcal{F};\mathrm{in}(\mathcal{A},\mathcal{B})$. Hence we have $f \in \mathcal{G};\mathrm{in}(\mathcal{A},\mathcal{B})$ and there exist $(g : C' \to A') \in \mathcal{G}$ and $\mathrm{in}(A', B') \in \mathrm{in}(\mathcal{A},\mathcal{B})$ with $f \trianglelefteq^m g;\mathrm{in}(A', B')$. We now choose $A'' \in \mathcal{A}$ above A, A' and $B'' \in \mathcal{B}$ above A, B', A'' and deduce the following from the above equation

$$\begin{array}{rcll} f;\mathrm{in}(A,B') & \sqsubseteq & \mathrm{in}(C,C');g;\mathrm{in}(A',B') & \Rightarrow \\ f;\mathrm{in}(A,B');\mathrm{in}(B',B'') & \sqsubseteq & \mathrm{in}(C,C');g;\mathrm{in}(A',B');\mathrm{in}(B',B'') & \Rightarrow \\ f;\mathrm{in}(A,B'') & \sqsubseteq & \mathrm{in}(C,C');g;\mathrm{in}(A',A'');\mathrm{in}(A'',B'') & \Rightarrow \\ f;\mathrm{in}(A,A'');\mathrm{in}(A'',B'') & \sqsubseteq & \mathrm{in}(C,C');g;\mathrm{in}(A',A'');\mathrm{in}(A'',B'') & \Rightarrow \\ f;\mathrm{in}(A,A'') & \sqsubseteq & \mathrm{in}(C,C');g;\mathrm{in}(A',A'') & \Rightarrow \\ f & \trianglelefteq^m & g;\mathrm{in}(A',A''). & \end{array}$$

As in the last step in the proof of Proposition 2.16, we get $g;\mathrm{in}(A',A'') \in \mathcal{G}$, and it follows that $f \in \mathcal{G}$ and the proof is complete.

(ii) $\overline{P}$ is a complete I-category: First we claim that $\mathcal{F}_1 \trianglelefteq^m \mathcal{F}_2$ iff $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Let $\mathcal{F}_1 : \mathcal{A}_1 \to \mathcal{B}_1$ and $\mathcal{F}_2 : \mathcal{A}_2 \to \mathcal{B}_2$. Suppose we have $\mathcal{F}_1 \subseteq \mathcal{F}_2$. By the definition of the domain and codomain of a morphism in $\overline{P}$, it follows that $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{B}_1 \subseteq \mathcal{B}_2$. From the definition of composition of morphisms in $\overline{P}$, one gets $\mathcal{G}_1 \subseteq \mathcal{G}_2$, $\mathcal{H}_1 \subseteq \mathcal{H}_2$, $\mathrm{cod}(\mathcal{G}_1) = \mathrm{dom}(\mathcal{H}_1)$ and $\mathrm{cod}(\mathcal{G}_2) = \mathrm{dom}(\mathcal{H}_2)$ which imply that $\mathcal{G}_1;\mathcal{H}_1 \subseteq \mathcal{G}_2;\mathcal{H}_2$. Hence, by combining these two observations, we obtain

$$\mathcal{F}_1;\mathrm{in}(\mathcal{B}_1,\mathcal{B}_2) = \mathrm{Id}(\mathcal{A}_1);\mathcal{F}_1;\mathrm{in}(\mathcal{B}_1,\mathcal{B}_2) \subseteq \mathrm{in}(\mathcal{A}_1,\mathcal{A}_2);\mathcal{F}_2;\mathrm{Id}(\mathcal{B}_2) = \mathrm{in}(\mathcal{A}_1,\mathcal{A}_2);\mathcal{F}_2$$

i.e. $\mathcal{F}_1 \trianglelefteq^m \mathcal{F}_2$. Suppose on the other hand that $\mathcal{F}_1;\mathrm{in}(\mathcal{B}_1,\mathcal{B}_2) \subseteq \mathrm{in}(\mathcal{A}_1,\mathcal{A}_2);\mathcal{F}_2$ and let $f_1 \in \mathcal{F}_1;\mathrm{in}(\mathcal{B}_1,\mathcal{B}_2)$ and hence there exists $\mathrm{in}(A_1,A_2) \in \mathrm{in}(\mathcal{A}_1,\mathcal{A}_2), f_2 \in \mathcal{F}_2$ with $f_1 \in \mathrm{in}(\mathcal{A}_1,\mathcal{A}_2);\mathcal{F}_2$. It follows that $f_1 \trianglelefteq^m f_2$ i.e. $f_1 \in \mathcal{F}_2$ and hence $\mathcal{F}_1 \subseteq \mathcal{F}_2$. This proves the claim. Now it is easy to check that an increasing chain of morphisms $\langle \mathcal{F}_i \rangle_{i \geq 0}$ in $\mathrm{Mor}_{\overline{P}}$ has lub $\bigcup \mathcal{F}_i$, with $\mathrm{dom}(\bigcup \mathcal{F}_i) = \bigcup \mathrm{dom}(\mathcal{F}_i)$, and $\mathrm{cod}(\bigcup \mathcal{F}_i) = \bigcup \mathrm{cod}(\mathcal{F}_i)$. Hence $\mathrm{Mor}_{\overline{P}}$ is a cpo. If $\mathcal{F}_i = \mathrm{in}(\mathcal{A}_i,\mathcal{B}_i)$, for all $i \geq 0$, then $\bigcup \mathcal{F}_i = \mathrm{in}(\bigcup \mathcal{A}_i, \bigcup \mathcal{B}_i)$, i.e. the inclusion morphisms form a subcpo of $\mathrm{Mor}_{\overline{P}}$. It is also easy to establish that for any two increasing chains of morphisms $\langle \mathcal{F}_i \rangle_{i \geq 0}$ and $\langle \mathcal{G}_i \rangle_{i \geq 0}$, with $\mathrm{cod}(\mathcal{F}_i) = \mathrm{dom}(\mathcal{G}_i)$, for all $i \geq 0$, we have $\bigcup(\mathcal{F}_i;\mathcal{G}_i) = (\bigcup \mathcal{F}_i);(\bigcup \mathcal{G}_i)$. This will complete the proof that $\overline{P}$ is a complete I-category.

(iii) $\overline{P}$ is ω-algebraic. That $(\mathrm{Mor}_{\overline{P}}, \overline{\trianglelefteq}^m)$ is ω-algebraic follows from the fact that it is the (directed) ideal completion of the countable poset $(\mathrm{Mor}_P, \trianglelefteq^m)$. The compact elements of $(\mathrm{Mor}_{\overline{P}}, \overline{\trianglelefteq}^m)$ are just the principal ideals of $(\mathrm{Mor}_P, \trianglelefteq^m)$. If $\downarrow f, f \in \mathrm{Mor}_P$, is such a compact element, we have $\mathrm{cod}(\downarrow f) = \downarrow \mathrm{cod}(f)$ which is a principal ideal in $(\mathrm{Obj}_P, \trianglelefteq)$ and hence a compact element of $(\mathrm{Obj}_{\overline{P}}, \overline{\trianglelefteq})$; i.e. Axiom 7(i) holds.

Next suppose $\downarrow A$ and $\downarrow B$, with $A, B \in \mathrm{Obj}_P$, are compact elements of $(\mathrm{Obj}_{\overline{P}}, \overline{\trianglelefteq})$ with $\downarrow A \subseteq \downarrow B$, then $A \trianglelefteq B$ and $\mathrm{in}(\downarrow A, \downarrow B) = \downarrow \mathrm{in}(A, B)$ which is compact. This gives Axiom 7(ii). Finally, if $\downarrow f$ and $\downarrow g$ are compact morphisms that can be composed, then $(\downarrow f);(\downarrow g) = \downarrow(f;g)$ which is compact, i.e. Axiom 7(iii) is also satisfied. $\Box$

Conversely, any ω-algebraic I-category is isomorphic to the completion of a countable I-category. To make this more precise we need :

Definition 2.18 The *base* of an ω-algebraic I-category, P, is the subcategory P^b of compact objects and compact morphisms of P, i.e. $\mathrm{Obj}(P^b) = \mathcal{K}_{\mathrm{Obj}_P}$, $\mathrm{Mor}(P^b) = \mathcal{K}_{\mathrm{Mor}_P}$.

Clearly, P^b is a countable category. We also have:

Proposition 2.19 *If P is an ω-algebraic I-category, then P is isomorphic to $\overline{P^b}$.*

Proof The functors $F : P \to \overline{P^b}$ and $G : \overline{P^b} \to P$ defined by

$$F_o : A \longmapsto \{B \in \mathrm{Obj}_{P^b} \mid B \trianglelefteq A\} \qquad F_m : f \longmapsto \{g \in \mathrm{Mor}_{P^b} \mid g \trianglelefteq^m f\}$$
$$G_o : \mathcal{A} \longmapsto \bigvee\nolimits^{\uparrow} \mathcal{A} \qquad G_m : \mathcal{F} \longmapsto \bigvee\nolimits^{\uparrow} \mathcal{F}$$

give an isomorphism between P and $\overline{P^b}$. $\Box$

In practice, many concrete ω-algebraic I-categories are obtained by completing I-categories with finite hom-sets. In fact we will see in the final section that this result holds for ω-algebraic information categories which are concrete ω-algebraic I-categories.

Before introducing the notion of information categories and presenting some basic examples, however, we are going to establish some fundamental results on solving domain equations in complete I-categories.

3 The Initial Algebra Theorem

In this section, we present one of our main results, i.e. the initial algebra theorem for a suitable class of endofunctors on a complete I-category. We begin with some definitions.

3.1 Strict morphisms

Definition 3.1 A morphism, $f : A \to B$, of an I-category P is *strict* if the diagram

$$\begin{tikzcd} A \arrow[dr, "f"] & \\ \Delta \arrow[u, tail] \arrow[r, tail] & B \end{tikzcd}$$

commutes.

In the case of Scott information systems, strict morphisms represent the usual notion of a strict function, i.e. one which preserves the least element.

Proposition 3.2 *Let P be an I-category.*

(i) *If $f \in Mor_P$ is strict and $g \trianglelefteq^m f$, then g is strict as well.*

(ii) *Inclusion morphisms of P are strict.*

(iii) *The composition of strict morphisms of P is strict.*

(iv) *If P is complete, then the lub of an increasing chain of strict morphisms of P is strict.*

Proof (i) Assume $f : A \to B$ and $g : C \to D$. Then, we have:

$$\begin{array}{lcl}
g \trianglelefteq^m f & \Rightarrow & \\
g; \mathrm{in}(D,B) \ \sqsubseteq \ \mathrm{in}(C,A); f & \Rightarrow & \\
\mathrm{in}(\Delta,C); g; \mathrm{in}(D,B) \ \sqsubseteq \ \mathrm{in}(\Delta,C); \mathrm{in}(C,A); f & \Rightarrow & \\
\mathrm{in}(\Delta,C); g; \mathrm{in}(D,B) \ \sqsubseteq \ \mathrm{in}(\Delta,A); f & \Rightarrow & \text{(by strictness of } f) \\
\mathrm{in}(\Delta,C); g; \mathrm{in}(D,B) \ \sqsubseteq \ \mathrm{in}(\Delta,B) & \Rightarrow & \\
\mathrm{in}(\Delta,C); g; \mathrm{in}(D,B) \ \sqsubseteq \ \mathrm{in}(\Delta,D); \mathrm{in}(D,B) & \Rightarrow & \text{(by Axiom 1(iii))} \\
\mathrm{in}(\Delta,C); g \ \sqsubseteq \ \mathrm{in}(\Delta,D) & \Rightarrow & \\
\mathrm{in}(\Delta,C); g \ = \ \mathrm{in}(\Delta,D) & & .
\end{array}$$

(ii) This follows from the transitivity of $\trianglelefteq$.

(iii) Easy.

(iv) Let $\langle (f_i : A_i \to B_i) \rangle_{i \geq 0}$ be an increasing chain of strict morphisms, i.e. $\mathrm{in}(\Delta, A_i); f_i = \mathrm{in}(\Delta, B_i)$. Taking the lubs of both sides and using the continuity of composition with respect to $\trianglelefteq^m$ we get: $\mathrm{in}(\Delta, \bigsqcup A_i); \bigsqcup f_i = \mathrm{in}(\Delta, \bigsqcup B_i)$. $\Box$

We now define, for an I-category P, the *subcategory of strict morphisms*, P^s, by:

$$\mathrm{Obj}_{P^s} = \mathrm{Obj}_P \ , \ \mathrm{Mor}_{P^s} = \{ f \in \mathrm{Mor}_P \mid f \text{ is strict} \}.$$

Using the above proposition, the following can now be shown.

Proposition 3.3 (i) *If P is an I-category, so is P^s.*

(ii) *If P is an (ω-algebraic) complete I-category, so is P^s.*

Proof (i) Axiom 1 holds as inclusion morphisms are strict by Proposition 3.2(ii), whereas Axiom 2 holds by Proposition 3.2(iii).

(ii) If P is a complete I-category, then $(\mathrm{Mor}_{P^s}, \trianglelefteq^m)$ will be a cpo by Proposition 3.2(iv), and the other axioms of a complete I-category will trivially hold for P^s. If P is ω-algebraic, then any strict morphism will be the lub of an increasing chain of compact morphisms, which will be strict as well by Proposition 3.2(i). This implies that P^s will be ω-algebraic. $\Box$

3.2 Standard functors

In dealing with functors between I-categories, it is natural to require that the functors preserve the class of inclusion morphisms.

Definition 3.4 Let P, P' be I-categories and P, P' complete I-categories.

(i) A functor $F : P \to P'$ is *standard* if it preserves the order in the hom-sets and the inclusion morphisms; it is *strictness preserving* if it preserves strict morphisms.

(ii) A functor $F : P \to P'$ is *continuous* if the induced map on morphisms $F_m : (\mathrm{Mor}_P, \trianglelefteq^m) \to (\mathrm{Mor}_{P'}, \trianglelefteq^m)$ is a continuous (cpo) function.

We will work with standard and continuous endofunctors on I-categories. Here is a list of their basic properties:

Proposition 3.5 *Let P, P' be I-categories and P, P' complete I-categories.*

(i) *A functor $F : P \to P'$ is standard iff $F(in(A,B)) = in(F(A), F(B))$ for all $in(A,B)) \in Inc_P$.*

(ii) *Composition of standard functors is standard.*

(iii) *A standard functor $F : P \to P'$ is strictness preserving.*

(iv) *If $F : P \to P'$ is a continuous endofunctor, then the induced map on objects $F_o : (Obj_P, \trianglelefteq) \to (Obj_{P'}, \trianglelefteq)$ is continuous.*

Proof (i) & (ii) Trivial.

(iii) Let $(f : A \to B) \in P^s$. Then, we have:

$$\begin{aligned} \mathrm{in}(\Delta, F(A)); F(f) &= \mathrm{in}(\Delta, F(\Delta)); \mathrm{in}(F(\Delta), F(A)); F(f) \\ &= \mathrm{in}(\Delta, F(\Delta)); \mathrm{in}(F(\Delta), F(B)) \\ &= \mathrm{in}(\Delta, F(B)). \end{aligned}$$

(iv) This follows easily by the continuity of Id_P and $\mathrm{dom}_{P'}$ (Proposition 2.7), and the observation that $F_o = \mathrm{dom}_{P'} \circ F_m \circ \mathrm{Id}_P$. $\square$

Since we will only be dealing with standard functors, *we assume from now on that all functors between I-categories are standard.* We now proceed to state our results for I-categories. Recall that we denote the inclusion morphism $\mathrm{in}(A,B)$ by $A \rightarrowtail B$.

Lemma 3.6 *The lub of any increasing chain of objects in a complete I-category P is a colimit of the chain in P. It is also a colimit of the chain in P^s.*

Proof Let $T = A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots$ be a chain in P. Then $r : T \to \bigsqcup A_i$ with $r_i = \mathrm{in}(A_i, \bigsqcup_i A_i)$ is trivially a cone. If $s : T \to E$ with $s_i \in \hom(A_i, E)$, $i \geq 0$, is another cone, i.e. $s_i = \mathrm{in}(A_i, A_j); s_j$ for all $j \geq i$, we find that $\langle s_i \rangle_{i \geq 0}$ is an increasing chain of morphisms with lub $\bigsqcup s_i$. Fixing $i \geq 0$ and taking the lub of the above equations with respect to j, we get:

$$s_i = \bigsqcup_j (\mathrm{in}(A_i, A_j); s_j) = \mathrm{in}(A_i, \bigsqcup A_j); \bigsqcup s_j.$$

To show that $h = \bigsqcup s_i$ is the unique morphism satisfying $s_i = \mathrm{in}(A_i, \bigsqcup A_j); h$, for all $i \geq 0$, assume that this equation holds for all $i \geq 0$. Taking the lub of both sides we obtain:

$$\bigsqcup s_i = \bigsqcup_i ((\mathrm{in}(A_i, \bigsqcup_j A_j); h) = \mathrm{in}(\bigsqcup_i A_i, \bigsqcup_j A_j); h = h.$$

This proves the first part. As for the second part, note that since inclusion morphisms are strict (Proposition 3.2(ii)) we only need to check that $h = \bigsqcup s_i$ is strict if s_i's are strict. But this follows from Proposition 3.2(iv). □

Corollary 3.7 *If $F : P \to P$ is a continuous functor on a complete I-category P, then the chain*

$$T = \Delta \rightarrowtail F(\Delta) \rightarrowtail F^2(\Delta) \rightarrowtail \cdots$$

has a colimit, D, in P with $F(D) = D$. D is also a colimit of the chain in P^s.

Proof Since, by Proposition 3.5(iv), F is continuous on $(\mathrm{Obj}, \trianglelefteq)$, the lub, D, of the increasing chain $\Delta \trianglelefteq F(\Delta) \trianglelefteq F^2(\Delta) \cdots$ satisfies $F(D) = D$. The rest follows from the lemma. □

3.3 Initial algebras

Recall that given an endofunctor $F : C \to C$ on a category C, the category of F-algebras has as objects the pairs (A, f), with $A \in \mathrm{Obj}_C$

and $f \in \hom(F(A), A)$, and as morphisms, between objects (A, f) and (B, g), those $h \in \hom(A, B)$ for which the following diagram:

$$\begin{tikzcd} A \arrow[d, "h"'] & F(A) \arrow[l, "f"'] \arrow[d, "F(h)"] \\ B & F(B) \arrow[l, "g"] \end{tikzcd}$$

commutes. The *initial algebra* of F is defined to be the initial object, if it exists, of the category of F-algebras; and we then say that F has an initial algebra, or a least fixed point, in C. Initial algebras give a canonical solution of domain equations and therefore play a fundamental role in computing science (see, for example, [MA86] for details). Remember that all functors between I-categories are assumed to be standard.

Theorem 3.8 *A continuous endofunctor on a complete I-category P has an initial algebra in P^s.*

Proof Let $D = \bigsqcup F^i(\Delta)$ as in Corollary 3.7. Then $F(D) = D$ and, therefore, $(D, \mathrm{Id}(D))$ is an F-algebra in P^s, since $\mathrm{Id}(D)$ is strict by Proposition 3.2(ii). Suppose (E, k) is any F-algebra in P^s i.e. $\mathrm{dom}(k) = F(E)$, $\mathrm{cod}(k) = E$ and k is strict. We must prove the existence of a unique strict morphism h with $h = F(h); k$ i.e. for which the diagram.

$$\begin{tikzcd} D \arrow[r, equal] \arrow[d, "h"'] & F(D) \arrow[d, "F(h)"] \\ E & F(E) \arrow[l, "k"'] \end{tikzcd}$$

commutes. *Existence of h*: We inductively define the morphisms g_i, $i \geq 0$, as follows:

$$g_0 = \mathrm{in}(\Delta, E) \qquad\qquad g_i = F(g_{i-1}); k\,.$$

We will show that $\langle g_i \rangle_{i \geq 0}$ is an increasing chain of strict morphisms, whose lub h is strict and satisfies $h = F(h); k$. To show that the g_i's are increasing, first note that, by Axiom 1(ii), and Proposition 2.3(iii) we have:

$$g_0 \trianglelefteq^m \mathrm{in}(\Delta, F(\Delta)); g_1 \trianglelefteq^m g_1.$$

Assume inductively that $g_i \trianglelefteq^m g_{i+1}$. Since F is monotonic on $(\mathrm{Mor}, \trianglelefteq^m)$ and composition of morphisms is a monotonic operation with respect to $\trianglelefteq^m$, it follows that $F(g_i); k \trianglelefteq^m F(g_{i+1}); k$ i.e. $g_{i+1} \trianglelefteq^m g_{i+2}$, proving the inductive step. To show that the g_i's are strict, note that g_0 is strict as it is an inclusion morphism. Assume that g_i is strict. Then $F(g_i)$ is strict since F is a standard functor (Proposition 3.5(iii)), and hence $g_{i+1} = F(g_i); k$ is strict, since the composition of strict morphisms is strict. It now follows by Proposition 3.2(iv) that h is strict as it is the lub of an increasing chain of strict morphisms. Furthermore, the continuity of F implies:

$$h = \bigsqcup g_i = \bigsqcup g_{i+1} = \bigsqcup (F(g_i); k) = (\bigsqcup F(g_i)); k = (F(\bigsqcup g_i)); k = F(h); k.$$

Therefore $(h, F(h))$ is a morphism of the category of F-algebras of P^s mediating between $(D, \mathrm{Id}(D))$ and (E, k).

Uniqueness of h: Let f be another strict morphism also satisfying $h = F(h); k$. We show, by induction, that $\mathrm{in}(F^i(\Delta), D); f = g_i$ for all $i \geq 0$. For $i = 0$ this equation is just the strictness condition for f. Assume that the equation holds for $i \geq 0$, we have:

$$\begin{array}{lll}
\mathrm{in}(F^i(\Delta), D); f = g_i & \Rightarrow & F(\mathrm{in}(F^i(\Delta), D); f) = F(g_i) \quad \Rightarrow \\
\mathrm{in}(F^{i+1}(\Delta), F(D)); F(f) = F(g_i) & \Rightarrow & \mathrm{in}(F^{i+1}(\Delta), D); F(f) = F(g_i) \quad \Rightarrow \\
\mathrm{in}(F^{i+1}(\Delta), D); F(f); k = F(g_i); k & \Rightarrow & \mathrm{in}(F^{i+1}(\Delta), D); f = g_{i+1}.
\end{array}$$

This completes the inductive proof. It now follows that:

$$\begin{aligned}
h &= \bigsqcup g_i = \bigsqcup (\mathrm{in}(F^i(\Delta), D); f) = (\bigsqcup (\mathrm{in}(F^i(\Delta), D)); f = (\mathrm{in}(\bigsqcup F^i(\Delta), D)); f \\
&= \mathrm{in}(D, D); f = f.
\end{aligned}$$

This completes the proof.□

We have therefore succeeded in obtaining an initial algebra theorem by essentially a cpo construction without the usual heavy category-theoretic machinery based on global cocontinuity of functors.

Example 3.9 Let $(Q, \sqsubseteq)$ be a cpo considered as a complete I-category. (See Example 2.8.) All continuous functions $f : Q \to Q$ induce continuous functors on Q, and the initial algebra theorem reduces to the statement that the least fixed point $D = \bigsqcup f^i(\bot)$ is the least pre-fixed point (where E is a pre-fixed point of f if $E \sqsubseteq f(E)$).

Example 3.10 Consider **Set-ISys**. (See Example 2.13.) Using the BNF notation, we can construct a set of endofunctors as follows:

$$F ::= \mathrm{ID} \mid F_A \mid \mathcal{P}_f \mid F_1 \times F_2 \mid F_1 + F_2 \mid F_{A_f} \to F \mid F_1 \circ F_2$$

where ID is the identity functor, F_A is the constant functor which maps all objects to $A \in \mathrm{Obj}$, $\mathcal{P}_f$ is the finite power set constructor, "$- \times -$" and "$- + -$" are the product and the disjoint sum functors, A_f denotes a finite set in Obj, "$- \to -$" is the function space constructor and $F_1 \circ F_2$ is the composition of F_1 and F_2. It is straightforward to check that this defines a set of continuous endofunctors. Note that in the function space constructor $- \to -$, we have to fix the first argument in order to get a covariant functor. Furthermore, the finiteness of the set A_f is necessary for continuity, as for example the endofunctor $\mathbb{N} \to \mathrm{ID}$ is not continuous: $\mathbb{N} \to \mathbb{N}$ is in fact strictly larger than $\bigcup_i (\mathbb{N} \to N_i)$, where N_i is the set of the first i natural numbers. As an example, we solve the domain equation $X = 1 + X$, in **Set-ISys**, where $1 = \{\mathrm{nil}\}$ is a singleton set. Writing $F_{1+}(X) = 1 + X$ we readily find that $D = \bigsqcup F^i_{1+}(\emptyset)$ is the least solution or colimit. To appreciate this, let's fix our notation for the disjoint sum of two sets and write:

$$A + B = \{(0, a) \mid a \in A\} \cup \{(1, b) \mid b \in B\}.$$

Therefore, we have:

$$F^0_{1+}(\emptyset) = \emptyset, \;\; F_{1+}(\emptyset) = 1 + \emptyset = \{(0, \mathrm{nil})\}$$

$$F^2_{1+}(\emptyset) = 1 + (1 + \emptyset) = \{(0, \mathrm{nil}), (1, (0, \mathrm{nil}))\}, \;\; \cdots$$

We then find, after dropping all brackets, that

$$D = \{0\,\mathrm{nil}, 10\,\mathrm{nil}, 110\,\mathrm{nil}, 1110\,\mathrm{nil}, \cdots\}$$

which can be identified with $\mathbb{N}$.

Remark 3.11 *In categories of information systems for domains an inclusion morphism corresponds to an embedding projection pair and the function space constructor induces a covariant standard and continuous functor with two arguments on the subcategory of inclusion morphisms. In this restricted subcategory, we can therefore apply the initial algebra theorem into functors involving the function space. This restriction is analogous to the corresponding treatment of the function space constructor in the categories of domains as in [SP82]. See Section 5.1.1 for detail.*

Finally, we note that a continuous endofunctor on P may not have an initial algebra in P itself. In fact, although the existence of h, in the proof of Theorem 3.8, is always guaranteed, its uniqueness will fail, in general, when we allow non-strict morphisms.

4 Effectiveness

In this section, we present an effective theory for ω-algebraic I-categories. The idea is to postulate a suitable recursive structure on the base P^b of the category P and define the computable objects and morphisms of P to be the lubs of effective chains of objects and morphisms of P^b respectively. A computable functor will then be defined as a standard functor which is computable as a cpo function on $(\mathrm{Mor}_P, \trianglelefteq^m)$. Our aim is to obtain an effective version of the initial algebra theorem. We will use the standard notions from classical recursion theory as, for example, in [Cut80]. In particular, ϕ_n, is the n^{th} partial recursive function in the standard enumeration and $\langle m_1, \cdots, m_n \rangle$ is the n-tupling function from $\mathbb{N}^n$ to $\mathbb{N}$.

4.1 Effective cpo's

We briefly recall the theory of computability of ω-algebraic cpo's. See [Plo81] for details. Let $(Q, \sqsubseteq)$ be any algebraic cpo with least element $\perp$ and let $e : \mathbb{N} \to \mathcal{K}_Q$ be any surjection representing an *enumeration* of its basis $\mathcal{K}_Q$. Write e_n for $e(n)$ and $x \uparrow y$ if $x, y \in Q$ have an upper bound. We say that Q is *effectively given with respect to* e with index $\langle a, r, s \rangle$ if the following three conditions hold:

(i) $e_a = \bot$

(ii) $e_m \uparrow e_n$ is recursive in m and n with index r.

(iii) $e_m \sqsubseteq e_n$ is recursive in m and n with index s.

An *effective chain* (of compact elements) of Q with index j with respect to e is an increasing chain $e_{\phi_j(1)} \sqsubseteq e_{\phi_j(2)} \sqsubseteq e_{\phi_j(3)} \sqsubseteq \cdots$. An element $d \in Q$ is said to be *computable* if there exists an effective chain with lub d. The index of d with respect to e is then defined to be the index of this effective chain. Denoting the poset of the computable elements of Q by C_Q, we can then obtain an enumeration $e' : \mathbb{N} \to C_Q$ of C_Q with respect to e. Given two ω-algebraic cpo's Q_1 and Q_2 with e_1, e_1' and e_2, e_2' as enumeration of their compact and computable elements respectively, we say that a continuous function $f : Q_1 \to Q_2$ is *computable* if there exists a recursive function $l : \mathbb{N} \to \mathbb{N}$ such that the diagram:

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "l"] \arrow[d, "e_1'"'] & \mathbb{N} \arrow[d, "e_2'"] \\
Q_1 \arrow[r, "f"] & Q_2
\end{tikzcd}$$

commutes. Any index for l is, then, said to be an index for f. Note that the composition of computable functions is computable, since the composition of recursive functions is recursive. We need the following simple results which can be found in [Plo81].

Lemma 4.1 *The lub of an increasing effective chain of computable elements is computable and an index for it can be effectively obtained from an index for the effective chain.*

Corollary 4.2 *The least fixed point of a computable function on an ω-algebraic cpo is computable and an index for it is effectively obtainable from an index for the function.*

4.2 Effective I-categories

We are now ready to define the notion of an effectively given ω-algebraic I-category.

Definition 4.3 Let P be an ω-algebraic I-category. An *enumeration* (A, f) of the base of P is given by an enumeration $A : \mathbb{N} \to \mathrm{Obj}_{P^b}$ and an enumeration $f : \mathbb{N} \to \mathrm{Mor}_{P^b}$ of its compact objects and morphisms (i.e. objects and morphisms of P^b) respectively. P is said to be *effectively given* with respect to (A, f) by index $\langle a, r, s, t, u, v, w\rangle$ if the following five conditions hold:

R1 $(\mathrm{Mor}_P, \trianglelefteq^m)$ is effectively given with respect to f with index $\langle a, r, s\rangle$;

R2 $\mathrm{dom}(f_n) = A_l$ is recursive in n and l with index t;

R3 $\mathrm{cod}(f_n) = A_l$ is recursive in n and l with index u;

R4 $f_n = \mathrm{in}(A_l, A_p)$ is recursive in n, l and p with index v;

R5 $f_n; f_l = f_p$ is recursive in n, l and p with index w.

Proposition 4.4 *Let P be an ω-algebraic I-category effectively given with respect to (A, f).*

(i) *$(Obj_P, \trianglelefteq)$ is effectively given with respect to A with an index which can be effectively obtained from an index for P with respect to (A, f).*

(ii) *The mappings dom, cod : $(Mor, \trianglelefteq^m) \to (Obj, \trianglelefteq)$ and the mapping $Id : (Obj, \trianglelefteq) \to (Mor, , \trianglelefteq^m)$ are computable and an index for each can be effectively obtained from an index for P with respect to (A, f).*

(iii) *If $B, C \in Obj_P$ are computable with indices i, j respectively and $B \trianglelefteq C$, then $in(B, C)$ is computable and an index for it can be effectively obtained from i, j.*

(iv) *If $g, h \in Mor_P$ are computable with $cod(g) = dom(h)$, then $g; h$ is also computable and an index for it can be effectively obtained from one for g and one for h.*

Proof (i) We have $f_a = \mathrm{Id}(\Delta)$. Using R4, let b be the least integer with $f_a = \mathrm{Id}(A_b)$. Then, $A_b = \Delta$. Next we show that $A_m \uparrow B_n$ is recursive in m, n with an index which can be effectively obtained from r. Clearly,

$$A_m \uparrow B_n \iff \mathrm{Id}(A_m) \uparrow \mathrm{Id}(B_n).$$

Let m', n' be the least integers with $f_{m'} = \mathrm{Id}(A_m)$ and $f_{n'} = \mathrm{Id}(A_n)$ respectively. Since $f_{m'} \uparrow f_{n'}$ is recursive with index r, it follows that $A_m \uparrow B_n$ is recursive with an index which can be effectively obtained from r. The predicate $A_m \sqsubseteq A_n$ is treated in a similar way.

(ii) Let $g \in (\mathrm{Mor}, \trianglelefteq^m)$ be computable with index j, say. Then, $g = \bigsqcup_n f_{\phi_j(n)}$ and hence $\mathrm{dom}(g) = \bigsqcup_n \mathrm{dom}(f_{\phi_j(n)})$. For each n, let $m(n)$ be the least integer such that $\mathrm{dom}(f_{\phi_j(n)}) = A_{m(n)}$. Then, m will be recursive by R2 and an index for m can be effectively obtained, via l say from j, i.e. $A_{m(n)} = A_{\phi_{l(j)}(n)}$ and $\mathrm{dom}(g) = \bigsqcup_n A_{\phi_{l(j)}(n)}$, where l is recursive. Therefore, dom is computable. Similarly, we show that cod and Id are computable mappings.

(iii) We have $B = \bigsqcup_n A_{\phi_i(n)}, C = \bigsqcup_n A_{\phi_j(n)}$. For each integer $n \geq 0$, let n' be the least integer with $n' \geq n$ and $A_{\phi_i(n)} \trianglelefteq A_{\phi_j(n')}$. Such n' exists since A_m's are compact and $B \trianglelefteq C$. Now let $l(n)$ be the least integer with $f_{l(n)} = \mathrm{in}(A_{\phi_i(n)}, A_{\phi_j(n')})$. Then l is recursive with an index which can be effectively obtained from i, j and we have $\mathrm{in}(B, C) = \bigsqcup_n f_{l(n)}$.

(iv) Let g, h have indices i, j respectively, i.e. $g = \bigsqcup_n f_{\phi_i(n)}$ and $h = \bigsqcup_n f_{\phi_j(n)}$. For each $n \geq 0$, let n' be the least integer with $n' \geq n$ and $\mathrm{cod}(f_{\phi_i(n)}) \trianglelefteq \mathrm{dom}(f_{\phi_j(n')})$ and let $l(n)$ be the least integer with

$$f_{l(n)} = f_{\phi_i(n)}; \mathrm{in}(\mathrm{cod}(f_{\phi_i(n)}), \mathrm{dom}(f_{\phi_j(n')})); f_{\phi_j(n')}.$$

We have $g; h = \bigsqcup_n f_{l(n)}$, and moreover, by applying R2-R5, we find that l is recursive with an index which is effectively computable from i and j. $\square$

Definition 4.5 (i) A continuous endofunctor F on an effectively given ω-algebraic I-category is *computable* if the induced cpo function $F_m : (\mathrm{Mor}_P, \trianglelefteq^m) \to (\mathrm{Mor}_P, \trianglelefteq^m)$ is computable.

(ii) An F-algebra (E, k) is *computable* if $k \in (\mathrm{Mor}_P, \trianglelefteq^m)$ is computable.

Proposition 4.6 *If F is a computable endofunctor on P, then the induced function $F_o : Obj_P \to Obj_P$ is computable.*

Proof This follows immediately from the equality $F_o = \mathrm{dom} \circ F_m \circ \mathrm{Id}$, since the composition of computable functions is computable. □

We can now present the effective version of the initial algebra theorem which first appeared in [Eda89].

Theorem 4.7 *Let F be a computable endofunctor on P. Then P has a computable initial algebra $(D, Id(D))$, an index for which is effectively obtainable from one for F. Moreover if (E, k) is a computable F-algebra, the unique morphism h satisfying $h = F(h); k$ is computable and an index for h can be effectively obtained from one for F and one for (E, k).*

Proof Let i be an index for F. By Corollary 4.2, $D = \bigsqcup F^n(\Delta)$ is computable and an index for it can be effectively obtained from i. This establishes the first part of the theorem. Now let (E, k) be a computable F-algebra with index j, say, and let g_i's be defined as in the proof of Theorem 3.8. We claim that $\langle g_i \rangle_{i \geq 0}$, whose lub is h, is an effective chain of computable morphisms with an index effectively obtainable from i and j. We show that the n^{th} element of the chain is computable with an index denoted by $l_{i,j}(n)$ which can be effectively obtained from i and j. In fact, $g_0 = \mathrm{in}(\Delta, E)$ is, by Proposition 4.4(iii), computable with an index effectively obtainable from j. Now suppose the induction hypothesis holds for g_n. Then, it follows by computability of F that $F(g_n)$ is computable with an index effectively obtainable from i and j. By Proposition 4.4(iv), therefore, $g_{n+1} = F(g_n); k$ is also computable with an index effectively obtainable from i and j and the inductive proof is complete. Finally, the function $n \to l_{i,j}(n)$ is recursive with an index effectively obtainable from i and j. This proves our claim. Since h is the lub of this chain, the result now follows from Lemma 4.1. □

We have therefore an effective initial algebra theorem and a satisfactory theory of computability for ω-algebraic I-categories.

5 Basic examples

We will briefly treat some further simple examples of I-categories in this paper.

5.1 Bounded complete information systems

A bounded complete information system is a structure $A = (|A|, \vdash, \wedge, \Delta)$, where $|A|$ is a countable set of tokens, $\vdash$ is a pre-order over $|A|$, and $\wedge$ is a partial binary operation over $|A|$ and Δ is a distinguished token satisfying:

(i) $a \vdash \Delta$ for all $a \in |A|$.

(ii) $a \wedge b$ exists iff we have $c \vdash a$ & $c \vdash b$ for some $c \in |A|$ and in that case $c \vdash a \wedge b$.

Note that this is the same as "propositional languages" of Fourman and Grayson in [FG82]. We require that bounded complete information systems have the *same* distinguished token Δ. A morphism or an approximable mapping $f : A \to B$ between two bounded complete information systems $A = (|A|, \vdash_A, \wedge_A)$ and $B = (|B|, \vdash_B, \wedge_B)$ is a relation $f \subseteq |A| \times |B|$ satisfying:

(i) $\Delta f \Delta$

(ii) $afb \ \&\ afb' \ \Rightarrow\ af(b \wedge_B b')$

(iii) $a \vdash_A a' \ \&\ a'fb' \ \&\ b' \vdash_B b \ \Rightarrow\ afb$

The identity morphism on an object A is given by the approximable mapping $\vdash_A$. The category of bounded complete information systems with approximable mappings, denoted by **BC-ISys**, is a full subcategory of Gunter's category of pre-orders and approximable mappings in [Gun87] and is equivalent to the category **S-Dom** of bounded complete (Scott) domains and continuous functions. The equivalence is given by the functors

$$\textbf{S-Dom} \underset{F}{\overset{G}{\rightleftarrows}} \textbf{BC-ISys}$$

where

$$\begin{array}{rl} F_o: & (D,\sqsubseteq) \longmapsto (\mathcal{K}_D,\vdash,\wedge) \text{ with } a \vdash a' \text{ iff } a' \sqsubseteq a \text{ and } a \wedge a' = a \sqcup a' \\ F_m: & f \longmapsto R_f \text{ with } aR_f b \text{ iff } b \sqsubseteq f(a) \\ \\ G_o: & (I,\vdash,\wedge) \longmapsto (\mathrm{Fil}(I),\subseteq) \\ G_m: & R \longmapsto f_R \quad \text{with} \quad f_R(x) = \{b \mid \exists a \in x.\, aRb\} \end{array}$$

and $\mathrm{Fil}(I)$ is the set of filters of I. It is routine to check that F and G in fact induce an equivalence of categories, and they preserve the order of homsets.

We define the set of inclusion morphisms Inc as follows. The inclusion morphism from A to B exists and is equal to $\vdash_B \cap(|A| \times |B|)$ iff A is a substructure of B, i.e. $|A| \subseteq |B|$ and $\vdash_A$ and $\wedge_A$ are respectively the restrictions of $\vdash_B$ and $\wedge_B$ to $|A|$. It can now be easily verified that $(\textbf{BC-ISys}, \mathrm{Inc}, \subseteq, \{\Delta\})$ is an I-category. Furthermore, the following proposition shows that embedding projection pairs between domains and inclusion morphisms between information systems fully capture each other.

Proposition 5.1 (i) *The functor G maps inclusion morphisms to embeddings.*

(ii) *If (e,p) is an embedding projection pair between Scott domains D and E, then there exist information systems A and B with $A \cong F(D)$, $B \cong F(E)$ and $A \trianglelefteq B$, such that the following diagram commutes:*

$$\begin{tikzcd} F(E) \arrow[r, "\cong"] & B \\ F(D) \arrow[u, "F(e)"] \arrow[r, "\cong"] & A \arrow[u] \end{tikzcd}$$

Proof (i) Let $A \trianglelefteq B$ and $I = \mathrm{in}(A,B)$. Define the approximable map $P : B \to A$ by bPa iff $a \vdash_B b$. Then $I;P = \mathrm{Id}(A)$ and $P;I \subseteq \mathrm{Id}(B)$. We conclude that $G(I)$ is an embedding with the corresponding projection $G(P)$.

(ii) Put $A = F(D)$ and let B be $F(E)$ with the proviso that for every compact element d of D, the corresponding compact element $e(d)$ of E is relabelled with d.$\Box$

Note further that the I-category **BC-ISys** is complete; given an increasing chain of objects $\langle A_k \rangle_{k \geq 0}$, its lub is the object

$$\bigcup A_k = (\bigcup |A_k|, \bigcup \vdash_k, \bigcup \wedge_k),$$

and an increasing chain of morphisms $\langle (f_k : A_k \to B_k) \rangle_{k \geq 0}$ has lub

$$\bigcup f_k : \bigcup A_k \to \bigcup B_k,$$

where $\bigcup f_k$ is just the union of f_k's.

Consider the full subcategory, **BC-ISys***, of objects in which $\vdash$ is a partial order. Since every object of **BC-ISys** is isomorphic to its Lindenbaum algebra (i.e. the quotient of the structure under the equivalence induced by $\vdash$), which is, upto a relabelling of tokens, an object of **BC-ISys***, it follows that these two categories are in fact equivalent. It can be easily shown that **BC-ISys*** is an ω-algebraic I-category with the nice feature that an object is compact if and only if its carrier set is finite, and compact morphisms are precisely the morphisms between compact objects. It is therefore convenient to work with **BC-ISys***. As regards functors, one usually defines them in **BC-ISys**, but it is also possible to define them in **BC-ISys***. We will illustrate this by constructing the function space in the latter category.

5.1.1 Function space constructor

Recall that the initial algebra theorem holds for covariant functors, but the general function space constructor

$$(-) \to (-) : \textbf{BC-ISys*}^{op} \times \textbf{BC-ISys*} \to \textbf{BC-ISys*}$$

is contravariant in its first argument. The situation is analogous to the one encountered in domain theory, where the problem is handled by restricting to the subcategory of projection-embeddings on which the function space constructor induces a covariant functor [Sco72, SP82].

Here, we follow a similar line and restrict to the subcategory of inclusion morphisms denoted by $\mathbf{BC\text{-}ISys^{*}}^E$, on which we can obtain a covariant functor

$$(-) \to^E (-) : \mathbf{BC\text{-}ISys^{*}}^E \times \mathbf{BC\text{-}ISys^{*}}^E \to \mathbf{BC\text{-}ISys}^E.$$

By Proposition 5.1, we will then have a treatment equivalent to the corresponding one in domain theory. We now describe the action of $(-) \to^E (-)$ on objects and inclusion morphisms. For convenience we will drop the superscript E. Given objects I and J of **BC-ISys***, $I \to J$ is defined as follows.

- $|I \to J|$ consists of finite sets f of pairs of elements of I and J, i.e. $f = \{(a_k, b_k) | a_k \in I, b_k \in J \setminus \{\Delta\}, k \in K, K \text{ finite } \}$ satisfying the following conditions:

  H(i) whenever $L \subseteq K$ and $\{a_l | l \in L\}$ is bounded, we have $\bigwedge_{l\in L} a_l = a_k$ for some $k \in K$;

  H(ii) whenever $(a, b), (a', b') \in f$ and $a \lneq a'$, we have $b \lneq b'$;

  H(iii) whenever $(a, b), (a, b') \in f$, we have $b = b'$.

- $f_1 \leq_{I\to J} f_2$ iff for all $(a, b) \in f_2$ there exists $(a', b') \in f_1$ with $a \leq a'$ and $b' \leq b$.

- Given bounded $f_1, f_2 \in |I \to J|$, i.e. $f \leq f_1$ and $f \leq f_2$ for some $f \in |I \to J|$, $f_1 \wedge f_2 = \mathrm{cl}(f_1 \cup f_2)$, where $\mathrm{cl}(f_1 \cup f_2)$ is the closure of $f_1 \cup f_2$ under the H conditions above. This closure is obtained from $f_1 \cup f_2$ by the following two steps:

  cl(i) We start with $f_1 \cup f_2$ and for any of its subsets $\{(a_l, b_l) | l \in L\}$ with $\{a_l | l \in L\}$ bounded below, we add the pair $(\bigwedge_{l\in L} a_l, \bigwedge_{l\in L} b_l)$ to it. (Note that $\bigwedge_{l\in L} b_l$ exists since f satisfies H(i).)

  cl(ii) We now remove redundancies by imposing H(ii) and H(iii).

- $\Delta = \emptyset$

It is routine to check that $I \to J$ is an object of **BC-ISys***. The conditions H(i)-(iii) ensure that a compact element of the function space of two Scott domains has a unique representation in the function space of the corresponding information systems and hence the latter space is in fact a partial order and not simply a pre-order. The functor $(-) \to (-)$ acts on inclusion morphisms as follows. If $I \trianglelefteq I'$ and $J \trianglelefteq J'$, it is readily seen that $(I \to J) \trianglelefteq (I' \to J')$. We therefore define

$$(\mathrm{in}(I, I') \to \mathrm{in}(J, J')) \;=\; \mathrm{in}((I \to J), (I' \to J')),$$

which makes the functor $(-) \to (-)$ covariant in both arguments and standard as desired. The reader can verify that this functor is in fact continuous and it captures the intended meaning of the function space. All the details for this section can be found in [ES91].

5.2 Continuous bounded complete posets

We can use **BC-ISys** to construct a complete I-category for continuous bounded complete posets. It is well known that the continuous bounded complete posets are exactly the projections of bounded complete (Scott) domains. This means that the category of continuous bounded complete posets and (Scott) continuous functions is equivalent to a full subcategory, **CBC-ISys**, of the *Karoubi envelope* (see for example [LS86, Page 100]) of **BC-ISys**. In more detail, the construction is as follows.

Let $P = (P, \mathrm{Inc}, \sqsubseteq, \Delta)$ be an I-category. Then the Karoubi envelop $\mathrm{K}(P)$ has as objects the retractions of P, that is, morphisms r of P such that $r;r = r$, while a morphism $f : r \to s$ of $\mathrm{K}(P)$ is a morphism $f : \mathrm{dom}(r) \to \mathrm{dom}(s)$ of P such that $f = r;f;s$. For notational convenience, we will denote an object r of $\mathrm{K}(P)$ more explicitly as (A, r), where $A = \mathrm{dom}(r) = \mathrm{cod}(r)$. The inclusion morphisms of $\mathrm{K}(P)$ are defined by putting $(A, r) \trianglelefteq (B, s)$ iff $A \trianglelefteq B$ and $r \trianglelefteq^m s$, with $\mathrm{in}((A, r), (B, s))$ as $r; \mathrm{in}(A, B); s$. The hom-sets of $\mathrm{K}(P)$ have the ordering induced by the hom-set orderings of P. The distinguished object $\Delta_{\mathrm{K}(P)}$ is $(\Delta, \mathrm{Id}(\Delta))$. Finally, we define $\mathrm{PK}(P)$ (informally, the category of projections of P) as the full subcategory of $\mathrm{K}(P)$ with objects (A, r) such that $r \sqsubseteq \mathrm{Id}(A)$.

Proposition 5.2 *Suppose that P is a complete I-category. Then $PK(P)$ is a complete I-category.*

Proof (outline) We consider just a few of the axioms that have to be verified.
Axiom 1(ii): The inclusion $\mathrm{in}((\Delta, \mathrm{Id}(\Delta)), (A, r))$ is $\mathrm{in}(\Delta, A); r$, and this is clearly the least element of $\hom((\Delta, \mathrm{Id}(\Delta)), (A, r))$. (Since projections are evidently strict, we actually have $\mathrm{in}(\Delta, A); r = \mathrm{in}(\Delta, A)$.)
Axiom 1(iii): Suppose that $f; \mathrm{in}((A, r), (B, s)) \sqsubseteq g; \mathrm{in}((A, r), (B, s))$. This means that $f; r; e; s \sqsubseteq g; r; e; s$, where we have written e for $\mathrm{in}(A, B)$. Since $r; e \sqsubseteq e; s$ and $s \sqsubseteq \mathrm{Id}(B)$, it follows that $f; r; r; e \sqsubseteq g; r; e$. Since P satisfies Axiom 1(iii), we can deduce that $f; r \sqsubseteq g; r$, that is, $f \sqsubseteq g$.
Axioms 3-5 (completeness of $\mathrm{PK}(P)$): The key observation here is that the global ordering $\trianglelefteq^m_K$ of $\mathrm{Mor}(\mathrm{PK}(P))$ is the restriction of $\trianglelefteq^m$. Suppose that $f : (A, r) \to (A', r')$ and $g : (B, s) \to (B', s')$ are morphisms of $\mathrm{PK}(P)$, where $(A, r) \trianglelefteq (B, s)$ and $(A', r') \trianglelefteq (B', s')$. Putting $e = \mathrm{in}(A, B)$ and $e' = \mathrm{in}(A', B')$, we have by assumption $r; e \sqsubseteq e; s$ and $r';' e \sqsubseteq e'; s'$. By definition, $f \trianglelefteq^m g$ iff $f; e' \sqsubseteq e; g$, while $f \trianglelefteq^m_K g$ iff $f; r'; e'; s' \sqsubseteq r; e; s; g$. Thus

$$f \trianglelefteq^m g \;\Rightarrow\; r; f; e'; s' \sqsubseteq r; e; g; s' \;\Rightarrow\; f \trianglelefteq^m_K g,$$

while

$$f \trianglelefteq^m_K g \;\Rightarrow\; f; r'; r'; e' \sqsubseteq e; s; s; g \;\Rightarrow\; f \trianglelefteq^m g,$$

as claimed. Also, it is easily verified that the lub (with respect to $\mathrm{Mor}(P)$) of an increasing sequence in $\mathrm{Mor}(\mathrm{PK}(P))$ is a morphism of $\mathrm{PK}(P)$; thus $\mathrm{Mor}(\mathrm{PK}(P))$ is a subcpo of $\mathrm{Mor}(P)$. It now readily follows that Axioms 3-5 hold for $\mathrm{PK}(P)$.□

Of course, **CBC-ISys** is PK(**BC-ISys**). Notice that it is only the presence of Axiom 1(iii) that forces us to use the construction $\mathrm{PK}(P)$ rather than the more standard $\mathrm{K}(P)$. For a treatment in terms of the standard Karoubi construction, with a notion of I-category lacking Axiom 1(iii), see [Eda91].

Turning to domain equations, we note that any standard endofunctor F on a (complete) I-category P, which preserves the order in each hom-set, gives rise to a standard endofunctor F' on $\mathrm{PK}(P)$ defined as follows. For an object (A, r), we have $F'(A, r) = (F(A), F(r))$ and for a morphism $f : (A, r) \to (B, s)$ we have $F'(f) = F(f)$. It can also be checked that F' is continuous if F is continuous. This means that all the usual functors on **BC-ISys** give rise to standard and continuous functors on **CBC-ISys** and that the initial algebra theorem can be used to solve domain equations in **CBC-ISys**.

Continuous information systems have recently attained some popularity, and we shall now briefly compare the preceding with other works on the subject. We have previously [ES91] used the Karoubi construction, to obtain continuous systems, in the context of (concrete) information categories. In that context, the objects of $\mathrm{K}(P)$ are obtained by adding a transitive, interpolative relation (or dense order) to the structures which are the objects of P. Some details of the ensuing treatment differ from the preceding, since the relation $\trianglelefteq$ understood as the substructure relation does not necessarily coincide with the categorically defined $\trianglelefteq$ considered above. Hoofman [Hoo93] has independently given an account of continuous information systems in terms of the Karoubi construction . His account resembles our treatment of the concrete case quite closely. A difference of detail is that he retains the use of $\vdash$ as an entailment relation between (finite) sets of tokens and individual tokens, rather than reducing it to a mere ordering of information as we have done. The applications which Hoofman develops for continuous systems are quite different from those which we consider, and he does not introduce a cpo of systems or study domain equations.

Apart from Karoubi constructions, one may have the idea of presenting continuous domains as simply as possible in terms of densely ordered sets of tokens. This has been done in [Smy77], using "R-structures" (an R-structure is a transitively ordered set of tokens such that the predecessors of each token from a directed set). The idea has recently been taken up by Vickers [Vic93], who works with an even simpler notion of "Infosys" (or densely ordered set) as well as with R-structures. As far as concerns the topics of the present paper, [Smy77] addresses the effective solution of domain equations via sequences of embeddings of R-structures (but does not have an explicit cpo of R-structures); the applications in [Vic93], on the other hand, have little bearing on the present paper.

5.3 Information systems for Boolean algebras

Consider the category of classical propositional logics, **CPL**, whose objects are given by tuples $(A, \vdash, \wedge, \vee, \neg, 0, 1)$, where $\vdash$ is a pre-order, $\wedge$ and $\vee$ are binary operations, $\neg$ is a unary operation and 0 and 1 are constants (representing falsity and truth respectively) satisfying:

- $0 \vdash a$ $\qquad a \vdash 1$
- $a \vdash a \vee b$ $\qquad b \vdash a \vee b$ $\qquad a \vdash c \,\&\, b \vdash c \Rightarrow a \vee b \vdash c$
- $a \wedge b \vdash a$ $\qquad a \wedge b \vdash b$ $\qquad c \vdash a \,\&\, c \vdash b \Rightarrow c \vdash a \wedge b$
- $(a \vee b) \wedge (a \vee c) \vdash a \vee (b \wedge c)$
- $1 \vdash (\neg a) \vee a$ $\qquad (\neg a) \wedge a \vdash 0$.

We write $a \equiv b$ iff $a \vdash b \,\&\, b \vdash a$. A morphism, $f : A \to B$, between two objects is a relation defined as follows:

- $\forall a \, \exists b. \; afb$ $\qquad afb \,\&\, afb' \Rightarrow b \equiv b'$ $\qquad a \equiv a' \,\&\, b \equiv b' \,\&\, afb \Rightarrow a'fb'$
- $0f0$ $\qquad 1f1$
- $afb \,\&\, a'fb' \Rightarrow (a \wedge a')f(b \wedge b')$ $\qquad afb \,\&\, a'fb' \Rightarrow (a \vee a')f(b \vee b')$.

Note that the identity morphism on A is the relation $\equiv_A$. We can now construct a category of information systems **CPL-ISys** for **CPL**. Given two objects A and B with $A \trianglelefteq B$, the restriction of $\equiv_B$ to $A \times B$ is clearly a morphism; it is the inclusion map of $|A|$ into $|B|$ i.e. $\mathrm{in}(A, B)$. We take equality as the trivial partial order on hom-sets and also put Δ to be the trivial object 2 with carrier set $\{0, 1\}$ and require that $2 \trianglelefteq A$ for all objects of **CPL-ISys**. We then obtain an I-category (**CPL-ISys**, Inc, $=$, 2) which can be easily shown to be complete.

Objects of **CPL-ISys** in which the pre-order $\vdash$ is in fact a partial order are Boolean algebras and, therefore by Stone duality [Hal63], **CPL-ISys** is dual to the category, **Stone** of (non-empty) Stone spaces (i.e. totally disconnected compact Hausdorff spaces) and continuous maps. Recall that the equivalence is given by the functors

$$\mathbf{Stone} \underset{F}{\overset{G}{\rightleftarrows}} \mathbf{CPL\text{-}ISys}^{op}$$

where F and G are defined as follows. For a Stone space X, $F(X)$ is the Boolean algebra of closed open sets of X, and for a continuous map $f : X \to Y$, $F(f)$ is the restriction of the frame map $\Omega f : \Omega Y \to \Omega X$. For an object A of **CPL-ISys**, $G(A)$ is the set of ultrafilters (an ultrafilter is a filter x such that for all $a \in A$ either $a \in x$ or $\neg a \in x$) of A with a base of topology given by the closed open sets $\hat{a} = \{x \mid a \in x\}$ for all $a \in A$, and for a morphism $g : A \to B$, the morphism $G(g) : G(B) \to G(A)$ is given by $G(g)(y) = \{a \in A \mid \exists b \in y.\, agb\}$.

We denote by **Bool-ISys** the full subcategory consisting of Boolean algebras with elements from a countable alphabet. The morphisms are now Boolean homomorphisms. Since in any Boolean algebra the partial order can be expressed in terms of $\vee$ by $a \leq b \iff b = a \vee b$, we can present the objects of **Bool-ISys** by $(A, \wedge, \vee, \neg)$. We obtain a complete I-category $(\mathbf{Bool\text{-}ISys}, \mathrm{Inc}, =, 2)$, which can be easily seen to be ω-algebraic.

5.3.1 Functors

We define a set of standard and continuous functors on **CPL-ISys** by:

$$F ::= \mathrm{ID} \mid F_A \mid F_1 \times F_2 \mid F_1 + F_2 \mid P \mid F_1 \circ F_2$$

where ID, F_A, and $F_1 \circ F_2$ are the identity functor, the constant functor, and the composition of F_1 and F_2 respectively; whereas $- \times -, - + -$ and P are the product, the disjoint sum and Vietoris functors, which we will now describe.

Product. Given objects A and B of **CPL-ISys**, the product $A \times B$ has, apart from 0 and 1, tokens of the form (a, b) with $a \in A$ and $b \in B$. The predicate $\vdash$ and all operations on $A \times B$ are defined componentwise with the extra requirement that $(0, 0) \vdash 0$ and $1 \vdash (1, 1)$. It is easy to check that this in fact gives the categorical product. Furthermore, the product of two morphisms is defined componentwise in the obvious way and it is easy to check that the functor $- \times - : \mathbf{CPL\text{-}ISys} \times \mathbf{CPL\text{-}ISys} \to \mathbf{CPL\text{-}ISys}$ is in fact standard and continuous.

Coproduct. The coproduct $A + B$ is defined as the algebra generated by the disjoint sum $|A| + |B| = \{(\ell, a) \mid a \in A\} \cup \{(r, b) \mid b \in B\}$ subject to the condition that $\{\ell\} \times A$, respectively $\{r\} \times B$, inherit the relations and operations from A, respectively B, and furthermore:

$$1 \vdash (\ell, 1) \qquad 1 \vdash (r, 1) \qquad (\ell, 0) \vdash 0 \qquad (r, 0) \vdash 0.$$

It is again simple to check that this gives the categorical coproduct, and that the functor $-+-:$ **CPL-ISys** $\times$ **CPL-ISys** $\to$ **CPL-ISys** is in fact standard and continuous.

Vietoris algebra. Given an object A of **CPL-ISys**, the Vietoris algebra $P(A)$ is the algebra generated by the set $\{\diamond a \mid a \in A\}$, subject to the conditions:

- $\diamond 0 \vdash 0$.
- $\diamond a \vdash \diamond a' \iff a \vdash a'$.
- $(\diamond a) \vee (\diamond a') \equiv \diamond(a \vee a')$.
- $(\neg \diamond \neg a) \wedge (\diamond a') \vdash \diamond(a \wedge a')$.

Given a morphism $f : A \to B$ of $\mathbf{CPL-ISys}$, the morphism $P(f) : P(A) \to P(B)$ is generated by the relations $(\diamond a)P(f)(\diamond b)$ iff afb. The functor P : **CPL-ISys**$\to$**CPL-ISys** is easily checked to be standard and continuous. It corresponds to the Vietoris endofunctor V on **CPL-ISys**, which is defined as follows. For a Stone space X, the space $V(X)$ is the set of all closed subsets of X with a base of topology determined by requiring that for any closed open set $a \subseteq X$, the subset $L_a \subseteq \mathrm{V}(X)$ consisting of those closed sets of X with nonempty intersection with a is a closed open set. For a continuous map $f : X \to Y$, the mapping $V(f) : V(X) \to V(Y)$ takes any closed set in X to its image under f. The relation between the functors P and V is given by the following.

Proposition 5.3 *The following diagram commutes up to isomorphism:*

$$\begin{tikzcd} \mathbf{Stone} \arrow[r, "V"] \arrow[d, "F"'] & \mathbf{Stone} \arrow[d, "F"] \\ \mathbf{CPL\text{-}ISys} \arrow[r, "P"] & \mathbf{CPL\text{-}ISys} \end{tikzcd}$$

Proof Let X be a Stone space. It is straightforward to check that the closed open sets L_a of $V(X)$, which generate the Boolean algebra $F(V(X))$, satisfy

- $L_\emptyset = \emptyset$.
- $L_a \subseteq L_b \iff a \subseteq b$.
- $L_a \cup L_b = L_{a\cup b}$.
- $\neg L_{\neg a} \cap L_b \subseteq L_{a\cap b}$.

where $\neg a$ is the complement of a. On the other hand, the generators $\diamond a$ of the Vietoris algebra $P(F(X))$ of the Boolean algebra $F(X)$ satisfy these same relations with L_a replaced by $\diamond a$. It follows that $F(V(X))$ and $P(F(X))$ are isomorphic.$\Box$

The duality between **Stone** and **CPL-ISys** implies that solving a domain equation in either of these categories is equivalent to solving the dual equation in the other, in particular an initial algebra in **CPL-ISys** corresponds to a final coalgebra in **Stone**. These domain equations can be quite interesting; see for example Abramsky's treatment of non-well founded sets [Abr88]. We can find the initial algebra of any endofunctor of the above type and all our construction can be made effective. As an example, we solve the domain equation $X = F(X) = 2 \times X$ in **Bool-ISys**. We readily find that $D = \bigcup F^i(2)$ is the least solution (colimit). Writing the first few terms in the above sum, we get:

$$F^0(2) = 2 : \begin{tikzcd} 1 \\ 0 \arrow[u] \end{tikzcd} \qquad F(2) = 2 \times 2 : \begin{tikzcd} & 1 & \\ 01 \arrow[ur] & & 10 \arrow[ul] \\ & 0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$$F^2(2) = 2 \times (2 \times 2) : \begin{tikzcd} & 1 & \\ 110 \arrow[ur] & 101 \arrow[u] & 01 \arrow[ul] \\ 10 \arrow[u] \arrow[ur] & 010 \arrow[ul] \arrow[u] \arrow[ur] & 001 \arrow[ul] \arrow[u] \\ & 0 \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

D will therefore consist of $0, 1$ and all finite strings ending in 01 or 10. Atoms of D are precisely the strings of 0's ending in 10. These can therefore be identified with $\mathbb{N}$. However unlike $F^i(2), i \geq 0$, which can be identified with the nonnegative integers less than or equal to i, D is not atomic. The ultrafilter containing all the strings of 0's ending in 1 corresponds to the point at ∞, since every such string dominates all the atoms (integers) which have at least the same number of initial 0's. D is in fact the Boolean algebra associated with the final (greatest) solution of the dual equation $Y = 1 + Y$ in the category of Stone spaces. It is the one point compactification of $\mathbb{N}$.

5.4 Concluding remarks: Information categories

The two I-categories **BC-ISys** and **CPL-ISys** treated in this section are examples of a much more general notion of a *concrete* I-category. These concrete I-categories have objects which are sets with some internal structure given by operations and predicates defined on the elements of the sets or on their finite subsets, i.e. they are weak second order structures in the terminology of [Bar77]. The partial order on objects corresponds to the substructure relation between objects; and morphisms are relations between elements or finite subsets of the carrier sets of objects. We will call these categories *information categories*, which like the abstract I-categories can be complete or ω-algebraic. In information categories the partial order on objects $A \trianglelefteq B$ corresponds to the notion that A is a substructure of B i.e. $|A| \subseteq |B|$ and the predicates and operations on A are the restrictions of those on B to tokens in A. The partial order on morphisms $f \trianglelefteq^m g$ simply reduces to $f \subseteq g$ i.e. the inclusion of relations. Similarly in complete information categories the lub of a chain of objects will be the union of the chain of structures and the lub of a chain of morphisms will simply be the set union of the relations representing the morphisms. Finally in ω-algebraic information categories compact objects will be precisely the finite objects i.e. objects with a finite carrier set and the compact morphisms will be precisely the relations between finite objects. These features make information categories conceptually simple, and easy to handle; the task of verifying that a certain category is an information category and therefore an I-category becomes more straightforward. Using the notion of information categories, a number of substantial examples of complete (or ω-algebraic) I-categories— including categories of information systems for SFP domains, dI-domains and continuous domains—have already been constructed in detail in [ES91]; and other examples, including a category of information systems for metric spaces, are in preparation.

Acknowledgements

This work is one of the results of the SERC "Foundational Structures for Computer Science" at Imperial College, London; the first author was funded by that project. The first author would like to thank Samson Abramksy, Achim Jung and Steve Vickers for discussions on information systems and domain theory. Thanks are also due to Achim Jung and Mark Ryan for their technical assistance in producing this paper in LaTeX. Paul Taylor's diagram macros have been used in this paper.

References

[Abr88] S. Abramsky. A Cooks tour of the finitary non-well founded sets (abstract). *EATCS Bulletin*, 36:233–234, 1988.

[Bar77] K. J. Barwise. An introduction to first order logic. In K. J. Barwise, editor, *The Handbook of Mathematical Logic*, Studies in Logic and Foundations of Mathematics, pages 5–46. North Holland, 1977.

[BC81] G. Berry and P.-L. Curien. Sequential algorithms on concrete data structures. Technical report, Report of Ecole Nationale Superieure des Mines de Paris, Centre de Mathematiques Appliquées, Sophia Antipolis, 1981.

[Cut80] N. J. Cutland. *Computability: An Introduction to Recursive Function Theory.* Cambridge University Press, 1980.

[Eda89] A. Edalat. Categories of information systems. Master's thesis, Imperial College, University of London, 1989.

[Eda91] A. Edalat. Self duality in information categories. Technical Report Doc 91/42, Department of Computing, Imperial College, 1991.

[ES91] A. Edalat and M. B. Smyth. Categories of information systems. Technical Report Doc 91/21, Department of Computing, Imperial College, 1991.

[FG82] M. P. Fourman and R. J. Grayson. Formal spaces. In A. S. Trolstra and D. van Dalen, editors, *The L.E.J.Brouwer Centenary Symposium*, pages 107–121. North Holland, 1982.

[Gun87] C. Gunter. Universal profinite domains. *Information and Computation*, 72(1):1–30, 1987.

[Hal63] P. R. Halmos. *Lectures on Boolean Algebras*. Number 1 in Van Nostrand Mathematical Studies. Princeton, 1963.

[Hoo93] R. Hoofman. Continuous information systems. *Information and Computation*, 105:42–71, 1993.

[LS86] J. Lambek and P. J. Scott. *Introduction to Higher Order Categorical Logic*. Cambridge Studies in Advanced Mathematics Vol. 7. Cambridge University Press, 1986.

[LW84] K. G. Larsen and G. Winskel. Using information systems to solve recursive domain equations effectively. In D. B. MacQueen G. Kahn and G. Plotkin, editors, *Semantics of Data Types*, pages 109–130, Berlin, 1984. Springer-Verlag. Lecture Notes in Computer Science Vol. 173.

[MA86] E. Manes and M. A. Arbib. *Algebraic Approaches to Program Semantics*. Springer-Verlag, 1986.

[Plo81] G. D. Plotkin. Post-graduate lecture notes in advanced domain theory (incorporating the "Pisa Notes"). Dept. of Computer Science, Univ. of Edinburgh. Available in LaTeX, 1981.

[Sco72] D. S. Scott. Continuous lattices. In F.W. Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, pages 97–136. Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics 274.

[Sco82] D. S. Scott. Domains for denotational semantics. In M. Nielson and E. M. Schmidt, editors, *Automata, Languages and Programming: Proceedings 1982*. Springer-Verlag, Berlin, 1982. Lecture Notes in Computer Science **140**.

[Smy77] M. B. Smyth. Effectively given domains. *Theoretical Computer Science*, 5:257–274, 1977.

[SP82] M. B. Smyth and G. D. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM J. Computing*, 11:761–783, 1982.

[Vic93] S. J. Vickers. Information systems for continuous posets. *Theoretical Computer Science*, 114:201–229, 1993.