Linear maps on $\mathbb{R}^n$
Linear structure of $\mathbb{R}^n$

- The vector space $\mathbb{R}^n$ has a linear structure with two features:
  - **Vector addition**: For all $u, v \in \mathbb{R}^n$ we have $u + v \in \mathbb{R}^n$ with the components $(u + v)_i = u_i + v_i$ for $1 \leq i \leq n$ with respect, say, to the standard basis vectors of $\mathbb{R}^n$.
  - **Scalar multiplication**: For all $u \in \mathbb{R}^n$ and all $c \in \mathbb{R}$, we have $cu \in \mathbb{R}^n$ with the components $(cu)_i = cu_i$ for $1 \leq i \leq n$.

- Note that these two structures exist independent of any particular basis we choose to represent the coordinates of vectors.
Convex combination of vectors

- The convex combination of any two vectors \( u, v \in \mathbb{R}^2 \) is the line segment \( au + bv \) joining \( u \) and \( v \), where \( 0 \leq a \leq 1 \), \( 0 \leq b \leq 1 \) and \( a + b = 1 \).

- Similarly, the convex combination of three vectors \( u, v, w \in \mathbb{R}^3 \) is the triangle \( au + bv + cw \) with vertices \( u, v \) and \( w \), where \( 0 \leq a, b, c \leq 1 \) and \( a + b + c = 1 \).
Scalar product of vectors

Given two vectors \( u, v \in \mathbb{R}^3 \), with coordinates

\[
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
\]

their scalar product is given by

\[
\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{3} u_i v_i = \| \mathbf{u} \| \| \mathbf{v} \| \cos \theta,
\]

where \( \| \mathbf{u} \| = \sqrt{u_1^2 + u_2^2 + u_3^2} \) and \( \theta \) is the angle between \( u \) and \( v \). Check this formula in \( \mathbb{R}^2 \)!
Scalar product and coordinates

- Note that the lengths $\|u\|$ and $\|v\|$ as well as the angle $\theta$ are independent of the three mutually perpendicular axes used to determine the coordinates of $u$ and $v$.

- It follows that if we move to another set of mutually perpendicular axes with respect to which $u$ and $v$ have coordinates $u = \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix}$ and $v = \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix}$, then, we have

$$u \cdot v = \sum_{i=1}^{3} u_i v_i = \|u\| \|v\| \cos \theta = \sum_{i=1}^{3} u_i' v_i'.$$
A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is called linear if it preserves the linear structure, i.e., if it satisfies:

1. $\forall u, v \in \mathbb{R}^n. \ f(u + v) = f(u) + f(v)$.
2. $\forall u \in \mathbb{R}^n \forall c \in \mathbb{R}. \ f(cu) = cf(u)$.

We can write the two conditions as a single condition:

$\forall u, v \in \mathbb{R}^n \forall a, b \in \mathbb{R}. \ f(au + bv) = af(u) + bf(v)$.

Example. For a fixed $u \in \mathbb{R}^n$, let $f_u : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f_u(v) = u \cdot v$$

It is easy to check that $f_u$ is a linear map.

If $\|u\| = 1$, then $f_u(v)$ is the projection of $v$ onto the direction of $u$. 

![Diagram showing vectors $u$, $v$, and their projection $f_u(v)$]
Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

Assume $e_1, e_2, \ldots, e_n$ form a basis of $\mathbb{R}^n$ and $d_1, d_2, \ldots, d_m$ form a basis of $\mathbb{R}^m$.

By linearity, the action of $f$ is completely determined by knowing the vectors $f(e_j) \in \mathbb{R}^m$ ($j = 1, \cdots, n$).

Since $d_1, \ldots, d_m$ form a basis of $\mathbb{R}^m$, each vector $f(e_j) \in \mathbb{R}^m$ can be expressed as

$$f(e_j) = \sum_{i=1}^{m} A_{ij} d_i,$$

for some coefficients $A_{ij} \in \mathbb{R}$ with $i = 1, \cdots, m$.

Given any vector $v = \sum_{j=1}^{n} v_j e_j$, we have by linearity:

$$f(v) = f(\sum_{j=1}^{n} v_j e_j) = \sum_{j=1}^{n} v_j f(e_j) = \sum_{j=1}^{n} v_j (\sum_{i=1}^{m} A_{ij} d_i).$$
Matrix representation of linear maps

- The $m \times n$ matrix matrix $A = (A_{ij})$ with $1 \leq i \leq m$, $1 \leq j \leq n$ is called the **matrix representation of $f$ with respect to the basis** $e_1, \ldots, e_n$ of $\mathbb{R}^n$ and the basis $d_1, \ldots, d_m$ of $\mathbb{R}^m$.

- Given any vector $v = \sum_{j=1}^{n} v_j e_j$, we can compute the vector representing $f(v) \in \mathbb{R}^m$ with respect to the basis $d_1, \ldots, d_m$:

  $$f(v) = f\left(\sum_{j=1}^{n} v_j e_j\right) = \sum_{j=1}^{n} v_j f(e_j) = \sum_{j=1}^{n} v_j \left(\sum_{i=1}^{m} A_{ij} d_i\right)$$

  $$= \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} v_j d_i = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} v_j d_i = \sum_{i=1}^{m} (Av)_i d_i$$

- Thus, $(Av)_i$ is the coefficient of $d_i$ in the expansion of $f(v)$ with respect to the basis $d_1, \ldots, d_m$, i.e,

- $Av$ represents $f(v)$ with respect to the basis $d_1, \ldots, d_m$. 
Extension to $\mathbb{C}^n$

- All the definitions and results for $\mathbb{R}^n$ we have seen here can be extended to the vector space $\mathbb{C}^n$, the complex vector space of dimension $n$.
- For example, $\mathbb{C}^2$ has vectors of the form $v = (v_1, v_2)^T$ with $v_1, v_2 \in \mathbb{C}$.
- The linear structure of $\mathbb{C}^n$ is like $\mathbb{R}^n$, except that we can now use scalar multiplication with complex numbers, i.e., for the vector $v$ above: $cv = (cv_1, cv_2)^T$ for $c \in \mathbb{C}$.
- Interestingly, the standard basis for $\mathbb{R}^n$ can be interpreted as a standard basis for $\mathbb{C}^n$.
- Note that even when we deal with real matrices, we cannot avoid using complex vectors.
- In fact, the eigenvalues of a real matrix can be complex numbers, which means that the corresponding eigenvectors would be complex vectors.
The inner product in $\mathbb{C}^n$

- How about the dot product and the norm of a vector in $\mathbb{C}^n$?
- If we simply copy the definition of the dot product from the real case for complex vectors as well, then the dot product of a complex vector with itself will not necessarily be a non-negative number and thus cannot be used to define the norm of the vector.
- For complex vectors $v, w \in \mathbb{C}^n$, we define the inner product $\langle v, w \rangle := \sum_{i=1}^{n} v_i^* \cdot w_i$.
- Recall that for $c = c_1 + ic_2 \in \mathbb{C}$ the complex conjugate of $c$ is given by $c^* = c_1 - ic_2$.
- Note that $c^* c = c_1^2 + c_2^2 \geq 0$ for any $c = c_1 + ic_2 \in \mathbb{C}$.
- Observe that the inner product is not commutative: $\langle v, w \rangle = \langle w, v \rangle^*$.
- The norm of $v \in \mathbb{C}^n$ is defined as $\|v\| := \sqrt{\langle v, v \rangle}$. 