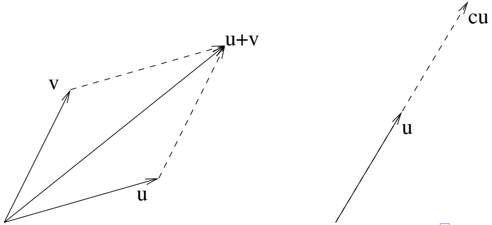
Linear maps on $\mathbb{R}^n$

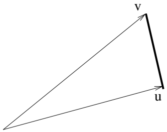
Linear structure of $\mathbb{R}^n$

- The vector space $\mathbb{R}^n$ has a linear structure with two features:
- **Vector addition:** For all $u, v \in \mathbb{R}^n$ we have $u + v \in \mathbb{R}^n$ with the components $(u + v)_i = u_i + v_i$ for $1 \leq i \leq n$ with respect, say, to the standard basis vectors of $\mathbb{R}^n$.
- **Scalar multiplication:** For all $u \in \mathbb{R}^n$ and all $c \in \mathbb{R}$, we have $cu \in \mathbb{R}^n$ with the components $(cu)_i = cu_i$ for $1 \leq i \leq n$.
- Note that these two structures exist independent of any particular basis we choose to represent the coordinates of vectors.

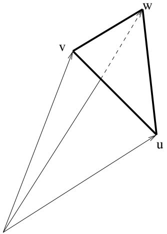


Convex combination of vectors

- The convex combination of any two vectors $u, v \in \mathbb{R}^2$ is the line segment $au + bv$ joining u and v, where $0 \le a \le 1$, $0 \le b \le 1$ and $a + b = 1$.



- Similarly, the convex combination of three vectors $u, v, w \in \mathbb{R}^3$ is the triangle $au + bv + cw$ with vertices u, v and w, where $0 \le a, b, c \le 1$ and $a + b + c = 1$.



Scalar product of vectors

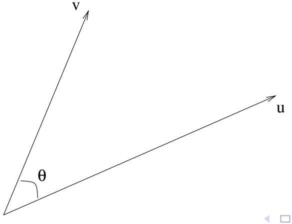
- Given two vectors $u, v \in \mathbb{R}^3$, with coordinates

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

their scalar product is given by

$$u \cdot v = \sum_{i=1}^{3} u_i v_i = \|u\| \|v\| \cos\theta,$$

where $\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$ and θ is the angle between u and v. Check this formula in $\mathbb{R}^2$!



Scalar product and coordinates

- Note that the lengths $\|u\|$ and $\|v\|$ as well as the angle θ are independent of the three mutually perpendicular axes used to determine the coordinates of u and v.
- It follows that if we move to another set of mutually perpendicular axes with respect to which u and v have coordinates

$$u = \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} \qquad v = \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix},$$

then, we have

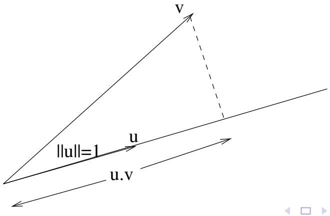
$$u \cdot v = \sum_{i=1}^{3} u_i v_i = \|u\| \|v\| \cos\theta = \sum_{i=1}^{3} u_i' v_i'.$$

Linear maps

- A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is called **linear** if it preserves the linear structure, i.e., if it satisfies:
- $\forall u, v \in \mathbb{R}^n.\ f(u + v) = f(u) + f(v)$.
- $\forall u \in \mathbb{R}^n \forall c \in \mathbb{R}.\ f(cu) = cf(u)$.
- We can write the two conditions as a single condition:
- $\forall u, v \in \mathbb{R}^n \forall a, b \in \mathbb{R}.\ f(au + bv) = af(u) + bf(v)$.
- **Example.** For a fixed $u \in \mathbb{R}^n$, let $f_u : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f_u(v) = u \cdot v$$

- It is easy to check that f_u is a linear map.
- If $\|u\| = 1$, then $f_u(v)$ is the projection of v onto the direction of u.



Linear maps and basis

- Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.
- Assume $e_1, e_2, \ldots, e_n$ form a basis of $\mathbb{R}^n$ and $d_1, d_2, \ldots, d_m$ form a basis of $\mathbb{R}^m$.
- By linearity, the action of f is completely determined by knowing the vectors $f(e_j) \in \mathbb{R}^m$ ($j = 1, \cdots n$).
- Since $d_1, \ldots, d_m$ form a basis of $\mathbb{R}^m$, each vector $f(e_j) \in \mathbb{R}^m$ can be expressed as

$$f(e_j) = \sum_{i=1}^{m} A_{ij} d_i,$$

  for some coefficients $A_{ij} \in \mathbb{R}$ with $i = 1, \cdots, m$.
- Given any vector $v = \sum_{j=1}^{n} v_j e_j$, we have by linearity:

$$f(v) = f(\sum_{j=1}^{n} v_j e_j) = \sum_{j=1}^{n} v_j f(e_j) = \sum_{j=1}^{n} v_j (\sum_{i=1}^{m} A_{ij} d_i)$$

Matrix representation of linear maps

- The $m \times n$ matrix matrix $A = (A_{ij})$ with $1 \le i \le m, 1 \le j \le n$ is called the **matrix representation of f with respect to the basis $e_1, \dots e_n$ of $\mathbb{R}^n$ and the basis $d_1, \dots, d_m$ of $\mathbb{R}^m$.**
- Given any vector $v = \sum_{j=1}^n v_j e_j$, we can compute the vector representing $f(v) \in \mathbb{R}^m$ with respect to the basis $d_1, \dots, d_m$:
- $$f(v) = f(\sum_{j=1}^n v_j e_j) = \sum_{j=1}^n v_j f(e_j) = \sum_{j=1}^n v_j (\sum_{i=1}^m A_{ij} d_i)$$
$$= \sum_{i=1}^n \sum_{j=1}^m A_{ij} v_j d_i = \sum_{i=1}^m \sum_{j=1}^n A_{ij} v_j d_i = \sum_{i=1}^m (Av)_i d_i$$
- Thus, $(Av)_i$ is the coefficient of d_i in the expansion of $f(v)$ with respect to the basis $d_1, \dots, d_m$, i.e,
- Av represents $f(v)$ with respect to the basis $d_1, \dots, d_m$.

Extension to $\mathbb{C}^n$

- All the definitions and results for $\mathbb{R}^n$ we have seen here can be extended to the vector space $\mathbb{C}^n$, the complex vector space of dimension n.
- For example, $\mathbb{C}^2$ has vectors of the form $v = (v_1, v_2)^T$ with $v_1, v_2 \in \mathbb{C}$.
- The linear structure of $\mathbb{C}^n$ is like $\mathbb{R}^n$, except that we can now use scalar multiplication with complex numbers, i.e., for the vector v above: $cv = (cv_1, cv_2)^T$ for $c \in \mathbb{C}$.
- Interestingly, the standard basis for $\mathbb{R}^n$ can be interpreted as a standard basis for $\mathbb{C}^n$.
- Note that even when we deal with real matrices, we cannot avoid using complex vectors.
- In fact, the eigenvalues of a real matrix can be complex numbers, which means that the corresponding eigenvectors would be complex vectors.

The inner product in $\mathbb{C}^n$

- How about the dot product and the norm of a vector in $\mathbb{C}^n$?
- If we simply copy the definition of the dot product from the real case for complex vectors as well, then the dot product of a complex vector with itself will not necessarily be a non-negative number and thus cannot be used to define the norm of the vector.
- For complex vectors $v, w \in \mathbb{C}^n$, we define the **inner product** $\langle v, w \rangle := \sum_{i=1}^{n} v_i^* w_i$.
- Recall that for $c = c_1 + ic_2 \in \mathbb{C}$ the **complex conjugate** of c is given by $c^* = c_1 - ic_2$.
- Note that $c^* c = c_1^2 + c_2^2 \geq 0$ for any $c = c_1 + ic_2 \in \mathbb{C}$.
- Observe that the inner product is not commutative: $\langle v, w \rangle = \langle w, v \rangle^*$.
- The **norm** of $v \in \mathbb{C}^n$ is defined as $\|v\| := \sqrt{\langle v, v \rangle}$.