Vector norms
The $\ell_p$ norms of a vector in $\mathbb{R}^n$

- For $p > 0$, the $\ell_p$ norm of any vector $v \in \mathbb{R}^n$ is defined as
  \[
  \|v\|_p = \left( \sum_{i=1}^{n} |v_i|^p \right)^{1/p}
  \]

- $p = 1$, $\ell_1$ norm: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

- $p = 2$, $\ell_2$ norm: $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$

- $p = \infty$, $\ell_\infty$ norm: $\|x\|_\infty = \max_{1 \leq i \leq m} |x_i|$

- Note that $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$.

- This follows from: $\max_{1 \leq i \leq n} |x_i|^2 \leq \sum_{i=1}^{n} x_i^2$, and $\sum_{i=1}^{n} x_i^2 \leq (\sum_{i=1}^{n} |x_i|)^2$. 
The $\ell_\infty$ norm

- As $p \to \infty$, we have $\|v\|_p \to \|v\|_\infty := \max_{1 \leq i \leq n} |v_i|$.
- If $v = 0$ then this is trivial, so assume $v \neq 0$.
- Let $m \in \{1, 2, 3, \ldots, n\}$ be such that $|v_m| = \max_{1 \leq i \leq n} |v_i|$.

$$\|v\|_p = |v_m| \left(\sum_{i=1}^{n} \frac{|v_i|^p}{|v_m|^p}\right)^{1/p}.$$

- We have $\frac{|v_i|^p}{|v_m|^p} \leq 1$ for $1 \leq i \leq n$ and at least one of them is one, since $\frac{|v_m|^p}{|v_m|^p} = 1$, i.e., the sum is between 1 and $n$.

- So, $|v_m| \leq |v_m| \left(\sum_{i=1}^{n} \frac{|v_i|^p}{|v_m|^p}\right)^{\frac{1}{p}} \leq |v_m|(n)^{\frac{1}{p}} \to |v_m|$ as $p \to \infty$, (since $n^{\frac{1}{p}} \to 1$ as $p \to \infty$).

- Thus, $\|v\|_p \to |v_m| = \|v\|_\infty$ as $p \to \infty$. 

Cauchy-Schwartz inequality

- For all $u, v \in \mathbb{R}^n$ we have
  \[ |u \cdot v|^2 \leq (u \cdot u)(v \cdot v), \text{ i.e., } |u \cdot v| \leq \|u\|_2 \|v\|_2. \]

- For a proof, consider the vector $\lambda u + v$ for any $\lambda \in \mathbb{R}$.

- Since the length of any vector is nonnegative, we have:
  \[ \forall \lambda. \ 0 \leq (\lambda u + v) \cdot (\lambda u + v) = (u \cdot u)\lambda^2 + 2(u \cdot v)\lambda + (v \cdot v) \]

- Thus, the above quadratic $a\lambda^2 + b\lambda + c$ in $\lambda$ with the three coefficients $a = u \cdot u$, $b = 2(u \cdot v)$ and $c = v \cdot v$ is always non-negative, i.e., it cannot have two distinct real roots.

- So we must have: $b^2 - 4ac \leq 0$.

- Thus, $|u \cdot v|^2 \leq (u \cdot u)(v \cdot v)$ as required.