The convex hull of finitely generable subsets and its predicate transformer

Mohammad Javad Davari
School of Computer Science
Institute for Research in Fundamental Sciences
Tehran, Iran
mjdavari@ipm.ir

Abbas Edalat
Department of Computing
Imperial College London
London, UK
a.edalat@imperial.ac.uk

André Lieutier
Dassault Systemes
Aix-en-Provence, France
andre.lieutier@3ds.com

***Abstract*—We consider the domain of non-empty convex and compact subsets of a finite dimensional Euclidean space to represent partial or imprecise points in computational geometry. The convex hull map on such imprecise points is given domain-theoretically by an inner and an outer convex hull. We provide a practical algorithm to compute the inner convex hull when there are a finite number of convex polytopes as partial points. A notion of pre-inner support function is introduced, whose convex hull gives the support function of the inner convex hull in a general setting. We then show that the convex hull map is Scott continuous and can be extended to finitely generable subsets, represented by the Plotkin power domain of the underlying domain. This in particular allows us to compute, for the first time, the convex hull of attractors of iterated function systems in fractal geometry. Finally, we derive a program logic for the convex hull map in the sense of the weakest pre-condition for a given post-condition and show that the convex hull predicate transformer is computable.**

***Keywords*—Domain Theory, Imprecision, Computability, Iterated Function Systems.**

1. Introduction

Convex hull computation is arguably the most important problem in computational geometry with a wide range of applications in many subjects including computer graphics, CAD and robotics. It has a huge literature and has been studied under different conditions and situations. The robustness of an algorithm in computational geometry is a fundamental property particularly when it is applied in critical situations. The impact of floating point system on numerical errors and robustness in algorithms is a well known problem.

Here we study a version of the convex hull problem when the location of input points are not precise. When there is uncertainty in input data the robustness of algorithms becomes crucial. Non-robustness may affect practical algorithms and therefore their outputs. A main source of non-robustness stems from limitations of the current model of floating point computation in computers. Computers run algorithms on a model which uses a limited representation model for numbers. The floating point system with its limited precision creates problems which may lead to undesirable results in various situations. The numerical errors may lead to inconsistent structures in computational geometry algorithms. There is also an additional source of uncertainty which comes from limitations of measuring devices.

Several techniques, such as exact predicates [1] or simulation of simplicity [2], allow us to design robust software for many situations arising in geometric computations. However, in the course of any long sequence of generic operations such as boolean operations or Minkowski sums, one encounters two rounding processes that are required to control the growth of the representation in practical geometric software and are the counterpart of numerical rounding in computer arithmetic. The first is related to the 64 bits floating point representation in computer arithmetic, and the second is related to the topology or the adjacency structures: for example tiny edges and triangle collapses. In this type of complex and iterative computations, lack of robustness of standard polyhedral representations remains a serious and costly issue in industry.

Our work is motivated by a long term attempt to explore other methods for representing geometry on computers. In particular, we ask whether a model with a uniform level of rounding, that manages uncertainties in data and unavoidable rounding procedures using a single paradigm, could be operational. To represent geometric objects, we use the robust model introduced in [3] which is based on domain theory introduced by Dana Scott as the mathematical foundation of computation [4]. In this context, Scott continuity of geometric operations is a prerequisite for building a sound model consistent with computability theory.

In this paper, we represent imprecise input points by non-empty, convex and compact subsets of the underlying Euclidean space. This data type generalises all existing representations of imprecise points in the literature: axis-aligned hyper-rectangles [5], [6], polytopes [7], spheres and ellipsoids [8]. We will then study properties of the so-called partial convex hull of these imprecise input points and prove the Scott continuity, and hence, in effect, the computability, of the convex hull function in this model. Today's modeling software include minerals, plants, animals or medical textures or structural representations that are sometimes modelled by fractals. Thus, in our model, we

978-1-7281-3608-0/19/$31.00

have also developed a method to compute the convex hull of attractors of iterated function systems that could have fractal structure [9].

Program Logic, the study of the properties of programs, was pioneered by the works of Floyd [10], Hoare [11] and Dijkstra [12]. Dijkstra introduced the notion of a predicate transformer and the weakest-precondition of a program with a post-condition. In this paper, we also develop the predicate transformer for the convex hull problem. A key basis of this work is grounded in domain theory. In [13], Scott showed how one can develop finitary structures to represent domains of computations and maps between them. These finitary structures and maps between them are called information systems and approximable mappings, respectively. The domain of computation for Scott's information systems are algebraic domains.

A topological view of predicate transformers was provided by Smyth using domain theory [14]. The idea is to consider open sets of a topological space as properties of a logic, points of the space as logical theories and a frame homomorphism, i.e., the action of the inverse of a continuous function on the lattice of open subsets of its range space, as predicate transformers. The underlying logic has been called Geometric Logic. In [15], Abramsky provided a comprehensive account of domain theory in logical form for stably locally compact algebraic domains, which have a topological basis of compact-open sets allowing a geometric logic with finitary operations. The domain of computation for classical Hausdorff spaces are continuous domains which do not have such a basis of compact-open sets. Smyth [16], Abramsky and Jung [17] and Vickers [18] have extended the notion of Scott's information systems to continuous domains. Jung and Sünderhauf have constructed a finitary geometric logic generated by so-called strong proximity lattices for representing stably locally compact spaces, which include basic domain structures with Euclidean spaces. We will employ a basic information system approach to study the logical properties of the convex hull problem and prove that the predicate transformer for the convex hull map on finitely generable sets is a computable predicate.

1.1. Related works

The algorithmic aspect of convex hull computation has been studied in depth by a number of authors [8], [19], [20], [21], [22]. Grünbaum studied properties of convex polytopes [23] and Klee studied topological properties of convex polytopes [24]. Kettner et. al demonstrated non-robustness of a set of computational geometry algorithms using experimental tests [25]. To overcome this problem several studies have used various methods to handle working with imprecise data [8], [26], [27], [28], [29], [30]. Some have employed interval geometry methods to approximate imprecise data. In this method an interval represents all possible choices for input. For example each point in $\mathbb{R}^N$ is represented by a hyper-rectangle which contains all possible values for a point [3], [31].

In the exact geometry model, it is assumed that all input data have exact (rational) values [32], [33]. Another method of working with imprecise data is to fatten objects e.g. points in $\mathbb{R}^2$ to circles or ellipses or polygons. In the epsilon geometry approach, each geometric object is enlarged by epsilon; for example a point in $\mathbb{R}^N$ becomes a hyper-sphere with its center at the given point. For more details on methods and algorithms in this model see [29]. Computing the convex hull of a set of imprecise points, when location of points are given by a probability function, is studied in [34], [35].

The solid domain model which uses domains to represent approximations of interior and exterior of a geometric object was introduced in [36], where it is shown that the solid domain is a robust model and also defines a model of computability for geometric objects. The solid domain model has been used to develop algorithms for the convex hull, Delaunay triangulation and Voronoi diagram when input points are partial points [6], [7], [37].

2. Preliminaries

In this section, we give a concise account of the basic domain theory notions, including the solid domain for geometric objects, that we require in this paper. We will also explain the previous domain-theoretic algorithms for the convex hull of imprecise points and extend them to a general setting in any finite dimensional Euclidean space. In this paper we denote the complement of a set $S \subset \mathbb{R}^N$ by S^c, its interior by S° and its closure by $\overline{S}$.

We use the terminology for domain theory as in [17], [38]. Recall that a poset $(D, \sqsubseteq)$ in which every directed subset has a supremum, denoted $\sup A$, is called a directed-complete partial order, or dcpo for short. The way-below relation $\ll$ in D is defined as $x \ll y$ if for any directed set $A \subset D$ the relation $y \sqsubseteq \sup A$ implies that there exists $a \in A$ with $x \sqsubseteq a$. An element $x \in D$ is called *compact* if $x \ll x$. A subset $B \subset D$ of a dcpo D is called a *basis* if for all $x \in D$, the set $\{a \in B : a \ll x\}$ is directed with supremum x. If D has a (countable) basis then it is called a (countably based) *continuous dcpo*. If it has a basis of compact elements then it is called an algebraic dcpo. We adopt the terminology in [38] from now on and refer to a continuous dcpo as a *domain* for short.

If $x \ll y$ in a domain then there exists a basis element z with $x \ll z \ll y$; this property is called the *interpolation property* of the way-below relation. A domain is *bounded complete* if any bounded set has a supremum: in particular every pair (a, b) of bounded elements has a supremum denoted by $a \sqcup b$ and the empty set has supremum $\perp$, the least element of the domain. The *Scott topology* ΩD of a domain D with basis B has basic open subsets of the form $\twoheaduparrow a = \{x \in D : a \ll x\}$ for any basis element $a \in B$. Note that Scott open sets are upward closed and the separation property of Scott topology is only T_0. If $f : D \to D$ is a Scott continuous function on a domain D with least element $\perp$, then it has a *least fixed point* $\mathsf{lfp}(f) = \sup_{n \geq 0} f^n(\perp)$. Given a basis $B \subset D$, the transitive order $(B, \ll)$ where $\ll$

is the restriction of the way-below relation to B, is called an *information system* for D. An *ideal* A of $(B, \ll)$ is a downward closed non-empty subset such that for $x, y \in A$ there exists $z \in A$ with $x, y \ll z$. The set of ideals of $(B, \ll)$ ordered by inclusion is a domain, called the *ideal completion* of $(B, \ll)$, isomorphic to D.

In this paper, the underlying domain we use is the bounded complete countably based domain $\mathbf{C}\mathbb{R}^N$ of all non-empty compact convex sets in $\mathbb{R}^N$, partially ordered by reverse inclusion (i.e., $A \sqsubseteq B$ if $B \subset A$) and augmented with $\mathbb{R}^N$ as its least element. In fact, $\mathbf{C}\mathbb{R}^N$ is a sub-domain of the upper space $\mathbf{U}\mathbb{R}^N$ of $\mathbb{R}^N$ [39] (containing all non-empty compact subsets ordered by reverse inclusion) and inherits its properties. Each non-empty compact convex set represents an imprecise point of $\mathbb{R}^N$ and its refinement to a smaller one contained in it represents more information about the imprecise point in question. In $\mathbf{C}\mathbb{R}^N$, the set of maximal elements is the set of singletons of $\mathbb{R}^N$, which with respect to the Euclidean topology and the Scott topology, is homeomorphic with $\mathbb{R}^N$. The supremum of a directed set is simply the intersection of all elements, i.e., the compact convex sets, in the set and the way-below relation is given by $A \ll B$ iff $B \subset A^\circ$. The sub-domain $\mathbf{I}\mathbb{R}^N \subset \mathbf{C}\mathbb{R}^N$ consists of all axis-aligned hyper-rectangles in $\mathbb{R}^N$.

Next we need to define the domain $(\mathbf{C}\mathbb{R}^N)^*$ of tuples of elements of $\mathbf{C}\mathbb{R}^N$ of any finite length. For any partially ordered set $(E, \sqsubseteq)$ and each positive integer $m \geq 2$, we obtain the partially ordered set E^m of all m-tuples of elements of E with component-wise ordering, i.e., $x = (x_1, \ldots x_m) \sqsubseteq y = (y_1, \ldots, y_m)$ is defined to hold if $x_i \sqsubseteq y_i$ for $1 \leq i \leq m$. Then the supremums, if they exist, of directed subsets are computed component-wise and it follows that if E is actually a domain then so is E^m with $x \ll y$ iff $x_i \ll y_i$ for $1 \leq i \leq m$, and $\Omega(E^m) = (\Omega E)^m$.

We then can define the partially ordered set E^* of all finite sequences of elements of $(E, \sqsubseteq)$ by putting

$$E^* = \bigcup_{m \geq 1} E^m$$

with the partial order $x \sqsubseteq y$ in E^* defined if there exists $m \geq 1$ such that $x, y \in E^m$ with $x \sqsubseteq y$ in E^m. The supremum, if it exists, of a directed set of elements of length $m \geq 1$ are then computed in E^m. If E is a domain, then so is E^* with $x \ll y$ in E^* iff $x, y \in E^m$ for some $m \geq 1$ and $x \ll y$ in E^m, and moreover $\Omega(E^*) = (\Omega E)^*$. We have defined E^* for a partial order E, but, clearly, E^* can also be defined when the order on E is only transitive, which is the case for $(B, \ll)$ as an information system.

We will give a brief introduction to the solid domain for $\mathbb{R}^N$ [36], which is the underlying structure for the basic domain of geometric objects we will use. The solid domain $\mathbf{S}\mathbb{R}^N$ of $\mathbb{R}^N$ is defined as the collection of all pairs (O_1, O_2) of disjoint open sets of $\mathbb{R}^N$ partially ordered component-wise. The idea is that O_1 and O_2 provide information about a *partial* or *imprecise* object, i.e., they respectively represent the interior and exterior (that is the interior of the complement) of a geometric object or subset of $\mathbb{R}^N$ at some given stage of computation. As we obtain more information about the object its interior and its exterior will both become larger open sets. In fact, $\mathbf{S}\mathbb{R}^N$ is a countably based bounded complete domain: The supremum of a directed set of partial objects is obtained by taking the unions of the open sets in the respective components and $(O_1, O_2) \ll (O_1', O_2')$ iff $\overline{O_i}$ is compact and $\overline{O_i} \subset O_i'$ for $i = 1, 2$. For a given geometric object $A \subset \mathbb{R}^N$, the supremum of all its partial geometric objects gives a pair of open sets which are the actual interior A° and exterior $(A^c)^\circ$ of the object.

2.1. Domain-theoretic convex hull algorithms

We first recall some basic definitions and properties of convex sets; see [40]. In this paper, the inner product of two vectors $v, x \in \mathbb{R}^N$ is written as $\langle v, x \rangle := \sum_{i=1}^N v_i x_i$ and the Euclidean norm of x as $\|x\| = \sqrt{\langle x, x \rangle}$.

A *convex combination* of a finite set of points $x_i \in \mathbb{R}^N$ for $i \in I$ is given by $\sum_{i \in I} w_i x_i$, where $w_i \geq 0$ with $\sum_{i \in I} w_i = 1$. A subset $X \subset \mathbb{R}^N$ is *convex* if any convex combination of any finite set of points in X belongs to X, or equivalently if for all points $y, z \in X$ the line segment joining y and z is contained in X. The description of the standard convex hull problem is as follows: Given a finite set of points $x_i \in \mathbb{R}^N$ for $i \in I$, find the smallest convex set that contains all these points. More generally, since the intersection of convex sets is convex, we can define the convex hull function $\Gamma : \mathscr{P}\mathbb{R}^N \to \mathscr{P}\mathbb{R}^N$ on the set $\mathscr{P}\mathbb{R}^N$ of subsets of $\mathbb{R}^N$ by

$$\Gamma(A) = \bigcap \{C : A \subset C, C \text{ convex}\}.$$

For a compact set $C \subset \mathbb{R}^N$ and $\epsilon > 0$, let

$$C_\epsilon := \{x \in \mathbb{R}^N : \exists y \in C. \|x - y\| \leq \epsilon\}$$

be the *closed ϵ-neighbourhood* of C. Let $(\mathcal{K}\mathbb{R}^n, d_H)$ be the space of non-empty compact subsets of $\mathbb{R}^N$ equipped with the Hausdorff metric d_H defined as

$$d_H(A, B) = \inf\{\epsilon > 0 : A \subset B_\epsilon \text{ and } B \subset A_\epsilon\}.$$

The following Lipschitz property of the convex hull is folklore in convex analysis. For a proof see [26, p. 80].

Lemma 1. *The convex hull map $\Gamma : \mathcal{K}\mathbb{R}^n \to \mathcal{K}\mathbb{R}^n$ is Lipschitz with Lipschitz constant* 1.

Recall that the *support function* of a convex set $C \subset \mathbb{R}^N$ is given by $S_C : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty, \infty\}$ with

$$S_C(v) = \sup_{x \in C} \langle v, x \rangle.$$

Note that for a convex set $C \subset \mathbb{R}^N$, we have $S_C = S_{\overline{C}}$, a property that we will frequently use in the rest of the paper. If v is a unit vector then $S_C(v)$ is the supremum of the orthogonal projection of C in the direction of v. The following provides the basic properties of the support function we require in this paper.

Proposition 1. (i) *For convex sets A and B, we have $S_A \leq S_B$ if $A \subset B$. If A and B are both closed (or both open) convex sets then $A \subset B$ if $S_A \leq S_B$.*

(ii) *If $A = \bigcup_{i \in I} A_i$ where $(A_i)_{i \in I}$ is a family of convex sets, which is directed with respect to inclusion, then $S_{\bigcup_{i \in I} A_i} = \sup_{i \in I} S_{A_i}$.*

Proof. The proof of (i) follows immediately from the definition of support function. To see (ii), let $v \in \mathbb{R}^N$. Then $\sup_{i \in I} S_{A_i}(v) = \sup_{i \in I} \sup_{x \in A_i} \langle v, x \rangle = \sup_{x \in \bigcup_{i \in I} A_i} \langle v, x \rangle = S_{\bigcup_{i \in I} A_i}(v)$. □

There is a well-known equivalent way of defining the convex hull of a non-empty compact convex set C as follows. A *half space* in $\mathbb{R}^N$ with *outer normal* $v \in \mathbb{R}^N$ is a subset of the form $\{x \in \mathbb{R}^n : \langle v, x \rangle \leq h\}$ for some $h \in \mathbb{R}$ and a non-zero $v \in \mathbb{R}^N$. Given a unit vector $v \in \mathbb{R}^N$, the half space $\{x : \langle v, x \rangle \leq S_C(v)\}$ is called the *supporting half space* of C in the direction v. Any non-empty compact and convex set is the intersection of its supporting half spaces. Furthermore, in this case, there always exists a point $x \in C$ such that $S_C(v) = \langle v, x \rangle$. Any such point is called a *supporting point* of C in the direction of v. See Figure 1 for examples of supporting half space and supporting point. If $A \subset \mathbb{R}^N$ is a convex polytope with k faces of dimension $N-1$, then A is the intersection of exactly k supporting half spaces each of which contains A and contains a face of A on its boundary: These half spaces are called the *generating half spaces* of A.

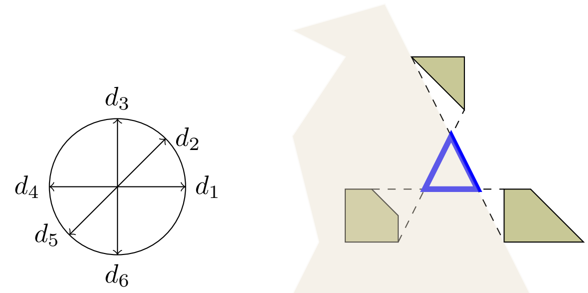
Assume we have a family $P = \{P_i : i \in I\}$ of imprecise points that are represented by non-empty, convex and compact subsets $P_i \subset \mathbb{R}^N$ for some indexing set I which may be infinite. We define $R(P) := \{\{p_i : i \in I\} : p_i \in P_i, i \in I\}$ as the collection of all possible subsets each containing exactly one point of each partial point in P.

Definition 1. *The* Convex Hull *function* CH *takes as input a family P of partial objects in $\mathbf{C}\mathbb{R}^N$ and returns an element $(\mathrm{CH}^-(P), \mathrm{CH}^+(P)) \in \mathbf{S}\mathbb{R}^N$ of the solid domain with* interior *or* inner convex hull *and* exterior *or* outer convex hull *given respectively by*

$$\begin{aligned} \mathrm{CH}^-(P) &= \left(\bigcap \Gamma(\{p : p \in R(P)\})\right)^\circ \\ \mathrm{CH}^+(P) &= \left(\bigcup \Gamma(\{p : p \in R(P)\})\right)^c. \end{aligned}$$

In words, $\mathrm{CH}^-(P)$ is the interior of the set whose points are in the convex hull of any selection $p \in R(P)$, while $\mathrm{CH}^+(P)$ consists of points that are in the complement of any such selection.

The partial convex hull function CH introduced in [6] is of type: $\mathrm{CH} : (\mathbf{I}\mathbb{R}^N)^m \to \mathbf{S}\mathbb{R}^N$ for $m \geq 1$. An $O(m \log m)$ algorithm for computing the interior and exterior of the partial convex hull of a set of m partial points in $\mathbf{I}\mathbb{R}^N$ for $N = 2, 3$ is given in [6]. The algorithm also works in $\mathbb{R}^N$ for $N > 3$, but the complexity may no longer be $O(m \log m)$. The algorithm computes the exterior by computing convex hull of the set of vertices of all partial points. The interior of the partial convex hull is the interior of the intersection of 2^N convex hulls, each of which is the convex hull of the vertices of the same type of all the partial points (i.e., lower or upper end of the projection of a partial point in each dimension).



(a) (b)

Figure 1: (a) A set of 2×3 unit vectors in $\mathbb{R}^2$ which define 6 cones. We here have $d_4 = -d_1$, $d_5 = -d_2$ and $d_6 = -d_3$. (b) Three partial points (grey) defined by the 6 directions and their inner convex hull (blue). The boundaries of the inner convex hull's three generating half spaces are shown with dashed lines. One of these three generating half spaces is shaded with light grey. In the direction of the outer normal to this generating half space, the support function of the inner convex hull has its supporting line which, in this case, contains an edge of the inner convex hull. The points of this edge are the supporting points of the support function in the outer normal direction of the generating half space.

In [7] an algorithm is proposed to compute the partial convex hull when inputs are imprecise points in $\mathbb{R}^2$ represented by compact convex polytopes. In this setting each partial point is defined by the intersection of a set of half spaces whose outer normals are from a given finite set of unit vectors. In this section, we present an extension of this algorithm to $\mathbb{R}^N$, which is therefore an extension of the work in [6] to polytopes as imprecise points.

Suppose $P = (P_1, ..., P_m)$ is a list of polytopes in $\mathbf{C}\mathbb{R}^N$. We first note that the outer convex hull of P is easy to compute since $\mathrm{CH}^+(P) = (\Gamma(\bigcup P))^c$, where $\bigcup P = \bigcup_{1 \leq i \leq m} P_i$; thus we can simply compute the convex hull of all vertices of all partial points P_i for $1 \leq i \leq m$. In the same way that an axis-aligned hyper-rectangle in $\mathbb{R}^N$ is given by a lower and an upper end in each dimension, we use a set of opposite directions to define the polytopes in P as follows. Let $d = (d_1, ..., d_{2n})$, $n \geq N$ be a list of $2n$ unit vectors in $\mathbb{R}^N$ with $d_j = -d_{j+n}$ for $1 \leq j \leq n$ with which all partial points in P are defined by the intersection of $2n$ half spaces, i.e., such that the outer normal of each half space is d_j for some $1 \leq j \leq 2n$. Each vertex of polytope P_i is the intersection of at least N adjacent generating half spaces. Therefore P_i has at most $\binom{2n}{N}$ vertices. We classify the vertices of P with outer normal vectors of their intersecting faces.

Suppose C is the convex hull of the unit vectors d_i for $1 \leq i \leq 2n$ in d. Then each face of C determines a cone in $\mathbb{R}^N$. We denote by CN the set of cones created by d. For each cone $c \in \mathrm{CN}$ there exists $-c \in \mathrm{CN}$ such that for each unit vector $x \in c$, we have $-x \in -c$. We can now characterize the vertices of the partial points of P using CN. For each $P_i \in P$ we define P_{ic} as the vertex of P_i

Algorithm 1: Inner convex hull algorithm

Input : An ordered list P of partial points in $\mathbf{C}\mathbb{R}^N$
Output: Inner convex hull of P
Let $T = \emptyset$;
foreach $c \in CN$ **do**
  Let pts $= \emptyset$;
  foreach $P_i \in P$ **do**
    Add P_{ic} to pts;
  end
  Let $\text{CH}_c =$ convex hull of pts;
  Add CH_c to T;
end
Compute and return intersection of polytopes in T

furthest away from the boundary of any half space with outer normal in c that contains P_i. It is easy to see that P_{ic} exists for each P_i and is unique. However two cones in CN may have the same corresponding vertices. Using this classification Algorithm 1 computes the intersection of a set of convex hulls of type-similar corners of the partial points. It provides an efficient way to compute the inner convex hull.

Algorithm 1 includes running of |CN| of the classic convex hull algorithm in $\mathbb{R}^N$ and then intersects |CN| convex polytopes. Thus the complexity of Algorithm 1 is $O(|CN|T_{\text{CH}}(m,N)+\text{Int}(\text{CN},m,N))$, in which $T_{\text{CH}}(m,N)$ is the complexity of computing convex hull in $\mathbb{R}^N$ for m points, and $\text{Int}(\text{CN},m,N)$ is the complexity to compute the intersection of |CN| polytopes with m vertices in N dimensions. When $N = 2,3$ computing the convex hull can be done efficiently in $O(m \log m)$, and there are linear algorithms to compute the intersection of a set of convex polygons [20]. Therefore when $N = 2,3$, Algorithm 1 has complexity $O(m \log m)$. For dimensions $N > 3$ the number of facets of the convex polytopes in the algorithm may increase exponentially in N. It follows that different representations of the convex hull, e.g., by vertex or facet description, have different sizes. For $N > 3$ computing the convex hull can be done efficiently in $O(m^{\lfloor \frac{N}{2} \rfloor})$ for m points in $\mathbb{R}^N$ [21].

3. Convex and compact sets as partial points

We consider a general geometric framework in which an imprecise or partial point in $\mathbb{R}^N$ is given by a non-empty, convex and compact subset of $\mathbb{R}^N$, i.e., elements of $\mathbf{C}\mathbb{R}^N$ represent our data type for points. Axis-aligned hyper-rectangles, as considered in Interval Analysis [5], [6], ellipsoids or spheres as considered in [8] and polytopes as considered in [7] are particular examples of this general data type.

It is convenient to use a sub-domain of the solid domain consisting of disjoint pairs of open sets (A_1, A_2), with A_1 and A_2^c both convex. The *solid convex domain* is defined as $\mathbf{SC}\mathbb{R}^N := \{(A_1, A_2) \in \mathbf{S}\mathbb{R}^N : A_1, A_2^c \text{ both convex}\}$. Then, $\mathbf{SC}\mathbb{R}^N$ inherits the properties of $\mathbf{S}\mathbb{R}^N$:

Proposition 2. *The sub-poset $\mathbf{SC}\mathbb{R}^N$ is itself a domain which is countably based and bounded complete with $(u_1, u_2) \ll (v_1, v_2)$ iff $\overline{u_i}$ is compact with $\overline{u_i} \subset v_i$ for $i = 1, 2$. The least element is $\bot = (\emptyset, \emptyset)$ and the bounded binary sup and binary inf are given as:*

$$(u_1, u_2) \sqcup (v_1, v_2) = (\Gamma(u_1 \cup v_1), u_2 \cup v_2)$$

$$(u_1, u_2) \sqcap (v_1, v_2) = ((u_1 \cap v_1), (\Gamma(u_2^c \cup v_2^c))^c).$$

Proof. The proof of the first statement is similar to the proof that $\mathbf{S}\mathbb{R}^N$ is a countably based and bounded complete domain [36]. It is easy to check that the bounded binary sup and binary inf are given as above. $\square$

3.1. Adding more imprecise points

Up to now we have restricted our attention to $\mathbf{C}\mathbb{R}^m$ for a given $m \geq 1$. The convex hull map $\text{CH} : (\mathbf{C}\mathbb{R}^N)^* \to \mathbf{SC}\mathbb{R}^N$ is clearly well defined and monotone since $(\mathbf{C}\mathbb{R}^N)^*$ is simply the union of $(\mathbf{C}\mathbb{R}^N)^m$ for $m \geq 1$.

Let us define the infix *concatenation* operator

$$((.) + +(.)) : (\mathbf{C}\mathbb{R}^N)^* \times (\mathbf{C}\mathbb{R}^N)^* \to (\mathbf{C}\mathbb{R}^N)^*.$$

If $P, Q \in (\mathbf{C}\mathbb{R}^N)^*$ then $P = (P_1, \ldots, P_m)$ and $Q = (Q_1, \ldots, Q_n)$ for some $m, n \geq 1$. we define $P + +Q = (P_1, \ldots, P_m, Q_1, \ldots, Q_n) \in (\mathbf{C}\mathbb{R}^N)^{m+n} \subset (\mathbf{C}\mathbb{R}^N)^*$. It is straightforward to check the following properties.

Proposition 3. *Suppose $P, Q \in (\mathbf{C}\mathbb{R}^N)^*$. Then*

$$\begin{array}{rcl} \text{CH}^-(P) \cup \text{CH}^-(Q) & \subset & \text{CH}^-(P + +Q) \\ \text{CH}^+(P) \cup \text{CH}^+(Q) & \supset & \text{CH}^+(P + +Q). \end{array}$$

Therefore, with the ordering in $(\mathbf{C}\mathbb{R}^N)^*$, we do not get a monotone map if we add new partial points to our input. We will show in Section 5 that by moving to the Plotkin power domain of $\mathbf{C}\mathbb{R}^N$ we have the right partial order to increase the number of partial points and yet have a Scott continuous convex hull map. We however need a toolkit to prove the Scott continuity of the convex hull map and the next section provides us with the required methods.

4. Pre-inner support function

In this section, we introduce the notion of pre-inner support function which gives the support function of the inner convex hull as in Theorem 1. In obtaining this result, we use ideas from convex polarity [40, sections 14,15,16]. However, deriving the theorem from the classical material is not straightforward and, in contrast, our elementary and self-contained proof offers a clear view without requiring the reader to delve in convexity theory.

Consider a bounded collection $P = \{P_i : i \in I\}$ of non-empty, convex and compact sets $P_i \subset \mathbb{R}^N$, i.e., there exists $K > 0$ such that the open ball of radius K centred at the origin contains P_i for all $i \in I$. Define the *pre-inner support function* of P as the map $S_P^* : \mathbb{R}^N \to \mathbb{R}$ with:

$$S_P^*(v) = \sup_{i \in I} \inf_{x \in P_i} \langle v, x \rangle = \sup_{i \in I} \min_{x \in P_i} \langle v, x \rangle. \tag{1}$$

Recall that a function $\phi : \mathbb{R}^N \to \mathbb{R}$ is *positively homogeneous* if

$$\forall \lambda \geq 0, \phi(\lambda X) = \lambda \phi(X).$$

From Equation (1) above, $S_P^*(v)$ is positively homogeneous and Lipschitz with Lipschitz constant K. Recall the definition of CH from Definition 1.

Lemma 2. *One has the following equivalence:*

$$\begin{aligned} z \in \mathrm{CH}^-(P) \quad &\iff \quad \forall v \neq 0, \langle z, v \rangle \ < \ S_P^*(v) \\ &\iff \quad \forall v, \|v\| = 1 \Rightarrow \langle z, v \rangle \ < \ S_P^*(v). \end{aligned}$$

Proof. For $v \neq 0$ and $i \in I$ there exists a point $p_i(v) \in P_i$ such that $\langle p_i(v), v \rangle = \min_{x \in P_i} \langle x, v \rangle$. Take $z \in \mathrm{CH}^-(P)$. From the definition of $\mathrm{CH}^-(P)$, one has $z \in (\Gamma(\{p_i(v) : i \in I\}))^\circ$ which implies:

$$\langle z, v \rangle < \sup_{i \in I} \langle p_i(v), v \rangle = \sup_{i \in I} \min_{x \in P_i} \langle x, v \rangle = S_P^*(v).$$

Thus, we have: $z \in \mathrm{CH}^-(P) \Rightarrow \forall v \neq 0, \langle z, v \rangle \ < \ S_P^*(v)$.

Next, consider the reverse implication. For each $i \in I$, pick a point $p_i \in P_i$. Since the convex hull of a set X is the intersection of all half spaces containing X one has:

$$(\Gamma(\{p_i : i \in I\}))^\circ = \bigcap_{v \neq 0} \left\{ z : \langle v, z \rangle < \max_i \langle v, p_i \rangle \right\}$$

and since $S_P^*(v) \leq \max_{i \in I} \langle v, p_i \rangle$, we have:

$$\Gamma\left(\{p_i : i \in I\}\right)^\circ \supset \bigcap_{v \neq 0} \{z : \langle v, z \rangle < S_P^*(v)\}$$

which gives the reverse inclusion. The second equivalence follows since $S_P^*(v)$ is positively homogeneous. ☐

Definition 2. *For a positively homogeneous function $\phi : \mathbb{R}^N \to \mathbb{R}$ the* centre *of ϕ is defined as*

$$\mathcal{I}(\phi) = \{z \in \mathbb{R}^N : \forall v \in \mathbb{R}^N, \langle z, v \rangle \leq \phi(v)\}.$$

Note that $\mathcal{I}(\phi)$ is convex and closed but can be empty. Taking $\phi = S_P^*$, Lemma 2 gives

$$\mathcal{I}(S_P^*) = \overline{\mathrm{CH}^-(P)}. \tag{2}$$

We now introduce the convex hull of a real-valued map, which is used later in Theorem 1.

Definition 3. *(Convex Hull of a Function) [41] [42]. Given a function $\phi : \mathbb{R}^N \to \mathbb{R}$, the* Convex Hull *of ϕ with type $\mathcal{H}(\phi) : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ is the supremum of convex functions below ϕ.*

Note that if there exists a convex function below ϕ then $\mathcal{H}(\phi)$ is itself a convex function. Otherwise, $\mathcal{H}(\phi)$ is the constant extended real-valued function with value $-\infty$. In addition, a convex function below ϕ exists if and only if there exists an affine function below ϕ. In particular, if $\phi(0) = 0$, then a convex function exists below ϕ if and only if there is a linear function below ϕ, in other words if there exists $h \in \mathbb{R}^N$ such that $\forall x \in \mathbb{R}^N, \langle x, h \rangle \leq \phi(x)$. Denote by $\mathrm{epi}(\phi)$ the epigraph (or supergraph or superlevel set) of ϕ:

$$\mathrm{epi}(\phi) = \{(X, t) \in \mathbb{R}^N \times \mathbb{R}, \ t \geq \phi(X)\}.$$

Observe that a function $F : \mathbb{R}^N \to \mathbb{R}$ is convex if and only if the set $\mathrm{epi}(F)$ is convex and $F \geq F'$ if and only if $\mathrm{epi}(F) \subset \mathrm{epi}(F')$. Therefore, since $\Gamma(\mathrm{epi}(\phi))$ is the minimal convex set (with respect to inclusion) containing $\mathrm{epi}(\phi)$, it is the epigraph of the largest convex function below ϕ, i.e.,

$$\Gamma(\mathrm{epi}(\phi)) = \mathrm{epi}(\mathcal{H}(\phi)). \tag{3}$$

Lemma 3. *Given a function $\phi : \mathbb{R}^N \to \mathbb{R}$, one has*

$$\mathcal{H}(\phi)(X) = \inf \left\{ \sum_{i \in I} w_i \phi(X_i) : X_i \in \mathbb{R}^N, \right.$$
$$\left. w_i \geq 0, \sum_{i \in I} w_i = 1, \sum_{i \in I} w_i X_i = X, I \textit{ finite} \right\}. \tag{4}$$

If moreover ϕ is positively homogeneous then:

$$\mathcal{H}(\phi)(X)$$
$$= \inf \left\{ \sum_{i \in I} \phi(X_i) : X_i \in \mathbb{R}^N, \sum_{i \in I} X_i = X, I \textit{ finite} \right\}.$$

Proof. It is easy to check that the map on the right hand side of Equation (4) which has type $\mathbb{R}^N \to \mathbb{R}$ with

$$X \mapsto \inf \left\{ \sum_{i \in I} w_i \phi(X_i) : X_i \in \mathbb{R}^N, \right.$$
$$\left. w_i \geq 0, \sum_{i \in I} w_i = 1, \sum_{i \in I} w_i X_i = X, I \text{ finite} \right\}$$

is the largest convex function below ϕ and its epigraph is $\mathcal{H}(\mathrm{epi}(\phi))$. Therefore Equation (4) follows from Equation (3). The second claim follows easily. ☐

Lemma 4. *For a positively homogeneous function $\phi : \mathbb{R}^N \to \mathbb{R}$, we have*

$$\mathcal{I}(\phi) = \mathcal{I}(\mathcal{H}(\phi)).$$

Proof. Since $\forall v \in \mathbb{R}^N, \mathcal{H}(\phi)(v) \leq \phi(v)$, the relation $\mathcal{I}(\phi) \supset \mathcal{I}(\mathcal{H}(\phi))$ is trivial. Consider now the reverse inclusion. If there exists $v \in \mathbb{R}^N$ with $\mathcal{H}(\phi)(v) = -\infty$, then by the comments after Definition 3, it follows that $\mathcal{I}(\phi) = \mathcal{I}(\mathcal{H}(\phi)) = \emptyset$. Assume now that $\mathcal{H}(\phi)(v) > -\infty$ for all $v \in \mathbb{R}^N$. Then from Lemma 3, for any $v \in \mathbb{R}^N$ and $\epsilon > 0$ there are a finite number of vectors $v_i \in \mathbb{R}^N$ for $i \in I$, with $\sum_{i \in I} v_i = v$ such that:

$$\mathcal{H}(\phi)(v) + \epsilon > \sum_{i \in I} \phi(v_i).$$

It follows that if $z \in \mathcal{I}(\phi)$ then

$$\mathcal{H}(\phi)(v) + \epsilon > \sum_{i \in I} \langle z, v_i \rangle = \langle z, v \rangle.$$

Since this is true for arbitrary small ϵ we have $\mathcal{H}(\phi))(v) \geq \langle z, v\rangle$ and therefore, since this is true for any $v \in \mathbb{R}^N$, we conclude that $z \in \mathcal{I}(\mathcal{H}(\phi))$. $\square$

4.1. Inner convex hull from pre-inner convex hull

In this section, we will provide an explicit expression for the support function of the centre $\mathcal{I}(\phi)$ of a positively homogeneous map $\phi : \mathbb{R}^n \to \mathbb{R}$.

Theorem 1. *Suppose $\phi : \mathbb{R}^N \to \mathbb{R}$ is a positively homogeneous function with a non-empty center. Then, we have:*

$$S_{\mathcal{I}(\phi)} = \mathcal{H}(\phi).$$

Proof. Since ϕ is positively homogeneous its epigraph $\mathrm{epi}(\phi)$ is a cone with apex 0. The convex hull of $\mathrm{epi}(\phi)$ in $\mathbb{R}^N \times \mathbb{R}$ is the intersection of all half spaces containing it. Denote points in $\mathbb{R}^N \times \mathbb{R}$ by (x, λ) with $x \in \mathbb{R}^N, \lambda \in \mathbb{R}$. Since ϕ is defined (i.e., it is real-valued) on $\mathbb{R}^N$, and since, by definition $\mathrm{epi}(\phi)$ is not bounded above (i.e. it is unbounded in the positive direction of the last coordinate), all half spaces in $\mathbb{R}^{N+1}$ containing $\mathrm{epi}(\phi)$ have an outer normal with a negative last coordinate. In addition, since $\mathrm{epi}(\phi)$ is a cone with apex $(0, 0)$, if a half space H contains $\mathrm{epi}(\phi)$ it contains $(0, 0)$. We claim moreover that if H is a half space that contains $\mathrm{epi}(\phi)$, then the half space H', with the same outer normal as H, whose boundary hyper-plane goes through $(0, 0)$ still contains $\mathrm{epi}(\phi)$. Indeed, assume that the line $0 \times \mathbb{R}$ in $\mathbb{R}^N \times \mathbb{R}$ cuts the boundary of H at some point $(0, \mu)$ with $\mu < 0$; note that if $\mu = 0$ then $H = H'$ and $\mu > 0$ would contradict $(0, 0) \in H$. Thus, $H = \{(x, \lambda) : \langle h, x\rangle + \lambda \geq \mu\}$ and $H' = \{(x, \lambda) : \langle h, x\rangle + \lambda \geq 0\}$ for some vector $h \in \mathbb{R}^N$. Assume, for a contradiction, that there exists $(x_0, \lambda_0) \in \mathrm{epi}(\phi)$ such that $(x_0, \lambda_0) \in H \setminus H'$. Then $\mu \leq \langle h, x_0\rangle + \lambda_0 < 0$. Now consider the point:

$$(x_1, \lambda_1) = \left(\frac{-2\mu}{-(\langle h, x_0\rangle + \lambda_0)} x_0, \frac{-2\mu}{-(\langle h, x_0\rangle + \lambda_0)} \lambda_0 \right).$$

By positive homogeneity, $(x_1, \lambda_1) \in \mathrm{epi}(\phi)$ with $\langle h, x_1\rangle + \lambda_1 = 2\mu < \mu$ and $(x_1, \lambda_1) \notin H$ which contradicts $\mathrm{epi}(\phi) \subset H$. This proves the claim. Denote by H_h the half space:

$$H_h = \{(x, \lambda) \in \mathbb{R}^{N+1} : \langle x, h\rangle + \lambda \geq 0\}.$$

From the claim, it follows that:

$$\Gamma(\mathrm{epi}(\phi)) = \bigcap_{\mathrm{epi}(\phi) \subset H_h} H_h.$$

We have:

$$\begin{array}{ll} \mathrm{epi}(\phi) \subset H_h & \iff \\ \forall x \in \mathbb{R}^N, \lambda \geq \phi(x) \Rightarrow \langle x, h\rangle + \lambda \geq 0 & \iff \\ \forall x \in \mathbb{R}^N, \langle x, h\rangle + \phi(x) \geq 0 & \iff \\ \forall x \in \mathbb{R}^N, \langle x, -h\rangle \leq \phi(x) & \iff \\ -h \in \mathcal{I}(\phi) & \end{array}$$

which gives:

$$\Gamma(\mathrm{epi}(\phi)) = \bigcap_{h \in \mathcal{I}(\phi)} H_{-h}. \tag{5}$$

Now consider $\mathrm{epi}(S_{\mathcal{I}(\phi)})$ the epigraph of the support function of the center $\mathcal{I}(\phi)$ of ϕ. We obtain:

$$\begin{array}{rcl} (x, \lambda) \in \mathrm{epi}(S_{\mathcal{I}(\phi)}) & \iff & \lambda \geq S_{\mathcal{I}(\phi)}(x) \\ & \iff & \forall h \in \mathcal{I}(\phi), \lambda \geq \langle x, h\rangle \\ & \iff & \forall h \in \mathcal{I}(\phi), (x, \lambda) \in H_{-h}. \end{array}$$

Therefore:

$$\mathrm{epi}\left(S_{\mathcal{I}(\phi)}\right) = \bigcap_{h \in \mathcal{I}(\phi)} H_{-h}.$$

Comparing this with (5), we deduce:

$$\Gamma\left(\mathrm{epi}(\phi)\right) = \mathrm{epi}\left(S_{\mathcal{I}(\phi)}\right).$$

By Equation (3), we obtain $\mathrm{epi}(\mathcal{H}(\phi)) = \Gamma(\mathrm{epi}(\phi)) = \mathrm{epi}(S_{\mathcal{I}(\phi)})$ and the result follows. $\square$

Using Equation (2), we now obtain a direct application of Theorem 1:

Corollary 1. *If $\mathrm{CH}^-(P)$ is non empty, the support function $S_{\mathrm{CH}^-(P)}$ of $\mathrm{CH}^-(P)$ is given by:*

$$S_{\mathrm{CH}^-(P)} = S_{\overline{\mathrm{CH}^-(P)}} = \mathcal{H}(S_P^*).$$

Next, consider a directed and bounded family $(\phi_j)_{j \in J}$ of maps $\phi_j : \mathbb{R}^N \to \mathbb{R}$, i.e., for each $i, j \in J$ there exists $k \in J$ such that $\phi_i, \phi_j \leq \phi_k$ and $\sup_{j \in J} \phi_j(x) < \infty$ for all $x \in \mathbb{R}^N$. From Definition (2), we easily obtain

$$\mathcal{I}(\sup_{j \in J} \phi_j) = \overline{\bigcup_{j \in J} \mathcal{I}(\phi_j)}. \tag{6}$$

Proposition 4. *If $(\phi_j)_{j \in J}$ is a directed and bounded family of real-valued maps on $\mathbb{R}^N$, then*

$$\sup_{j \in J} S_{\mathcal{I}(\phi_j)} = S_{\mathcal{I}(\sup_{j \in J} \phi_j)}, \quad \sup_{j \in J} \mathcal{H}(\phi_j) = \mathcal{H}(\sup_{j \in J} \phi_j).$$

Proof. By monotonicity, $\sup_{j \in J} S_{\mathcal{I}(\phi_j)} \leq S_{\mathcal{I}(\sup_{j \in J} \phi_j)}$ and $\sup_{j \in J} \mathcal{H}(\phi_j) \leq \mathcal{H}(\sup_{j \in J} \phi_j)$. Using Equation (6), the first equation follows from Proposition 1(ii); the second equation then follows from Theorem 1. $\square$

Corollary 2. *Suppose P_j for each $j \in J$ is a family of non-empty convex and compact subsets. If $(S_{P_j}^*)_{j \in J}$ is a directed family in the function space $(\mathbb{R}^N \to (\mathbb{R}, \leq))$, then*

$$\sup_{j \in J} S_{\mathrm{CH}^-(P_j)} = S_{\bigcup_{j \in J} \mathrm{CH}^-(P_j)}.$$

Proof. Put $\phi_j := S_{P_j}^*$ in Proposition 4. Then, we obtain:

$$\begin{array}{rcll} \sup_{j \in J} S_{\mathrm{CH}^-(P_j)} & = & \sup_{j \in J} S_{\mathcal{I}(S_{P_j}^*)} & \text{Lemma 2} \\ & = & S_{\mathcal{I}(\sup_{j \in J} S_{P_j}^*)} & \text{Proposition 4} \\ & = & S_{(\bigcup_{j \in J} \mathcal{I}(S_{P_j}^*)} & \text{Equation (6)} \\ & = & S_{\bigcup_{j \in J} \mathrm{CH}^-(P_j)} & \text{Lemma 2.} \end{array}$$

$\square$

Using Proposition 4, we can show that the domain-theoretic convex hull map is Scott continuous:

Proposition 5. *The convex hull map* $\mathrm{CH} : (\mathbf{C}\mathbb{R}^N)^m \to \mathbf{SC}\mathbb{R}^N$ *for any* $m \geq 1$ *is Scott continuous.*

Proof. The monotonicity of CH follows immediately from the Definition 1. Since $\mathbf{C}\mathbb{R}^N$ is a countably based bounded complete domain it is sufficient to show that the supremum of an increasing sequence $(P_n)_{n\geq 0}$ with $P_n \in (\mathbf{C}\mathbb{R}^N)^m$ for all $n \geq 0$ is preserved. Put $P = \sup_{n\geq 0} P_n \in (\mathbf{C}\mathbb{R}^N)^m$.

First consider $\mathrm{CH}^+(P)$. Then $\bigcup P = \bigcap_{n\geq 0} \bigcup P_n$. (Recall that for a set A of subsets of $\mathbb{R}^N$, $\bigcup A$ denotes the union of subsets in A.) Therefore, $d_H(\bigcup P, \bigcup P_n) \to 0$ as $n \to \infty$. Thus, by the Lipschitz property of Γ in Lemma 1, $d_H(\Gamma(\bigcup P), \Gamma(\bigcup P_n)) \to 0$ as $n \to \infty$. Hence, $\Gamma(\bigcup P) = \Gamma(\bigcap_{n\geq 0} \bigcup P_n) = \bigcap_{n\geq 0} \Gamma(\bigcup P_n)$ and $\mathrm{CH}^+(P) = (\Gamma(\bigcup P))^c = (\Gamma(\bigcap_{n\geq 0} \bigcup P_n))^c = (\bigcap_{n\geq 0} \Gamma(\bigcup P_n))^c = \bigcup_{n\geq 0} (\Gamma(\bigcup P_n))^c = \bigcup_{n\geq 0} \mathrm{CH}^+(P_n)$.

Next consider $\mathrm{CH}^-(P)$. It follows from the definition of the pre-inner support function that $(S^*_{P_n})_{n\geq 0}$ is an increasing sequence of bounded functions in $(\mathbb{R}^n \to \mathbb{R})$ with $\sup_{n\geq 0} S^*_{P_n} = S^*_P$. In Proposition 4, let $\phi_n := S^*_{P_n}$. We have that $\overline{\mathrm{CH}^-(P_n)} = \mathcal{I}(S^*_{P_n})$ and $\overline{\mathrm{CH}^-(P)} = \mathcal{I}(S^*_P)$ by Equation (2). Also $\mathcal{I}(\sup_{n\geq 0} S^*_{P_n}) = \bigcup_{n\geq 0} \mathcal{I}(S^*_{P_n})$. Thus, $\sup_{n\geq 0} S_{\mathrm{CH}^-(P_n)} = \sup_{n\geq 0} S_{\overline{\mathrm{CH}^-(P_n)}} = S_{\overline{\mathrm{CH}^-(P)}} = S_{\mathrm{CH}^-(P)}$ and hence $\mathrm{CH}^-(P) = \bigcup_{n\geq 0} \mathrm{CH}^-(P_n)$. $\square$

Corollary 3. *The convex hull map* $\mathrm{CH} : (\mathbf{C}\mathbb{R}^N)^* \to \mathbf{SC}\mathbb{R}^N$ *is Scott continuous.*

5. Finitely generable subsets

In this section, we show that the domain-theoretic convex hull algorithm can be extended to finitely generable subsets as in non-deterministic semantics.

Recall that given a countably based domain D with a countable basis $B \subset D$, the *Plotkin power domain* $\mathbf{P}D$, sometimes called the *convex power domain*, of D can be constructed using finitely generable subsets of D as follows [43]. The construction is similar to the construction of the Plotkin power domain for algebraic domains [44], [45]. Consider any finitely branching tree T whose branches are all infinite and whose nodes are elements of D and each node is below its children in the information ordering of D. Then any infinite branch of T gives rise to an increasing sequence of elements of D with a lub in D. Then the set A of lubs of all infinite branches of T is called a *finitely generable set* of D. One can construct another finitely branching tree T' that gives rise to the same set A such that every node of T' is way-below its children. Denote the set of all finitely generable subsets of D by $\mathscr{F}(D)$. Any finite subset of D is trivially a finitely generable subset of D and we have $\mathscr{P}_{\mathrm{f}}(B) \subset \mathscr{P}_{\mathrm{f}}(D) \subset \mathscr{F}(D)$, where $\mathscr{P}_{\mathrm{f}}(S)$ denotes the set of finite subsets of the set S. For a finite subset $A \in \mathscr{P}_{\mathrm{f}}(D)$ and any finitely generable set $C \in \mathscr{F}(D)$ we define the relation $A \ll_{\mathrm{EM}} C$ if

$$\forall x \in A \exists y \in C. x \ll y. \;\&\; \forall y \in C \exists x \in A. x \ll y.$$

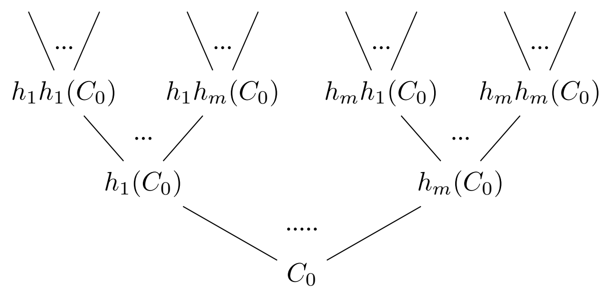


Figure 2: The IFS Tree

We can then define the *Egli-Milner* pre-order $\sqsubseteq_{\mathrm{EM}}$ on finitely generable subsets $\mathscr{F}(D)$ by stipulating that $C_1 \sqsubseteq_{\mathrm{EM}} C_2$ if for all finite subsets $A \in \mathscr{P}_{\mathrm{f}}(D)$ with $A \ll_{\mathrm{EM}} C_1$ we have $A \ll_{\mathrm{EM}} C_2$. The Plotkin power domain $\mathbf{P}D$ of D is then defined to be the quotient $(\mathscr{F}(D)_{/\equiv}, \sqsubseteq_{\mathrm{EM}}/_{\equiv})$ where $C_1 \equiv_{\mathrm{EM}} C_2$ iff $C_1 \sqsubseteq_{\mathrm{EM}} C_2$ and $C_2 \sqsubseteq_{\mathrm{EM}} C_1$. The countable basis $B \subset D$ provides a countable basis $\mathscr{P}_f(B)$ for $\mathbf{P}D$. Thus, $\mathbf{P}D$ is a countably based domain. If D is bounded complete and $C \in \mathscr{F}(D)$ is a finitely generable set consisting of maximal elements of D, then the equivalence class of C will have only one element namely C which is itself a maximal element of $\mathbf{P}D$. If E is a dcpo then any monotone $g : \mathscr{P}_{\mathrm{f}}(D) \to E$ induces a Scott continuous map on the Plotkin power domain $\hat{g} : \mathbf{P}D \to E$ which is given by: $\hat{g}(X) = \sup\{g(Y) : Y \in \mathscr{P}_{\mathrm{f}}(D), Y \ll_{\mathrm{EM}} X\}$.

Consider the Plotkin power domain $\mathbf{PC}\mathbb{R}^n$ of $\mathbf{C}\mathbb{R}^N$. We now introduce the notion of a finitely generable set.

Definition 4. *A subset* $A \subset \mathbb{R}^N$ *is* finitely generable *if it is an element of an equivalence class of* $\mathbf{PC}\mathbb{R}^N$*, i.e., if it is a finitely generable subset of the domain* $\mathbf{C}\mathbb{R}^N$.

Let T be a tree in the construction of an element of $\mathbf{PC}\mathbb{R}^N$, i.e., a finitely generable subset of $\mathbf{C}\mathbb{R}^N$. Let T_n denote the set of nodes of T on level $n \geq 0$ and T_ω the set of lubs of the infinite branches of T. Then, $\bigcup T_n$ is the finite union of non-empty compact sets and is compact, and thus $\bigcup T_\omega = \bigcap_{n\geq 0} \bigcup T_n$ is also compact. In addition, the following property shows the connection between finitely generable sets and compact sets.

Proposition 6. *If* $C \subset \mathbb{R}^N$ *is any non-empty compact set, then the set containing all singleton sets contained in* C *(i.e.,* $\{\{x\} : x \in C\}$*) is a finitely generable set, a maximal element of* $\mathbf{PC}\mathbb{R}^n$.

Proof. Using compactness of C, construct the level n of a finitely branching tree T by induction as follows. Let $\mathcal{C}_0$ be a finite open covering of C with open balls of radius less than $1 = 1/2^0$, which is minimal, i.e., there is no proper subset of $\mathcal{C}_0$ which covers C. Put $T_0 = \{\overline{O} : O \in \mathcal{C}_0\}$. Inductively, suppose we have a finite open covering $\mathcal{C}_n$ of C, by open balls of radius less than $1/2^n$, for $n \geq 1$, with $T_n = \{\overline{O} : O \in \mathcal{C}_n\}$ and $T_{n-1} \sqsubseteq_{\mathrm{EM}} T_n$. For each $x \in C$, take an open ball centred at x with radius less than $1/2^{n+1}$ that is contained in an element of $\mathcal{C}_n$. By compactness there is a minimal finite covering $\mathcal{C}_{n+1}$ which by construction refines $\mathcal{C}_n$, i.e., for each $O \in \mathcal{C}_{n+1}$, there exists $O' \in \mathcal{C}_n$ such that $O \subset O'$. Put $T_{n+1} = \{\overline{O} : O \in \mathcal{C}_{n+1}\}$. Then $T_n \sqsubseteq_{\text{EM}} T_{n+1}$ for $n \geq 0$ and $T_\omega = \{\{x\} : x \in C\}$ which is clearly a maximal element of $\mathbf{PC}\mathbb{R}^N$. □

5.1. Iterated Function Systems

A large class of finitely generable subsets is given by fractal sets induced as the attractors of Iterated Function Systems (IFS) consisting of affine maps [9], [46]. We use a general domain-theoretic formulation here as in [43]. Let $C_0 \subset \mathbb{R}^N$ be, say, a closed ball centred at the origin and let $\mathbf{C}(C_0)$ be the sub-domain of $\mathbf{C}\mathbb{R}^N$ consisting of non-empty, convex and compact subsets of C_0. Consider a finite family of Scott continuous functions $h_i : C_0 \to C_0$, for $i \in I$, that map C_0 into itself. Then the map $V : \mathbf{C}(C_0) \to \mathbf{C}(C_0)$ with $V(X) = \bigcup\{h_i(X) : i \in I\}$ is Scott continuous and thus has a least fixed point $\mathsf{lfp}(V) = \sup_{n\geq 0} V^n(C_0) \in \mathbf{C}(C_0)$. Define $H : (\mathscr{P}_{\text{f}}(\mathbf{C}(C_0)), \ll_{\text{EM}}) \to \mathbf{PC}(C_0)$ by

$$H : \{A_j : j \in J\} \mapsto \{h_i(A_j) : i \in I, j \in J\}.$$

It is easy to check that H is monotone and thus extends to a map $\hat{H} : \mathbf{PC}(C_0) \to \mathbf{PC}(C_0)$ with

$$\hat{H}(X) = \{V(Y) : Y \ll_{\text{EM}} X\}.$$

Using the Scott continuity of h_i for $i \in I$, it follows that $\hat{H}(X) = V(X)$ for $X \in \mathscr{P}_{\text{f}}(\mathbf{C}(C_0))$ and we thus write H for $\hat{H}$ for convenience. We obtain a least fixed point for $H : \mathbf{PC}(C_0) \to \mathbf{PC}(C_0)$ given by $\mathsf{lfp}(H) = \sup_{n\geq 0} H^n(C_0)$.

The increasing sequence $(H^n(C_0))_{n\geq 0}$ provides a finitary branching tree, called the *IFS tree*, whose level n is given by $H^n(C_0)$ for $n \geq 0$. This is depicted in Figure 2 for $I = \{1, \ldots, m\}$. The finitely generable set of the IFS tree consists of the intersection $\bigcap_{n\geq 1} h_{i_1} h_{i_2} \ldots h_{i_n}(C_0)$ of the nodes in any infinite branch of the tree with $i_j \in I$ for $j \geq 1$. It is easy to check that $V^n(C_0) = \bigcup H^n(C_0)$ and thus $\mathsf{lfp}(V) = \bigcup \mathsf{lfp}(H)$.

5.2. The convex hull of finitely generable sets

Using the Plotkin power domain $\mathbf{PC}\mathbb{R}^N$ of $\mathbf{C}\mathbb{R}^N$, we can allow, as in non-determinism of programs, for a partial point $P \in \mathbf{C}\mathbb{R}^N$ at a given stage of computation to be refined to several partial points $P_1, \ldots, P_m$ say with $P \sqsubseteq P_i$ for $1 \leq i \leq m$. We will now show that the domain-theoretic convex hull map can be extended to $\mathbf{PC}\mathbb{R}^N$. Let B be any basis of $\mathbf{C}\mathbb{R}^N$ including the case $B = \mathbf{C}\mathbb{R}^N$. The map $\text{CH} : (\mathbf{C}\mathbb{R}^N)^* \to \mathbf{SC}\mathbb{R}^N$ clearly does not depend on the order of the partial points P_i for $1 \leq i \leq m$ in an input $(P_1, \ldots, P_m) \in (\mathbf{C}\mathbb{R}^N)^*$. In other words, this map is also well-defined on $\mathscr{P}_f(B)$ and in this section we consider it as a map $\text{CH} : \mathscr{P}_f(B) \to \mathbf{SC}\mathbb{R}^N$.

Proposition 7. *If $P, Q \in \mathscr{P}_f(B)$, then $P \sqsubseteq_{EM} Q$ implies $\text{CH}^-(P) \subset \text{CH}^-(Q)$ and $\text{CH}^+(P) \subset \text{CH}^+(Q)$.*

Proof. Let $P = \{P_i : i \in I\}$ and $Q = \{Q_j : j \in J\}$. Since for each $j \in J$ there exists some $i \in I$ such that $P_i \supset Q_j$, it follows that $\bigcup P \supset \bigcup Q$ and thus $\Gamma(\bigcup P) \supset \Gamma(\bigcup Q)$. Hence, $\text{CH}^+(P) = (\Gamma(\bigcup P))^c \subset (\Gamma(\bigcup Q))^c = \text{CH}^+(Q)$. Next, let $q_j \in Q_j$ for each $j \in J$ and define $p_i \in P_i$ for each $i \in I$ as follows. For each $i \in I$, choose $j \in J$ with $P_i \supset Q_j$ and put $p_i := q_j$. Then $\{p_i : i \in I\} \subset \{q_j : j \in J\}$ and thus $\Gamma(\{p_i : i \in I\}) \subset \Gamma(\{q_j : j \in J\})$. Hence,

$$\begin{aligned} \text{CH}^-(P) &= \left(\bigcap_{p_i \in P_i} \Gamma(\{p_i : i \in I\})\right)^\circ \\ &\subset \left(\bigcap_{q_j \in Q_j} \Gamma(\{q_j : j \in J\})\right)^\circ = CH^-(Q) \end{aligned}$$

□

Therefore, $\text{CH} : (\mathscr{P}_f(B), \sqsubseteq_{\text{EM}}) \to \mathbf{SC}\mathbb{R}^N$ is monotone and thus it induces a Scott continuous map on the Plotkin power domain $\widehat{\text{CH}} : \mathbf{PC}\mathbb{R}^N \to \mathbf{SC}\mathbb{R}^N$ by defining: $\widehat{\text{CH}}(X) = \sup\{\text{CH}(Y) : Y \in \mathscr{P}_f(\mathbf{C}\mathbb{R}^N), Y \ll_{\text{EM}} X\}$.

Proposition 8. *Let $P \in \mathbf{PC}\mathbb{R}^N$ be a finitely generable set constructed using a finitary branching tree T with $P = \sup_{n\geq 0} T_n$. Then $S^*_{T_n}$ is an increasing sequence for $n \geq 0$ with $S^*_P = \sup_{n\geq 0} S^*_{T_n}$.*

Proof. It follows from the definition of S^*_X that $S^*_X \leq S^*_Y$ if $X \sqsubseteq_{\text{EM}} Y$. Thus, the sequence of functions $(S^*_{T_n})_{n\geq 0}$ is increasing for $n \geq 0$ with $S^*_P \geq \sup_{n\geq 0} S^*_{T_n}$. Let $P = \{P_i : i \in I\}$ and $v \in \mathbb{R}^N$ be non-zero. Put $a := S^*_P(v) = \sup_{i\in I} \min_{x\in P_i} \langle v, x\rangle$. Given $\epsilon > 0$, there exists $i \in I$ such that $\min_{x\in P_i} \langle v, x\rangle > a - \epsilon/2$. Since $P_i \in P$, by definition, there exist $A_n \in T_n$ for $n \geq 0$ such that $P_i = \bigcap_{n\geq 0} A_n$. Hence, there exists $n \geq 0$ such that $\min_{x\in A_n} \langle v, x\rangle > \min_{x\in P_i} \langle v, x\rangle - \epsilon/2 > a - \epsilon$. It follows that $S^*_{T_n}(v) = \sup_{A\in T_n} \min_{x\in A} \langle v, x\rangle > a - \epsilon$. Since $\epsilon > 0$ is arbitrary, $\sup_{n\geq 0} S^*_{T_n}(v) \geq a$ and the result follows. □

We can now show our main result in this section.

Theorem 2. *The map $\widehat{\text{CH}} : \mathbf{PC}\mathbb{R}^N \to \mathbf{SC}\mathbb{R}^N$ computes the convex hull, i.e., $\widehat{\text{CH}} = \text{CH}$.*

Proof. The proof is similar to that of Proposition 5 even though the input domains $(\mathbf{C}\mathbb{R}^N)^m$ and $\mathbf{PC}\mathbb{R}^N$ are different. Let $X \in \mathbf{PC}\mathbb{R}^N$ be a finitely generable set constructed using a finitary branching tree T with $T_n \ll_{\text{EM}} X$ and $X = \sup_{n\geq 0} T_n$. Note that X can be an element of $\mathscr{P}_f(\mathbf{C}\mathbb{R}^N)$. By definition,

$$\begin{aligned} \widehat{\text{CH}}(X) &= (\widehat{\text{CH}}^-(X), \widehat{\text{CH}}^+(X)) \\ &= \sup_{n\geq 0} \text{CH}(T_n) \\ &= \left(\bigcup_{n\geq 0} \text{CH}^-(T_n), \bigcup_{n\geq 0} \text{CH}^+(T_n)\right). \end{aligned}$$

First consider $\sup_{n\geq 0} \text{CH}^+(T_n)$. Since $(\bigcup T_n)_{n\geq 0}$ is a decreasing sequence of compact sets with $\bigcup X = \bigcap_{n\geq 0} \bigcup T_n$ and $\lim_{n\geq\infty} d_h(\bigcup X, \bigcup T_n) = 0$, it follows from Lemma 1 that $\lim_{n\geq 0} d_h(\Gamma(\bigcup X), \Gamma(\bigcup T_n)) = 0$. i.e., $\Gamma(\bigcup X) = \bigcap_{n\geq 0} \Gamma(\bigcup T_n)$. Thus, $\text{CH}^+(X) = (\Gamma(\bigcup X))^c = (\bigcap_{n\geq 0} \Gamma(\bigcup T_n))^c = \bigcup_{n\geq 0} \text{CH}^+(T_n)$. We conclude that

$\widehat{\mathrm{CH}}^+(X) = \mathrm{CH}^+(X)$. Next consider $\sup_{n\geq 0} \mathrm{CH}^-(T_n)$. By Proposition 8,

$$S_X^* = \sup_{n\geq 0} S_{T_n}^*. \tag{7}$$

We therefore have:

$$\begin{array}{rcll} S_{\mathrm{CH}^-(X)} & = & \mathcal{H}(S_X^*) & \text{Corollary 1} \\ & = & \mathcal{H}(\sup_{n\geq 0} S_{T_n}^*) & \text{Equation (7)} \\ & = & \sup_{n\geq 0} \mathcal{H}(S_{T_n}^*) & \text{Proposition 4} \\ & = & \sup_{n\geq 0} S_{\mathrm{CH}^-(T_n)} & \text{Corollary 1.} \end{array}$$

Thus, from Proposition 1(ii), $\mathrm{CH}^-(X) = \bigcup_{n\geq 0} \mathrm{CH}^-(T_n)$, i.e., $\widehat{\mathrm{CH}}^-(X) = \mathrm{CH}^-(X)$. We conclude that $\widehat{\mathrm{CH}} = \mathrm{CH}$. □

Note that, as for any element of $\mathscr{P}_f(\mathbf{C}\mathbb{R}^N)$, the outer convex hull of any finitely generable set $X \in \mathbf{PC}\mathbb{R}^N$ is given by $\mathrm{CH}^+(X) = (\Gamma(X))^c$. But in general $\Gamma(X)$ may be too complicated to compute directly and Theorem 2 tells us that it can be computed by taking the countable intersection of the shrinking sets $\Gamma(\bigcup T_n)$ for $n \geq 0$, i.e., the intersection of the convex hulls of its finitary branching tree levels. In addition, as for any element of $\mathscr{P}_f(\mathbf{C}\mathbb{R}^N)$, if a finitely generable set $X \in \mathbf{PC}\mathbb{R}^N$ consists of singletons then it follows directly from the definition that $\mathrm{CH}^-(X) = (\Gamma(\bigcup X))^\circ$. Otherwise, by Theorem 2, $\mathrm{CH}^-(X)$ can be computed as the increasing union of the inner convex hulls $\mathrm{CH}^-(T_n)$ of its finitary branching tree levels.

5.3. Two subclasses of IFS

In this subsection, based on our notations in Subsection 5.1 and 5.2, we study two subclasses of IFS and see how our previous results can be used to find the inner convex hull of the finitely generable sets induced by IFS.

5.3.1. IFS with contracting affine maps. Suppose $h_i : \mathbb{R}^N \to \mathbb{R}^N$ is affine (i.e., linear up to addition of a constant vector) and contracting for $1 \leq i \leq m$, i.e., there exists K with $1 > K \geq 0$ such that $\|h_i(x) - h_i(y)\| \leq K\|x - y\|$ for all $x, y \in \mathbb{R}^N$ and $1 \leq i \leq m$. This is called a *hyperbolic* IFS [9], [46]. In this case, we can always find a large enough closed ball C_0 centred at the origin with radius r that is mapped into itself by all h_i for $1 \leq i \leq m$. In fact, it can be checked that we can take $r = \max\{\|h_i(0)\| : 1 \leq i \leq m\}/(1 - K)$ [47].

Because the image of a compact and convex set under an affine map is both compact and convex, it follows that h_i extends to a map $\hat{h_i} : \mathbf{C}(C_0) \to \mathbf{C}(C_0)$ with $\hat{h_i}(X) = h_i[X]$, the forward image of $X \in \mathbf{C}(C_0)$. For convenience, by an abuse of notation, we write $\hat{h_i}$ simply as h_i $1 \leq i \leq m$. Because of contractivity, the lub of any infinite branch of the IFS is a singleton. Thus, in this class of examples, the finitely generable set $C = \mathrm{lfp}(H)$, using the notation in Section 5.1, consists of singletons only and hence $\mathrm{CH}(C) = ((\Gamma(\bigcup C))^\circ, (\Gamma(\bigcup C)^c)$. Most of the examples of IFS treated in [9] are of this form.

If $N = 1$ and $m = 2$, with $h_1(x) = x/3$ and $h_2(x) = (x+2)/3$, we can take $C_0 = [0, 1]$ and the finitely generable set given is the points of the classical Cantor set. If $N = 2$, $m = 3$ with $h_1, h_2, h_3 : \mathbb{R}^2 \to \mathbb{R}^2$ given by $h_1 : x \mapsto x/2$, $h_2 : x \mapsto x/2 + (1/2, 0)$ and $h_3 : x \mapsto x/2 + (1/4, \sqrt{3}/4)$, then the points of the finitely generable set are the points of the Sierpinski triangle. Figure 3 shows the generation of the inner convex hull for the levels of the IFS tree.

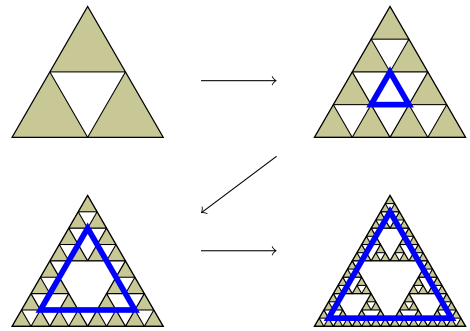


Figure 3: The partial points in the ternary tree of the Sierpinski triangle. Each grey triangle is a partial point; the inner convex hull is depicted by blue boundaries.

5.3.2. IFS with condensation. Another major class of IFS is called IFS with condensation. We provide a more general framework for this class here than in [9]. Suppose $h_i : \mathbb{R}^N \to \mathbb{R}^N$ for $1 \leq i \leq m$, are as in the previous section i.e., affine and contracting that map C_0 into itself. Assume further that we have a finite number of constant maps $f_j : \mathbf{C}(C_0) \to \mathbf{C}(C_0)$ with, say, constant values C_j for $1 \leq j \leq k$. Then

$$\{h_i, f_j : 1 \leq i \leq m, 1 \leq j \leq k\}$$

provides an IFS on $\mathbf{C}(C_0)$, which induces a Scott continuous map $H : \mathbf{PC}(C_0) \to \mathbf{PC}(C_0)$ that is given on an input element $\{A_t : t \in I\} \in \mathscr{P}_f(\mathbf{C}\mathbb{R}^N)$ by

$$\{h_i(A_t), f_j(A_t) : t \in I, 1 \leq i \leq m, 1 \leq j \leq k\}.$$

In general, the finitely generable set contains subsets that are not singletons, unless all the constant values of f_j for $1 \leq j \leq k$ are themselves singleton. In fact, it can be easily seen that $\mathrm{lfp}(H)$ is given by

$$\{h_{i_1} \circ \ldots \circ h_{i_n}(C_j) : 1 \leq i_1, \ldots, i_n \leq m, n \geq 1, 1 \leq j \leq k\}.$$

Putting $P_n = H^n(C_0)$ for $n \geq 0$, we obtain a bounded increasing family $(S_{P_n}^*)_{n\geq 0}$ of pre-inner support functions, and by Corollary 2, it follows that $\mathrm{CH}^-(\mathrm{lfp}(H)) = \bigcup_{n\geq 0} \mathrm{CH}^-(H^n(C_0))$. In the simplest case when $k = 1$, we have an *IFS with condensation*, as defined in [9]. Then, the constant value C_1, say, of f_1 and its recursive image under maps h_i for $1 \leq i \leq m$ are retained in the output of H. In fact, it is easy to check that the inner convex hull of $\mathrm{lfp}(H)$ will be given by $\mathrm{CH}^-(\mathrm{lfp}(H)) = \bigcup_{n\geq 0} \mathrm{CH}^-(H^n(C_1))$.

Consider a simple example with $N = 3$ and $m = 1$ in which the value C_1 of the constant map f_1 is the sphere

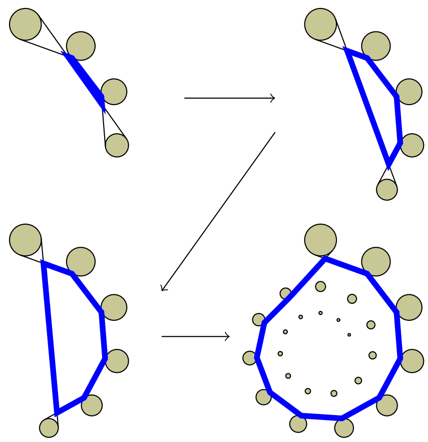


Figure 4: The sequence of partial points as spiral circles created by IFS with a condensation map. The inner convex hull for each iteration is depicted with the blue boundaries.

of radius 1 with centre at $(0, 4, z_0)$ with $z_0 \geq 0$, while $h_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the linear map represented by the matrix:

$$\alpha \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $\alpha = 0.95$ and $\theta = -2\pi/11$. The finitely generable set $\mathrm{lfp}(H)$ consists of an infinite sequence of spiralling and shrinking spheres around the origin, whose centres, for $z_0 > 0$, decrease in height z and converge to the origin. A 2d vertical view of this on the (x, y) plane can be seen in Figure 4, In fact, the figure presents the actual finitely generable set when $z_0 = 0$ and the spheres are represented as circles. In this case, $\mathrm{CH}^-(H^n(C_1)) = \mathrm{CH}^-(H^{10}(C_1))$ for $n \geq 10$ as seen in the figure.

When $z_0 > 0$, however, $H^n(C_1)$ is a strictly increasing sequence of subsets in $\mathbb{R}^3$ for $n \geq 0$ and $\mathrm{CH}^-(\mathrm{lfp}(H))$ will no longer be a polytope. However, by Scott continuity, it can be incrementally approximated by $\mathrm{CH}^-(\mathrm{lfp}(H)) = \bigcup_{n\geq 0} \mathrm{CH}^-(H^n(C_1))$.

6. Predicate transformer of the convex hull

Given an input $P \in \mathbf{PC}\mathbb{R}^N$ and a Scott open neighbourhood $O \in \Omega(\mathbf{SC}\mathbb{R}^N)$ of the output $\mathrm{CH}(P)$, we ask the question: What Scott open neighbourhood of P will contain elements in $\mathbf{PC}\mathbb{R}^N$ that are mapped into O? Equivalently, given a Scott open subset O of $\mathbf{SC}\mathbb{R}^N$, what Scott open set in $\mathbf{PC}\mathbb{R}^N$ is mapped into O? In the semantics of programming languages this is referred to as the question of predicate transformer for CH.

We will use a key approach to the study of logic of programmes, namely the notion of a predicate transformer for determining the weakest precondition for a given post-condition of a programme as introduced by Dijkstra [12]. In our setting, we employ the topological view of predicate transformers using topology and domain theory developed by Smyth [14]: open sets represent properties of points of topological spaces, points of the space represent logical theories and frame homomorphisms between the lattices of open sets of the topological spaces correspond to predicate transformers. The logic induced by topological spaces in this way has been aptly called geometric logic [15], [48].

Given a topological space X the propositions of the geometric logic are open sets of this space. Thus, every $a \in \Omega(X)$ defines a proposition P_a and every $x \in X$ is a model of this logic, P_a is interpreted as true iff $x \in a$, i.e.

$$x \models P_a \iff x \in a.$$

Moreover, propositions of geometric logic satisfy [49]:

- $a \subseteq b \Longrightarrow P_a \vdash P_b$.
- If S is a family of open sets: $P_{\bigcup S} \vdash \bigvee_{a\in S} P_a$.
- If S is a finite family of open sets: $\bigwedge_{a\in S} P_a \vdash P_{\bigcap S}$.

Next we will consider geometric logic in the context of domains as topological spaces. Let $f : D \to E$ be a Scott continuous map of countably based domains D and E with countable bases B_D and B_E respectively. The frame homomorphism $\Omega f : \Omega E \to \Omega D$ between the lattices ΩE and ΩD of open sets of E and D is given by $(\Omega f)(O) = f^{-1}(O)$, which preserves finite intersections and arbitrary unions. Since any point of D, respectively E, can be obtained as the supremum of elements of B_D, respectively B_E, way below it, we can use the information systems $(B_D, \ll_D)$ and $(B_E, \ll_E)$, i.e., the set of basis elements with ordering induced by the way-below relation.

In terms of predicates, this means that we can restrict ourselves to the countable collection of open sets $\twoheaduparrow x$ for $x \in B_D$ and $\twoheaduparrow y$ for $y \in B_E$. To capture the frame homomorphism Ωf in this setting, we make use of the fact that ΩD, a continuous lattice, is also a domain with a countable basis given by finite unions of open sets of the form $\{\twoheaduparrow x : x \in B_D\}$, which means $(\Omega f)(\twoheaduparrow y) = \{\twoheaduparrow x : x \in B_D, \twoheaduparrow x \ll_{\Omega D} f^{-1}(\twoheaduparrow y)\}$. Thus, we represent the predicate transformer $\Omega f : \Omega E \to \Omega D$ by the relation, called an *approximable mapping*, $R_f \subset B_D \times B_E$ given by $R_f = \{(x, y) : \twoheaduparrow x \ll_{\Omega D} f^{-1}(\twoheaduparrow y)\}$. We now need to derive several basic properties in domain theory.

Lemma 5. *Suppose D is a domain and $x, y \in D$ then:*

$$x \ll y \Longrightarrow \twoheaduparrow y \ll_{\Omega D} \twoheaduparrow x.$$

Proof. By the interpolation property, there exists $c \in D$ with $x \ll c \ll y$. Recall that, as a continuous lattice, the Scott topology of D has a basis consisting of finite unions of open sets of the form $\twoheaduparrow a$ for $a \in D$. Suppose $\bigcup_{i\in I} \bigcup_{j\in J_i} \twoheaduparrow a_{ij}$, where J_i is a finite set for each $i \in I$, is a directed family of basic open sets with $\twoheaduparrow x \subset \bigcup_{i\in I} \bigcup_{j\in J_i} \twoheaduparrow a_{ij}$. Since $c \in \twoheaduparrow x$, it follows that there exists $i \in I$ and $j \in J_i$ such that $c \in \twoheaduparrow a_{ij}$ and hence $y \in \twoheaduparrow a_{ij}$ as the latter set is upward closed. Using the upward closure of $\twoheaduparrow a_{ij}$ again, we obtain $\twoheaduparrow y \subset \twoheaduparrow a_{ij}$, i.e., $\twoheaduparrow y \subset \bigcup_{j\in J_i} \twoheaduparrow a_{ij}$ as required. $\square$

Given a domain D and an element $x \in D$ such that $\twoheaduparrow x \neq \emptyset$, we say x is *semi-compact* if for any open subset

$O \in \Omega D$ the relation $\Uparrow x \ll_{\Omega D} O$ implies $x \in O$. Clearly, any compact element of a domain is semi-compact.

Proposition 9. *(i) Suppose $f : D \longrightarrow E$ is a Scott continuous function of domains D and E with $x \in D$ and $y \in E$. Then*

$$f(x) \gg y \Longrightarrow \Uparrow x \ll_{\Omega D} f^{-1}(\Uparrow y).$$

(ii) If in addition, $x \in D$ is semi-compact then

$$\Uparrow x \ll_{\Omega D} f^{-1}(\Uparrow y) \Longrightarrow f(x) \gg y.$$

Proof. (i) Suppose $f(x) \gg y$. By the interpolation property, there exists $c \in E$ such that $f(x) \gg c \gg y$. Since f preserves the sup of directed sets and $x = \sup\{z : z \ll x\}$ where the latter set is directed, there exists $z \in D$ such that $z \ll x$ and $y \ll c \sqsubseteq f(z)$. Thus, $f(z) \in \Uparrow y$. Therefore $z \in f^{-1}(\Uparrow y)$, i.e., $\Uparrow z \subset f^{-1}(\Uparrow y)$. By Lemma 5, we have $\Uparrow x \ll_{\Omega D} \Uparrow z \subset f^{-1}(\Uparrow y)$ as required.

(ii) Since f is Scott continuous, $f^{-1}(\Uparrow y)$ is open and thus, since x is semi-compact, the relation $\Uparrow x \ll_{\Omega D} f^{-1}(\Uparrow y)$ implies $x \in f^{-1}(\Uparrow y)$, i.e., $f(x) \gg y$. □

Recall that a compact set $C \in \mathbf{U}\mathbb{R}^N$ is regular if $\overline{C^\circ} = C$ and that regular elements of $\mathbf{U}\mathbb{R}^N$ together with the bottom element form a basis for $\mathbf{U}\mathbb{R}^N$. For a compact set $C \in \mathbf{U}\mathbb{R}^N$ and $\epsilon > 0$, the *ϵ-inner closed set* of C is defined as $C_{\epsilon^-} := \{x \in C : D_\epsilon(x) \subset C\}$, where $D_\epsilon(x)$ is the closed ball of radius ϵ centred at x.

Proposition 10. *A non-bottom element $C \in \mathbf{U}\mathbb{R}^N$ is semi-compact if C is regular.*

Proof. Suppose $C \in \mathbf{U}\mathbb{R}^N$ is non-bottom and regular with $\Uparrow C \ll_{\Omega\mathbf{U}\mathbb{R}^N} O$ for some $O \in \Omega\mathbf{U}\mathbb{R}^N$. By the interpolation property there exist regular $C_i \in \mathbf{U}\mathbb{R}^N$ with $i \in I$, where I is finite, such that $\Uparrow C \ll_{\Omega\mathbf{U}\mathbb{R}^N} \bigcup_{i\in I} \Uparrow C_i \ll_{\Omega\mathbf{U}\mathbb{R}^N} O$. Then $\bigcup_{\epsilon>0} \bigcup_{i\in I} \Uparrow((C_i)_{\epsilon^-}) = \bigcup_{i\in I} \Uparrow C_i$ and thus there exists $\epsilon > 0$ such that $\Uparrow C \subset \bigcup_{i\in I} \Uparrow((C_i)_{\epsilon^-})$. Because I is finite, there exists $i \in I$ such that $\Uparrow C \subset \Uparrow((C_i)_{\epsilon^-})$, and hence, $C^\circ \subset (C_i)_{\epsilon^-}$, which, C being regular, implies $C = \overline{C^\circ} \subset (C_i)_{\epsilon^-}$. Thus $C_i \ll C$ and therefore $C \in \Uparrow C_i \subset O$. □

The converse of Proposition 10 also holds, though it is not required in this paper. Now consider the Scott continuous convex hull map CH in the domain-theoretic setting. Any non-bottom element $C \in \mathbf{C}\mathbb{R}^N$ with non-empty interior is regular and inherits the semi-compact property from $\mathbf{U}\mathbb{R}^N$ in Proposition 10. Similarly, any basis element $\{C_i : 1 \le i \le n\} \in \mathscr{P}\mathbf{C}\mathbb{R}^N$, where $C_i \in \mathbf{C}\mathbb{R}^N$ are non-bottom elements with non-empty interior for $1 \le i \le n$, can be shown to inherit the semi-compact property. We consider the basis $B_{\mathbf{C}\mathbb{R}^N}$ of $\mathbf{C}\mathbb{R}^N$ consisting of the set of convex compact polytopes with rational vertices and non-empty interior to get the two information systems $B_{(\mathbf{C}\mathbb{R}^N)^*} := ((B_{\mathbf{C}\mathbb{R}^N})^*, \ll)$ and $B_{\mathbf{P}\mathbf{C}\mathbb{R}^N} := (\mathscr{P}_{\mathrm{f}}(B_{\mathbf{C}\mathbb{R}^N}), \ll_{\mathrm{EM}})$, all whose elements are semi-compact. Let $B^\circ_{\mathbf{C}\mathbb{R}^N}$ denote the collection of convex open polytopes with rational vertices. Then we get an information system $(B_{\mathbf{S}\mathbf{C}\mathbb{R}^N}, \ll)$ for $\mathbf{S}\mathbf{C}\mathbb{R}^N$ where $B_{\mathbf{S}\mathbf{C}\mathbb{R}^N} = (B^\circ_{\mathbf{C}\mathbb{R}^N} \times B^\circ_{\mathbf{C}\mathbb{R}^N}) \cap \mathbf{S}\mathbf{C}\mathbb{R}^N$.

By Proposition 9, the predicate transformers of the maps $\mathrm{CH} : (\mathbf{C}\mathbb{R}^N)^* \to \mathbf{S}\mathbf{C}\mathbb{R}^N$ and $\mathrm{CH} : \mathbf{P}\mathbf{C}\mathbb{R}^N \to \mathbf{S}\mathbf{C}\mathbb{R}^N$ are equivalent respectively to the two relations

$$\begin{aligned} R^1_{\mathrm{CH}} &= \{(C,O) \in B_{(\mathbf{C}\mathbb{R}^N)^*} \times B_{\mathbf{S}\mathbf{C}\mathbb{R}^N} : \mathrm{CH}(C) \gg O\} \\ R^2_{\mathrm{CH}} &= \{(C,O) \in B_{\mathbf{P}\mathbf{C}\mathbb{R}^N} \times B_{\mathbf{S}\mathbf{C}\mathbb{R}^N} : \mathrm{CH}(C) \gg O\}. \end{aligned}$$

Note that R^1_{CH} and R^2_{CH} are syntactically the same relations if we identify a finite tuple of elements of $\mathbf{C}\mathbb{R}^N$ with the finite set containing the components of the tuple, which, as we have mentioned previously, have the same image under CH. Since the way-below relation in $\mathbf{S}\mathbf{C}\mathbb{R}^N$ is given by $(O_1, O_2) \ll (O'_1, O'_2)$ iff $O_i \ll O'_i$ for $i = 1, 2$, it follows that for both convex hull maps the predicate transformer is equivalent to the decidable test whether a convex polytope A with rational vertices is contained in the interior of another such polytope B, i.e., that $A \subset B$ and A does not have any common points with the boundary of B. We conclude that the convex hull predicate transformer is computable.

Given two convex polytopes C_1 and C_2 with respectively m and n vertices in $\mathbb{R}^N$, there is an obvious $O(n(\mathrm{LP}(N, m)))$ algorithm that checks for containment of each vertex of C_2 in C_1, and therefore decides the way-below relation. Here, $\mathrm{LP}(N, m)$ is the time complexity of solving a linear programming problem with m variables and N constraints. For $N = 2$, there is a simple linear algorithm, i.e., with $O(m+n)$ complexity, to check if a convex polygon is contained in the interior of another convex polygon [50].

7. Concluding remarks

We have presented a general data type for representing imprecise or partial points in $\mathbb{R}^N$ by non-empty convex and compact subsets, unifying all the different notions of partial points. We have formulated in this context an algorithm to find the inner convex hull of a finite set of partial points represented by non-empty compact and convex polytopes in $\mathbb{R}^N$. In order to derive the Scott continuity of the domain-theoretic convex hull map, we developed the notion of the pre-inner support function, whose convex hull gives the support function of the inner convex hull. We then extended the domain-theoretic convex hull map to the Plotkin power domain of the domain of non-empty convex and compact subsets of $\mathbb{R}^N$ in order to allow for computing the convex hull of finitely generable sets i.e., for non-determinism in the computation of partial points. We showed that this can be used to find the inner and outer convex hull of the fixed points or attractors of iterated function systems in different settings. Finally, we characterised the predicate transformer for the convex hull map in the general setting of the Plotkin power domain and showed that it is a computable predicate.

Future work in this area includes finding a logical characterisation for the inner convex hull of a finite number of partial points and for developing a more refined version of the predicate transformer for the convex hull by showing that the solid convex domain is a stably locally compact space and thus can be represented logically in a finitary way using semi-strong proximity lattices [51], [52].

References

[1] O. Devillers, A. Fronville, B. Mourrain, and M. Teillaud, "Algebraic methods and arithmetic filtering for exact predicates on circle arcs," *Computational Geometry*, vol. 22, no. 1-3, pp. 119–142, 2002.

[2] H. Edelsbrunner and E. P. Mücke, "Simulation of simplicity: a technique to cope with degenerate cases in geometric algorithms," *ACM Transactions on Graphics (tog)*, vol. 9, no. 1, pp. 66–104, 1990.

[3] A. Edalat and A. Lieutier, "Foundation of a computable solid modeling," in *Proceedings of the fifth ACM symposium on Solid modeling and applications*. ACM, 1999, pp. 278–284.

[4] D. S. Scott, "Outline of a mathematical theory of computation," in *4th Annual Princeton Conference on Information Sciences and Systems*, 1970, pp. 169–176.

[5] R. E. Moore, *Interval Analysis*. Englewood Cliffs: Prentice-Hall, 1966.

[6] A. Edalat, A. Lieutier, and E. Kashefi, "The convex hull in a new model of computation," in *Proc. 13th Canad. Conf. Comput. Geom.*, 2001, pp. 93–96.

[7] A. Edalat, A. Khanban, and A. Lieutier, "Computational geometry with imprecise input data," *Lecture Notes in Computer Science*, vol. 3526, pp. 117–127, 2005.

[8] M. Löffler and M. van Kreveld, "Largest and smallest convex hulls for imprecise points," *Algorithmica*, vol. 56, no. 2, p. 235, 2010.

[9] M. F. Barnsley, *Fractals Everywhere*, 2nd ed. Academic Press, 1993.

[10] R. W. Floyd, "Assigning meanings to programs," in *Mathematical Aspects of Computer Science*. American Mathematical Society, 1967, pp. 19–32, proceedings of Symposium on Applied Mathematics. 19.

[11] C. A. R. Hoare, "An axiomatic basis for computer programming," *Communications of the ACM*, vol. 12, pp. 576–580, 1969.

[12] E. W. Dijkstra, "Guarded commands, nondeterminacy and formal derivation of programs," *Commun. ACM*, vol. 18, no. 8, pp. 453–457, Aug. 1975. [Online]. Available: http://doi.acm.org/10.1145/360933.360975

[13] D. S. Scott, "Domains for denotational semantics," in *International Colloquium on Automata, Languages, and Programming*. Springer, 1982, pp. 577–610.

[14] M. B. Smyth, "Power domains and predicate transformers: a topological view," in *Automata, Languages and Programming*, J. Diaz, Ed. Berlin: Springer-Verlag, 1983, pp. 662–675, lecture Notes in Computer Science Vol. 154.

[15] S. Abramsky, "Domain theory in logical form," *Annals of Pure and Applied Logic*, 1991.

[16] M. B. Smyth, "Effectively given domains," *Theoretical Computer Science*, vol. 5, pp. 257–274, 1977.

[17] S. Abramsky and A. Jung, "Domain theory," in *Handbook of Logic in Computer Science*, S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, Eds. Clarendon, 1994, vol. 3.

[18] S. J. Vickers, "Information systems for continuous posets," *Theoretical Computer Science*, vol. 114, pp. 201–229, 1993.

[19] T. M. Chan, "Optimal output-sensitive convex hull algorithms in two and three dimensions," *Discrete & Computational Geometry*, vol. 16, no. 4, pp. 361–368, 1996.

[20] B. Chazelle, "An optimal algorithm for intersecting three-dimensional convex polyhedra," *SIAM Journal on Computing*, vol. 21, no. 4, pp. 671–696, 1992.

[21] ——, "An optimal convex hull algorithm in any fixed dimension," *Discrete & Computational Geometry*, vol. 10, no. 4, pp. 377–409, 1993.

[22] C. B. Barber, D. P. Dobkin, and H. Huhdanpaa, "The quickhull algorithm for convex hulls," *ACM Transactions on Mathematical Software (TOMS)*, vol. 22, no. 4, pp. 469–483, 1996.

[23] B. Grünbaum, V. Klee, M. A. Perles, and G. C. Shephard, *Convex polytopes*. Springer, 1967.

[24] V. L. Klee, "Some topological properties of convex sets," *Transactions of the American Mathematical Society*, vol. 78, no. 1, pp. 30–45, 1955.

[25] L. Kettner, K. Mehlhorn, S. Pion, S. Schirra, and C. Yap, "Classroom examples of robustness problems in geometric computations," *Computational Geometry*, vol. 40, no. 1, pp. 61–78, 2008.

[26] A. A. Khanban, "Basic algorithms in computational geometry with imprecise input," Ph.D. dissertation, University of London, 2005.

[27] J. S. Ely and A. P. Leclerc, "Correct Delaunay triangulation in the presence of inexact inputs and arithmetic," *Reliable Computing*, vol. 6, no. 1, pp. 23–38, 2000.

[28] M. Gavrilova, H. Ratschek, and J. G. Rokne, "Exact computation of Delaunay and power triangulations," *Reliable Computing*, vol. 6, no. 1, pp. 39–60, 2000.

[29] D. Salesin, J. Stolfi, and L. Guibas, "Epsilon geometry: building robust algorithms from imprecise computations," in *Proceedings of the fifth annual symposium on Computational geometry*. ACM, 1989, pp. 208–217.

[30] J. W. Jaromczyk and G. W. Wasilkowski, "Computing convex hull in a floating point arithmetic," *Computational Geometry*, vol. 4, no. 5, pp. 283–292, 1994.

[31] M. Segal and C. H. Séquin, "Consistent calculations for solids modeling," in *Proceedings of the first annual symposium on Computational geometry*. ACM, 1985, pp. 29–38.

[32] C.-K. Yap, "Towards exact geometric computation," *Computational Geometry*, vol. 7, no. 1-2, pp. 3–23, 1997.

[33] G. Mei, J. C. Tipper, and N. Xu, "Numerical robustness in geometric computation: An expository summary," *Applied Mathematics & Information Sciences*, vol. 8, no. 6, p. 2717, 2014.

[34] X. Lian and L. Chen, "Probabilistic ranked queries in uncertain databases," in *Proceedings of the 11th international conference on Extending database technology: Advances in database technology*. ACM, 2008, pp. 511–522.

[35] P. K. Agarwal, S. Har-Peled, S. Suri, H. Yıldız, and W. Zhang, "Convex hulls under uncertainty," *Algorithmica*, vol. 79, no. 2, pp. 340–367, 2017.

[36] A. Edalat and A. Lieutier, "Foundation of a computable solid modelling," *Theoretical Computer Science*, vol. 284, no. 2, pp. 319–345, 2002.

[37] A. A. Khanban, A. Edalat, A. Lieutier *et al.*, "Computability of partial Delaunay triangulation and voronoi diagram," *Electronic Notes in Theoretical Computer Science*, vol. 66, no. 1, pp. 1–13, 2002.

[38] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *Continuous Lattices and Domains*. Cambridge University Press, 2003.

[39] A. Edalat, "Dynamical systems, measures and fractals via domain theory," *Information and Computation*, vol. 120, no. 1, pp. 32–48, 1995.

[40] R. T. Rockafellar, "Convex analysis. princeton landmarks in mathematics," 1997.

[41] J. Benoist and J.-B. Hiriart-Urruty, "What is the subdifferential of the closed convex hull of a function?" *SIAM Journal on Mathematical Analysis*, vol. 27, no. 6, pp. 1661–1679, 1996.

[42] M. H. van der Vlerk, "Stochastic programming with integer recourse," Ph.D. dissertation, PhD thesis, University of Groningen, The Netherlands, 1995.

[43] A. Edalat, "Power domains and iterated function systems," *Information and Computation*, vol. 124, pp. 182–197, 1996.

[44] M. B. Smyth, "Power domains," *Journal of Computer and System Sciences*, vol. 16, no. 1, pp. 23–36, Feb. 1978.

[45] V. Stoltenberg-Hansen, I. Lindström, and E. R. Griffor, *Mathematical Theory of Domains*, ser. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1994, vol. 22.

[46] J. E. Hutchinson, "Fractals and self-similarity," *Indiana University Mathematics Journal*, vol. 30, pp. 713–747, 1981.

[47] A. Edalat, D. W. N. Sharp, and R. L. While, "Bounding the attractor of an IFS," *Information Processing Letters*, vol. 64, no. 4, pp. 197–202, 1997.

[48] S. J. Vickers, *Topology Via Logic*, ser. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1988.

[49] ——, "Geometric logic in computer science," in *Theory and Formal Methods 1993*. Springer, 1993, pp. 37–54.

[50] J. O'Rourke, C.-B. Chien, T. Olson, and D. Naddor, "A new linear algorithm for intersecting convex polygons," *Computer Graphics and Image Processing*, vol. 19, no. 4, pp. 384–391, 1982.

[51] A. Jung and P. Sünderhauf, "On the duality of compact vs. open," *Annals of the New York Academy of Sciences*, vol. 806, no. 1, pp. 214–230, 1996. [Online]. Available: http://dx.doi.org/10.1111/j.1749-6632.1996.tb49171.x

[52] A. Edalat and M. Maleki, "Differentiation in logical form," in *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE, 2017, pp. 1–12.