Domain Theory and Multi-Variable Calculus

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Abstract

We introduce a domain-theoretic model for multi-variable differential calculus, extending the existing framework for functions of one variable. The domain-theoretic derivative is shown to give the best hyper-rectangular approximation to the generalized (Clarke's) gradient of a locally Lipschitz function. We construct a continuous Scott domain for locally Lipschitz functions of several variables, which also serves as a domain for piecewise differentiable functions of several variables. This domain is constructed by pairing consistent information about the locally Lipschitz function and its differential properties. We then introduce a domain-theoretic notion of path integral, which we use to establish a main result of this paper: a necessary and sufficient condition for an interval-valued Scott continuous vector function to be integrable, extending the classical Green's Theorem for a vector field to be a gradient. The main result of the paper is to show that consistency is decidable on the rational basis elements of the domain for locally Lipschitz functions, giving an effective framework for multi-variable differential calculus.

1. Introduction

A domain-theoretic framework for differential calculus of a function of one variable was introduced in [4], with a number of essential ingredients including the dual notions of domain-theoretic derivative and integral, a fundamental theorem of calculus for interval-valued functions, a domain for locally Lipschitz and C^1 functions and a domain-theoretic version of Picard's theorem. In this paper, we develop a domain-theoretic framework for multi-variable differential calculus, which extends the above notions and results to functions of several variables. The basic idea is to collect together the local differential properties of the function by developing a generalization of the concept of a Lipschitz constant to an interval vector Lipschitz constant. The collection of these local differentiable properties are then used to define the domain-theoretic derivative and integral giving rise to a fundamental theorem of calculus for interval-valued functions. It is also used to construct a domain for locally Lipschitz functions by pairing together consistent information about the values of the function and its differential properties.

The domain-theoretic derivative is closely related to the so-called generalized (or Clarke's) gradient, which is a key tool in nonsmooth analysis, control theory and optimization theory [2, 3]. For any locally Lipschitz function, the domain-theoretic derivative at a point gives the smallest hyper-rectangle, with sides parallel to the coordinate planes, which contains the Clarke's gradient.

As in classical differential multi-variable calculus, an interval-valued function may fail to be integrable. Based on a new concept of a "direction dependent" scalar product of vectors, we introduce the notion of an interval-valued path integral for interval-valued vector fields, and obtain a necessary and sufficient condition for a step function to be integrable: zero must be contained in the path integral of the interval-valued vector field with respect to any closed path. This generalizes Green's theorem in classical differential calculus [7, pages 286-291,] to interval-valued vector fields.

The main result of the paper is to prove that consistency is decidable on the rational basis elements of the domain for locally Lipschitz functions, i.e., pairs of finitary function information and derivative information. First it is shown that the integrability condition is decidable on the derivative information. Having decided that the derivative information, regarded as an interval-valued vector field, does have a collection of integral primitives, the function and derivative information are jointly used to construct, respectively, a minimal and a maximal piecewise linear surface consistent with both information. These surfaces are obtained by, respectively, maximalizing and minimalizing the lower and the upper path integrals of the derivative information over piecewise linear paths. The maximalization and minimization are achieved, in each case, for a path among several homotopically non-equivalent locally minimal paths, which can all be effectively constructed. The decidability of consistency is then reduced to checking whether the minimal surface is below the maximal surface, a task that can be done in finite time. Thus, we obtain an effectively given domain for locally Lipschitz functions, which in particular provides an effective way to approximate Clarke's gradient.

We envisage two main applications of this work: (i) a domain-theoretic approach to solve partial differential equations and, (ii) a domain-theoretic effective version of the implicit function theorem, which combined with the domain-theoretic framework for computational geometry developed in [5], can be used in geometric modelling. The interval-valued path integral will also open the road to develop domain-theoretic frameworks for differential geometry and differential topology.

There is related work in computable analysis. Pour-El and Richards [9] relate the computability of a function with the computability of its derivative. The scheme employed by Weihrauch [12] has some similarity to ours: a representation of a space of differentiable functions also includes the representation of the derivatives. This leads to partially defined representations, but there is no general result on the decidability of the domain of the induced representations. Interval analysis [8] also provides a framework for verified numerical computation. There, differentiation is performed by symbolic techniques [10] in contrast to our sequence of approximations of the functions.

We use the standard notions of domain theory as in [1]. Let $D^0[0,1]^n = [0,1]^n \to \mathbf{I}\mathbb{R}$ be the domain of all Scott continuous functions of type $[0,1]^n \to \mathbf{I}\mathbb{R}$; we often write D^0 for $D^0[0,1]^n$. A function $f \in D^0$ is given by a pair of respectively lower and upper semi-continuous functions $f^-, f^+ : [0,1]^n \to \mathbb{R}$ with $f(x) = [f^-(x), f^+(x)]$. Given a domain A, we denote by A^n_s the smash product, i.e., $a \in A^n_s$ if $a = (a_1, \cdots, a_n) \in A^n$ with $a_i \neq \perp$ for all $i = 1, \cdots, n$ or $a = \perp$. Let $(\mathbf{I}\mathbb{R})^{m\times n}_s$ denote the set of all $m \times n$ matrices with entries in $\mathbf{I}\mathbb{R}$, where for such a matrix either all components are non-bottom or the matrix itself is bottom. We use standard operations of interval arithmetic on interval matrices. For $a = [\underline{a}, \overline{a}] \in \mathbf{I}\mathbb{R}$ we write $w(a) = \overline{a} - \underline{a}$. We identify the real number $r \in \mathbb{R}$ with the singleton $\{r\} \in \mathbf{I}\mathbb{R}$. Similarly for interval vectors and functions. We will use the *sign* function given by the multiplicative group homomorphism $\sigma : \mathbb{R} \to \{-, 0, +\}$ with

$$\sigma(r) = \begin{cases} + & r > 0 \\ - & r < 0 \\ 0 & r = 0. \end{cases}$$

We write $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ for the standard Euclidean norm of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Recall that the derivative of a map $f : [0,1]^n \to \mathbb{R}$ at $y \in [0,1]$, when it exists, is defined as the linear map $T : \mathbb{R}^n \to \mathbb{R}$ with

$$\lim_{\|x-y\|\to 0} \frac{|f(x) - f(y) - T(x-y)|}{\|x-y\|} = 0.$$

The linear map T is denoted by $f'(y)$. Let ∇f denote the gradient of f, when it exists, i.e.,

$$(\nabla f)_i(x) = \frac{\partial f}{\partial x_i} =$$

$$\lim_{x_i' \to x_i} \frac{f(x_1, \ldots, x_i, \ldots, x_n) - f(x_1, \ldots, x_i', \ldots, x_n)}{x_i - x_i'},$$

for $1 \leq i \leq n$. Recall that if the derivative exists at a point then the gradient also exists at that point and has the same value. We will reserve the notation $\frac{df}{dx}$ exclusively in this paper for the domain-theoretic derivative which will be introduced later. The interior of a set $A \subset \mathbb{R}^n$ is denoted by A° and its closure by $\mathrm{cl}(A)$.

A number of proofs which generalize the corresponding proofs for dimension one in [4] are given in the Appendix. The long proof of Proposition 6.5 is also given there. Alternatively the full paper can be obtained from [6].

2. Ties of functions of several variables

Definition 2.1 The continuous function $f : [0,1]^n \to \mathbf{I}\mathbb{R}$ has *an interval derivative* $b \in (\mathbf{I}\mathbb{R})^{1\times n}_s$ in $a \in (\mathbf{I}[0,1])^n$ if for all $x, y \in a^\circ$ we have: $b(x-y) \sqsubseteq f(x) - f(y)$. The *single-step tie* $\delta(a,b) \subseteq D^0[0,1]$ of a with b is the collection of all functions in $D^0[0,1]$ which have an interval derivative b in a.

For a single-step tie $\delta(a,b)$, one can think of b as a Lipschitz interval vector constant for the family of functions in $\delta(a,b)$. A classical Lipschitz would require $k = \overline{b_i} = -\underline{b_i} \geq 0$ for all $i = 1 \cdots n$. By generalizing the concept of a Lipschitz constant in this way, we are able to obtain essential information about the differential properties of the function, which includes what the classical Lipschitz constants provide:

Proposition 2.2 *If $f \in \delta(a,b)$ for $a^\circ \neq \emptyset$ and $b \neq \perp$, then $f(x)$ is maximal for each $x \in a^\circ$ and the induced function $f : a^\circ \to \mathbb{R}$ is Lipschitz: for all $u, v \in a^\circ$ we have $|f(u) - f(v)| \leq k\|u - v\|$, where $k = \max_{1\leq i\leq n}(|\underline{b}_i|, |\overline{b}_i|)$.*

The following proposition justifies our definition of interval derivative.

Proposition 2.3 *For $f \in C^1[0,1]^n$, the following conditions are equivalent.*

(i) $f \in \delta(a,b)$.

(ii) $\forall z \in a^\circ .\ f'(z) \in b$.

(iii) $a \searrow b \sqsubseteq f'$.

When the components of a and b are rational intervals $\delta(a,b)$ is a family of functions in D^0 with a finitary differential property. Note that

(i) $f \sqsubseteq g \ \Rightarrow\ (f \in \delta(a,b) \ \Rightarrow\ g \in \delta(a,b))$, i.e. $\delta(a,b)$ is an upper subset of $[0,1]^n \to \mathbf{I}\mathbb{R}$.

(ii) If $f = g + \{c\}$ for some $c \in \mathbb{R}$, then $f \in \delta(a,b) \iff g \in \delta(a,b)$.

We will see at the end of this section that $\delta(a,b)$ is defined as the primitives of $a \searrow b$. However, in order to define the primitives of a Scott continuous function in general, we first need to study and generalize the notion of single-step ties. First we note that single-step ties share some common properties with single-step functions.

Proposition 2.4 *Suppose $a^\circ \neq \emptyset$ and $b \neq \bot$. We have $\delta(a,b) \supseteq \delta(c,d)$ iff $c \sqsubseteq a$ and $b \sqsubseteq d$.*

Corollary 2.5 *Suppose $a^\circ, c^\circ \neq \bot$ and $b, d \neq \bot$. We have $\delta(a,b) = \delta(c,d) \iff a = c\ \&\ b = d$. Furthermore $\delta(a,b) \supseteq \delta(c,d)$ iff $a \searrow b \sqsubseteq c \searrow d$.*

For the rest of this section, we assume we are in dimension $n \geq 2$.

Definition 2.6 A *step tie* of D^0 is any finite intersection $\bigcap_{1\leq i\leq n} \delta(a_i,b_i) \subset D^0$. A *tie* of D^0 is any intersection $\Delta = \bigcap_{i\in I} \delta(a_i,b_i) \subset D^0$. The *domain* of a non-empty tie Δ is defined as $\mathrm{dom}(\Delta) = \bigcup_{i\in I}\{a_i^\circ \mid b_i \neq \bot\}$.

A non-empty step tie with rational intervals gives us a family of functions with a *finite* set of consistent differential properties, and a non-empty general tie gives a family of functions with a consistent set of differential properties. Similar to Proposition 2.2, we have the following result. Recall that a function $f : O \to \mathbb{R}$ defined on the open set $O \subseteq \mathbb{R}^n$ is *locally Lipschitz* if it is Lipschitz in a neighborhood of any point in O.

Proposition 2.7 *If $\Delta \subset D^0$ is a tie and $f \in \Delta$, then $f(x)$ is maximal for $x \in dom(\Delta)$ and f is locally Lipschitz on $dom(\Delta)$.*

We now collect some simple properties of step ties, which we will use later.

Proposition 2.8 *For any indexing set I, the family of step functions $(a_i \searrow b_i)_{i\in I}$ is consistent if $\bigcap_{i\in I} \delta(a_i,b_i) \neq \emptyset$.*

Corollary 2.9 *The family $(a_i \searrow b_i)_{i\in I}$ is consistent if for any finite subfamily $J \subseteq I$ we have $\bigcap_{i\in J} \delta(a_i,b_i) \neq \emptyset$.*

Proposition 2.10 $\delta(a,b) \supseteq \bigcap_{i\in I} \delta(a_i,b_i)$ *if* $a \searrow b \sqsubseteq \bigsqcup_{i\in I} a_i \searrow b_i$.

Corollary 2.11 $\bigcap_{i\in I} \delta(a_i,b_i) \supseteq \bigcap_{i\in J} \delta(a_i,b_i)$ *if* $\bigsqcup_{i\in I} a_i \searrow b_i \sqsubseteq \bigsqcup_{i\in J} a_i \searrow b_i$.

Let $(T^1[0,1], \supseteq)$ be the dcpo of ties of D^0 ordered by reverse inclusion. We are finally in a position to define the primitives of a Scott continuous function; in fact now we can do more and define:

Definition 2.12 The *primitive* map $\int : ([0,1]^n \to (\mathbf{I}\mathbb{R})_s^{1\times n}) \to T^1$ is defined by $\int(\bigsqcup_{i\in I} a_i \searrow b_i) = \bigcap_{i\in I} \delta(a_i,b_i)$. We usually write $\int(f)$ as $\int f$ and call it the *primitive maps* of f.

Proposition 2.13 *The primitive map is well-defined and continuous.*

For $n \geq 2$, as we are assuming here, the primitive map will have the empty tie in its range, a situation which does not occur for $n = 1$.

Example 2.14 Let $g \in [0,1]^2 \to (\mathbf{I}\mathbb{R})_s^{1\times 2})$ be given by $g = (g_1, g_2) = (\lambda x_1.\lambda x_2.1, \lambda x_1.\lambda x_2.x_1)$. Then $\frac{\partial g_1}{\partial x_2} = 0 \neq 1 = \frac{\partial g_2}{\partial x_1}$, and it will follow that $\int g = \emptyset$.

Proposition 2.15 *The primitive map is onto.*

Proof By Proposition 2.8 and Example 2.14. $\square$

Therefore, we have the following important notion in dimensions $n \geq 2$.

Definition 2.16 A map $g \in [0,1]^n \to (\mathbf{I}\mathbb{R})^{1\times n}$ is said to be *integrable* if $\int g \neq \emptyset$.

3. Domain-theoretic derivative

Given a Scott continuous function $f : [0,1]^n \to \mathbf{I}\mathbb{R}$, the relation $f \in \delta(a,b)$, for some intervals a and b, provides, as we have seen, finitary information about the local differential properties of f. By collecting all such local information, we obtain the complete differential properties of f, namely its derivative.

Definition 3.1 The *derivative* of a continuous function $f : [0,1]^n \to \mathbf{I}\mathbb{R}$ is the map

$$\frac{df}{dx} = \bigsqcup_{f\in\delta(a,b)} a \searrow b : [0,1]^n \to (\mathbf{I}\mathbb{R})_s^{1\times n}.$$

Theorem 3.2 (i) *$\frac{df}{dx}$ is well-defined and Scott continuous.*

(ii) *If $f \in C^1[0,1]^n$ then $\frac{df}{dx} = f'$.*

(iii) $f \in \delta(a,b)$ *iff* $a \searrow b \sqsubseteq \frac{df}{dx}$.

We obtain the generalization of Theorem 3.2(iii) to ties, which provides a duality between the domain-theoretic derivative and integral and can be considered as a variant of the fundamental theorem of calculus.

Corollary 3.3 $f \in \int g$ *iff* $g \sqsubseteq \frac{df}{dx}$.

The following proposition relates the domain theoretic derivative to its classical counterpart.

Proposition 3.4 (i) *Let $f : [0,1]^n \to \mathbf{I}\mathbb{R}$ be Scott continuous. Suppose for some $z \in [0,1]^n$, $f(z)$ is not maximal, then $\frac{df}{dx}(z) = \bot$.*

(ii) *If $\frac{df}{dx}(y) = c \in (\mathbf{I}\mathbb{R})_s^{1\times n}$ is maximal, then f sends elements to maximal elements in a neighborhood U of y and the derivative of the induced restriction $f : U \to \mathbb{R}$ exists at y and $f'(y) = c$.*

We now develop another characterization of the domain-theoretic derivative. First, recall that for a function $f : [0,1]^n \to \mathbb{R}$, the Dini's lower and upper partial derivatives are defined respectively as

$$(\nabla f)^l_i(x) = \tag{1}$$

$$\underline{\lim}_{x'_i \to x_i} \frac{f(x_1,\ldots,x_i,\ldots,x_n) - f(x_1,\ldots,x'_i,\ldots,x_n)}{x_i - x'_i}$$

$$(\nabla f)^u_i(x) = \tag{2}$$

$$\overline{\lim}_{x'_i \to x_i} \frac{f(x_1,\ldots,x_i,\ldots,x_n) - f(x_1,\ldots,x'_i,\ldots,x_n)}{x_i - x'_i}.$$

Note that the Dini's lower and upper partial derivatives always exist as extended real numbers.

Proposition 3.5 *The components of the domain-theoretic derivative are give by:*

$$(\frac{df}{dx})_i = \left[\underline{\lim}_{y\to x} (\nabla f)^l_i(y), \overline{\lim}_{y\to x} (\nabla f)^u_i(y)\right],$$

when the two limits are finite for all $i = 1,\cdots,n$ and $\frac{df}{dx} = \bot$ otherwise.

Proof If $f \in \delta(a,b)$ for some $a \in (\mathbf{I}[0,1])^n$ with $x \in a^\circ$ and $b \in (\mathbf{I}\mathbb{R})^n_s$ with $b \neq \bot$, then $\underline{b_i} \leq (\nabla f)^l_i(y) \leq (\nabla f)^u_i(y) \leq \overline{b_i}$ for $y \in a^\circ$, and thus $\underline{b_i} \leq \underline{\lim}_{y\to x}(\nabla f)^l_i(y) \leq \overline{\lim}_{y\to x}(\nabla f)^u_i(y) \leq \overline{b_i}$. It follows that

$$\left[\underline{\lim}_{y\to x}(\nabla f)^l_i(y), \overline{\lim}_{y\to x}(\nabla f)^u_i(y)\right] \subseteq (\frac{df}{dx})_i.$$

On the other hand, if $\underline{\lim}_{y\to x}(\nabla f)^l_i(y)$ and $\overline{\lim}_{y\to x}(\nabla f)^u_i(y)$ are finite for all $i = 1,\cdots,n$, then for any $\epsilon > 0$ there exists $a \in (\mathbf{I}[0.1])^n$ containing x in its interior such that, for all $y \in a^\circ$,

$$\underline{\lim}_{y\to x}(\nabla f)^l_i(y) - \epsilon < (\nabla f)^l_i(y)$$

$$(\nabla f)^u_i(y) < \overline{\lim}_{y\to x}(\nabla f)^u_i(y) + \epsilon.$$

It follows that $f \in \delta(a,b)$ for some $b \in (\mathbf{I}\mathbb{R})^n$ with $\underline{\lim}_{y\to x}(\nabla f)^l_i(y) - \epsilon < \underline{b_i}$ and $\overline{b_i} < \overline{\lim}_{y\to x}(\nabla f)^u_i(y) + \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that

$$\left[\underline{\lim}_{y\to x}(\nabla f)^l_i(y), \overline{\lim}_{y\to x}(\nabla f)^u_i(y)\right] \supseteq (\frac{df}{dx})_i. \qquad \square$$

Corollary 3.6 *If $f : [0,1]^n \to \mathbb{R}$ is differentiable at $x \in [0,1]^n$, then $f'(x) \in \frac{df}{dx}$.*

4. Relation with generalized gradient

Recall that by Rademacher's theorem [3, p 148], a function $f : [0,1]^n \to \mathbb{R}$ which is Lipschitz in an open neighborhood of $[0,1]^n$ is differentiable almost everywhere with respect to the n-dimensional Lebesgue measure in that neighborhood. Let $\Omega_f \subseteq [0,1]^n$ denote the set of points, where f is not differentiable.

Definition 4.1 Generalized gradient (Clarke's gradient)[2, page 63] Let $f : [0,1]^n \to \mathbb{R}$ be Lipschitz near $x \in [0,1]^n$, then the *generalized gradient* (or *Clarke's gradient*) of f at x is defined as

$$\partial f = \mathrm{co}\{\lim \nabla f(x_m) : x_m \to x,\ x_m \notin \Omega_f\}, \tag{3}$$

where for a subset $A \subset \mathbb{R}$ the convex set generated by A is denoted by co$A$. The meaning of the above expression is as follows: consider any sequence $x_m \in [0,1]^n \setminus \Omega_f$, $m \in \mathbb{N}$, tending to x such that the sequence $\nabla f(x_m)$ converges, then the convex hull of all such limits is the generalized gradient of f at x.

Note that, in the above definition, since f is locally Lipschitz at x, it is differentiable almost everywhere in a neighborhood of x and thus there are plenty of sequences $(x_m)_{m\geq 0}$ such that $\lim_{m\to\infty} x_m = x$ and $\lim_{m\to\infty} \nabla f(x_m)$ exists.

The generalized gradient at a point where the function is locally Lipschitz is a compact convex non-empty subset of $\mathbb{R}^n$ [2, page 27]. We now establish the connection between the domain-theoretic derivative and the generalized gradient.

Theorem 4.2 *For any function $f : [0,1]^n \to \mathbb{R}$, the domain-theoretic derivative at a point where the function is locally Lipschitz is the smallest hyper-rectangle with sides parallel to the coordinate planes that contains the generalized gradient at that point.* $\square$

Proof Suppose f is locally Lipschitz at $x \in [0,1]^n$. From Equations (1), (2), Proposition 3.5 and Definition 3, it follows directly that

$$\partial f(x) \subseteq \frac{df}{dx}. \tag{4}$$

We show that $\frac{df}{dx}$ is the smallest hyper-rectangle with sides parallel to the coordinate planes, which contains $\partial f(x)$. Assume $\epsilon > 0$ is given, $1 \leq i \leq n$ and let $B \subset \mathbb{R}^n$ be the unit closed ball centered at the origin. For $1 \leq i \leq n$, let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the projection to the i coordinate axis and consider the pointwise extension of π_i to compact subsets of $\mathbb{R}^n$. Note that

$$\overline{\pi_i(\partial f(x))} = \overline{\lim}\{(\nabla f)_i(y) : y \to x,\, y \notin \Omega_f\}.$$

Thus, there exists $\delta > 0$ such that for all $y \in x + \delta B$, we have $(\nabla f)_i(y) \leq \overline{\pi_i(\partial f(x))} + \epsilon$. Assume also that δ is small enough so that Ω_f has zero n-dimensional Lebesgue measure in $x + \delta B$. Consider the line segment $L_y = \{y + te_i : 0 < t < \delta/2\}$, where e_i is the unit vector in the direction of the ith coordinate axis. Since Ω_f has zero n-dimensional Lebesgue measure in $x+\delta B$, it follows from Fubini's theorem that for almost all $y \in x + \frac{\delta}{2}B$, the line segment L_y meets Ω_f in a set of zero one-dimensional Lebesgue measure. If y is such a point and $0 < t < \delta/2$, we obtain:

$$f(y + te_i) - f(y) = \int_0^t (\nabla f)_i(y + se_i)\, ds,$$

since, by Rademacher's theorem [3, p 148], f' exists almost everywhere on L_y. On the other hand, $(\nabla f)_i(y + se_i) \leq \overline{\pi_i(\partial f(x))} + \epsilon$, since $\|y + se_i - x\| < \delta$ for $0 < s < t$. Thus,

$$f(y + te_i) - f(y) \leq t(\overline{\pi_i(\partial f(x))} + \epsilon). \tag{5}$$

Equation 5 holds for almost all y within $\delta/2$ of x and for all $t \in (0, \delta/2)$. Since f, being Lipschitz, is continuous it follows that Equation 5 holds for all y within $\delta/2$ of x and for all $t \in (0, \delta/2)$. Thus, $(\nabla f)^u_i(y) \leq \overline{\pi_i(\partial f(x))} + \epsilon$ for all y within $\delta/2$ of x and we conclude that

$$\overline{(\frac{df}{dx})_i} = \overline{\lim}_{y\to x}(\nabla f)^u_i(y) \leq \overline{\pi_i(\partial f(x))}. \tag{6}$$

Similarly,

$$\underline{(\frac{df}{dx})_i} = \underline{\lim}_{y\to x}(\nabla f)^l_i(y) \geq$$

$$\underline{\lim}\{(\nabla f)_i(y) : y \to x,\, y \notin \Omega_f\} = \underline{\pi_i(\partial f(x))}. \tag{7}$$

Comparing Equations (6) and (7) with Equation (4), it follows that $\partial f(x)$ touches all the $2n$ sides of the hyper-rectangle $\frac{df}{dx}$ and the proof is complete. □

Example 4.3 Let $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x_1, x_2) = a|x_1| + b|x_2|$ where $a \geq 0$ and $b \geq 0$. Then

$$\frac{df}{dx} = \begin{cases} (\sigma(x_1)a, \sigma(x_2)b) & x_1 \neq 0, x_2 \neq 0 \\ (\sigma(x_1)a, [-b, b]) & x_1 \neq 0, x_2 = 0 \\ ([-a, a], \sigma(x_2)b) & x_1 = 0, x_2 \neq 0 \\ ([-a, a], (-b, b]) & x_1 = 0, x_2 = 0, \end{cases}$$

where, we recall, $\sigma(k)$ is the sign of the real number k and, for any real number $a \in \mathbb{R}$, we write $\{a\}$ as a. For the Clarke gradient we get the same values in this case, i.e., $\partial f(x) = \frac{df}{dx}$ for all $x \in \mathbb{R}^2$.

Example 4.4 Let $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x_1, x_2) = \max\{\min\{x_1, -x_2\}, x_2 - x_1\}$. Put

$$\begin{aligned} A &= \{(x_1, x_2) : x_2 \leq 2x_1 \text{ and } x_2 \leq -x_1\} \\ B &= \{(x_1, x_2) : x_2 \leq x_1/2 \text{ and } x_2 \geq -x_1\} \\ C &= \{(x_1, x_2) : x_2 \leq 2x_1 \text{ or } x_2 \leq x_1/2\}. \end{aligned}$$

Then $A \cup B \cup C = \mathbb{R}^2$ and A, B, C have disjoint interiors with their boundaries given by the three line segments:

$$A \cap B = \{(a, -a) : a \geq 0\} \qquad A \cap C = \{(a, 2a) : a \leq 0\}$$

$$B \cap C = \{(a, a/2) : a \geq 0\}.$$

We have:

$$\frac{df}{dx} = \begin{cases} (1, 0) & x \in A^\circ \\ (0, -1) & x \in B^\circ \\ (-1, 1) & x \in C^\circ \\ ([0, 1], [-1, 0]) & x \in A \cap B \setminus \{(0, 0)\} \\ ((-1, 0], [-1, 1]) & x \in B \cap C \setminus \{(0, 0)\} \\ ([-1, 1], [0, 1]) & x \in A \cap C \setminus \{(0, 0)\} \\ ([-1, 1], [-1, 1]) & x = (0, 0) \end{cases},$$

while for the Clarke gradient we obtain:

$$\partial f(x) = \begin{cases} \frac{df}{dx} & x \neq (0, 0) \\ \mathrm{co}\{(1, 0), (0, -1), (-1, 1)\} & x = (0, 0) \end{cases}.$$

The rectangle $([-1, 1], [-1, 1])$ is the smallest rectangle with sides parallel to the coordinate axes, containing the triangle $\mathrm{co}\{(1, 0), (0, -1), (-1, 1)\}$. While the domain-theoretic derivative here gives the best possible rectangular approximation to the Clarke's gradient and thus can be used for its approximation, one can develop a more sophisticated differential calculus, in which the domain-theoretic derivative would be the same as the Clarke's derivative. This will be based on the super-domain $[0, 1]^n \to \mathbf{C}(\mathbb{R}^n)$ of $[0, 1]^n \to (\mathbf{I}\mathbb{R})^{1\times n}_s$, where $\mathbf{C}(\mathbb{R}^n)$ is the collection of nonempty, convex and compact subsets of $\mathbb{R}^n$.

5. Domain for Lipschitz functions

We will construct a domain for locally Lipschitz functions and for $C^1[0,1]^n$. Note that the domain $[0,1]^n \to (\mathbf{I}\mathbb{R})_s^{1\times n}$ is isomorphic to the smash product $(D^0)_s^n$; we can write $g \in [0,1]^n \to (\mathbf{I}\mathbb{R})_s^{1\times n}$ as $g = (g_1, \cdots, g_n)$ with $\mathrm{dom}(g) = \mathrm{dom}(g_i)$ for all $i = 1, \cdots, n$. Consider the relation

$$\mathsf{Cons} \subset D^0 \times (D^0)_s^n,$$

defined by $(f,g) \in \mathsf{Cons}$ if $\uparrow f \cap \int g \neq \emptyset$. We will show that this relation is Scott closed.

Proposition 5.1 *Let $g \in (D^0)^n$ and $(f_i)_{i\in I}$ be a non-empty family of functions $f_i : dom(g) \to \mathbb{R}$ with $f_i \in \int g$ for all $i \in I$. If $h_1 = \inf_{i\in I} f_i$ is real-valued then $h_1 \in \int g$. Similarly, if $h_2 = \sup_{i\in I} f_i$ is real-valued, then $h_2 \in \int g$.*

Let $L[0,1]$ be the set of partial maps of $[0,1]$ into the extended real line. Consider the two dcpo's $(L[0,1], \leq)$ and $(L[0,1], \geq)$. Define the maps $s : D^0 \times (D^0)^n \to (L, \leq)$ and $t : D^0 \times (D^0)^n \to (L, \geq)$ by

$$s : (f,g) \mapsto \inf\{h : \mathrm{dom}(g) \to \mathbb{R} \,|\, h \in \int g \;\&\; h \geq f^-\}$$

$$t : (f,g) \mapsto \sup\{h : \mathrm{dom}(g) \to \mathbb{R} \,|\, h \in \int g \;\&\; h \leq f^+\}.$$

We use the convention that the infimum and the supremum of the empty set are ∞ and $-\infty$, respectively. Note that given a connected component A of $\mathrm{dom}(g)$ with $A \cap \mathrm{dom}(f) = \emptyset$, then $s(f,g)(x) = -\infty$ and $t(s,f)(x) = \infty$ for $x \in A$.

Proposition 5.2 *The following are equivalent:*

(i) $(f,g) \in \mathsf{Cons}$.

(ii) $s(f,g) \leq t(f,g)$.

(iii) *There exists $h : dom(g) \to \mathbb{R}$ with $h \in \int g$ and $f^- \leq h \leq f^+$ on $dom(g)$.*

To see that Cons is Scott closed, we need the following:

Proposition 5.3 *The maps s and t are Scott continuous.*

Corollary 5.4 *The relation* Cons *is Scott closed.*

Corollary 5.5 *Let $(f,g) \in$ Cons. Then in each connected component O of the domain of definition of g which intersects the domain of definition of f, there exist two locally Lipschitz functions $s : O \to \mathbb{R}$ and $t : O \to \mathbb{R}$ such that $s, t \in \uparrow f \cap \int g$ and for each $u \in \uparrow f \cap \int g$, we have with $s(x) \leq u(x) \leq t(x)$ for all $x \in O$.*

We now can define a central notion of this paper:

Definition 5.6 Define

$$D^1 = \{(f,g) \in D^0 \times (D^0)_s^n : (f,g) \in \mathsf{Cons}\}.$$

Corollary 5.7 *The poset D^1 is a continuous Scott domain, i.e. a bounded complete countably based continuous dcpo.*

Let $C^0[0,1]^n$ be the collection of real-valued continuous function. Let $\Gamma^1 : C^0[0,1]^n \to D^1[0,1]^n$ be defined by $\Gamma^1(f) = (f, \frac{df}{dx})$.

Theorem 5.8 *The map Γ^1 is an embedding into the set of maximal elements of D^1.*

Furthermore, Γ_1 restricts to give an embedding for locally Lipschitz functions (where $\frac{df}{dx} \neq \perp$ for all x) and for piecewise C^1 functions (where $\frac{df}{dx}$ is maximal except for a finite set of points). This establishes D^1 as a domain for locally Lipschitz functions.

6. Integrability, decidability of consistency

In this section, we will derive a necessary and sufficient condition for integrability and consider the question of decidability of the consistency predicate. The collection of step functions of the form $(f,g) \in D^0 \times (D^0)_s^n$, where $f \in D^0$ and $g \in (D^0)_s^n$ are step functions, forms a basis of D^1. The *rational* basis of D^1 is the collection of all rational step functions (f,g), i.e., those whose domains and values are defined over rational numbers.

We show that $(f,g) \in \mathsf{Cons}$ i.e. $\uparrow f \cap \int g \neq \emptyset$ is decidable on the rational basis elements of $D^0 \times (D^0)_s^n$. In order to prove this, we show that given a pair of rational step functions $(f,g) \in D^0 \times (D^0)_s^n$, we can decide if $\int g \neq \emptyset$ and then explicitly construct $s(f,g)$ and $t(f,g)$, which will enable us to decide if $s(f,g) \leq t(f,g)$.

Let $g = (g_1, \ldots, g_n) \in (D^0)_s^n$ be a step function. Recall that a *crescent* is the intersection of an open set and a closed set. The domain $\mathrm{dom}(g)$ of g is partitioned into a finite set of disjoint crescents $\{C_j : j \in I_i\}$, in each of which the value of g_i is constant, where we assume that the indexing sets I_i are pairwise disjoint for $i = 1, \ldots, n$. The collection

$$\{\bigcap_{1\leq i\leq n} C_{k_i} : k_i \in I_i, 1 \leq i \leq n\}$$

of crescents partition $\mathrm{dom}(g)$ into regions in which the value of g is a constant interval vector; they are called the associated crescents of g. Each associated crescent has boundaries parallel to the coordinate planes and these boundaries intersect at points, which are called the *vertices* of the crescent. Vertices of the crescents which are on the boundary of $\mathrm{dom}(g)$ are called the *corners* of $\mathrm{dom}(g)$. In Figure 1, an example of a step function g is given with its associated crescents, the interval in each crescent gives the value of g in that crescent. A solid line on the boundary of a crescent indicates that the boundary is in the crescent, whereas a broken line indicates that it is not.

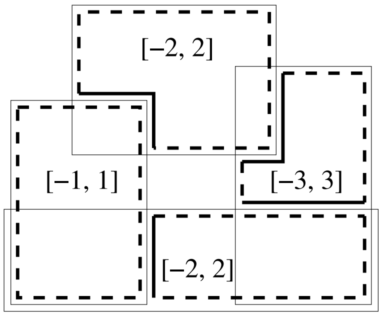

Figure 1. Crescents of a step function g

Recall that a path in a connected region $R \subset \mathbb{R}^n$ is a continuous map $p : [0,1] \to R$ with endpoints $p(0)$ and $p(1)$. If p is piecewise C^1, respectively piecewise linear, then the path is called a piecewise C^1, respectively piecewise linear. The space $P(R)$ of piecewise C^1 paths in R is equipped with the C^1 norm:

$$\|p\| = \max\{\max_{r\in[0,1]} \|p(r)\|, \max_{r\neq r_0,\cdots,r_k} \|p'(r)\|\}. \tag{8}$$

A path p is *non-self-intersecting* if $p(r) = p(r')$ for $r < r'$ implies $r = 0$ and $r' = 1$. We will be mainly concerned with piecewise linear paths in this paper. For these paths, there exists a strictly increasing sequence of points $(r_i)_{0\leq i\leq k}$ for some $k \in \mathbb{N}$ with $0 = r_0 < r_1 < \cdots r_{k-1} < r_k = 1$ such that p is linear in $[r_i, r_{i+1}]$ for $0 \leq i \leq k-1$. The points $p(r_i)$ for $i = 0, \cdots, k$, are said to be the *nodes* of p; the nodes $p(r_i)$ for $i = 1, \cdots, k-1$ are called the *inner* nodes. The line segment $\{p(r) : r_i \leq r \leq r_{i+1}\}$ is denoted by $p([r_i, r_{i+1}])$. If $p(0) = p(1)$, the path is said to be *closed*. The *length* of the path p is $\sum_{i=0}^{k} \|p(r_{i+1}) - p(r_i)\|$.

A *simple* path in a region $R \subset \mathbb{R}^n$ is a non-self-intersecting piecewise linear map. We now consider simple paths in the closure $\mathrm{cl}(O)$ of a connected component $O \subseteq \mathrm{dom}(g)$. We say that a simple path is *locally minimal* if its length is minimal among all the simple paths in its homotopic equivalence class in $\mathrm{cl}(O)$. Figure 2 shows that for the domain of the step function g of Figure 1 there are precisely two locally minimal paths for each pair of points $(x, y) \in \mathrm{cl}(\mathrm{dom}(g))$. When $x = y$, one of these paths is the trivial constant path with value x and the other is a closed path around the hole in $\mathrm{cl}(\mathrm{dom}(g))$.

Recall that given a vector field $F : R \to \mathbb{R}^n$ in a region $R \subset R^n$ and a piecewise C^1 path $p : [0,1] \to R$, the path integral of F with respect to p from 0 to $w \in [0,1]$ is defined as $\int_0^1 F(p(r)) \cdot p'(r)\,dr$, when the integral exists.

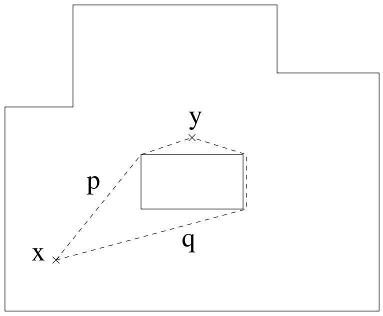

Figure 2. Two locally minimal paths

Here, $u \cdot v = \sum_{i=1}^n u_i v_i$ denotes the usual scalar product of two vectors $u, v \in \mathbb{R}^n$.

We define a generalization of the notion of scalar product for vectors of type: $u \in (\mathbf{I}\mathbb{R})^n$ and $v \in \mathbb{R}^n$. For $a = [\underline{a}, \overline{a}] \in (\mathbf{I}\mathbb{R})^n$, let $a^- = \underline{a}$, $a^+ = \overline{a}$ and $a^0 = 1$. We define the *direction dependent scalar product* as the strict map

$$- \odot - : (\mathbf{I}\mathbb{R})^n_s \times \mathbb{R}^n \to \mathbb{R}_\perp$$

with $u \odot v = \sum_{i=1}^n u_i^{\sigma(v_i)} v_i$ for $u \neq \perp$. Note that the direction dependent scalar product is not symmetric and is only linear in its first component. We can now define a notion of path integral of the interval-valued vector function $g = [g^-, g^+] \in (D^0)^n_s$ with respect to any piecewise C^1 path from y to x in $\mathrm{cl}(O)$, where O is a connected component of $\mathrm{dom}(g)$. For each $i = 1, \cdots n$, the ith component of g is given by $g_i = [g_i^-, g_i^+]$.

Definition 6.1 Given a step function $g \in (D^0)^n_s$ and a piecewise C^1 path p in the closure of connected component O of the domain of g, the *upper path integral* of g over p from 0 to $w \in [0,1]$ is defined as:

$$\mathsf{U}\int_{p[0,w]} g(r)\,dr = \int_0^w g(p(r)) \odot p'(r)\,dr.$$

The *lower path integral* of g over p from 0 to $w \in [0,1]$ is similarly defined as

$$\mathsf{L}\int_{p[0,w]} g(r)\,dr = -\int_0^w g(p(r)) \odot (-p'(r))\,dr.$$

Thus, if the jth component of the path, for some j with $1 \leq j \leq n$, is increasing locally at some $r \in [0,1]$, i.e. $p_j' > 0$ in a neighborhood of r, then $g_j^{-\sigma(p_j(r))} = g_j^-$ will contribute locally to the jth component of the sum in the lower integral, while if $p_j' < 0$ in a neighborhood of r, then $g_j^{-\sigma(p_j(r))} = g_j^+$ will contribute. In case the path is locally perpendicular to the jth axis at r, i.e. $p_j'(r) = 0$ in a local neighborhood of r, then there will be zero contribution for the jth component in the sum. For the upper integral the contributions of g_i^- and g_i^+ are reversed.

The geometric interpretation of the lower and upper path integrals is as follows. We regard $g \in (D^0)^n_s$ as an interval-valued vector field in $[0,1]^n$. For any continuous vector field $F : \mathrm{dom}(g) \to \mathbb{R}^n$ with $F(x) \in g(x)$ for all $x \in \mathrm{dom}(g)$ and any piecewise C^1 path $p \in P(O)$ in a connected component O of $\mathrm{dom}(g)$, the classical path integral map $w \mapsto \int_0^w F(p(r))p'(r)\,dr : [0,1] \to \mathbb{R}$ induces a continuous curve

$$c : w \to (p(w), \int_0^w F(p(r))p'(r)\,dr) : [0,1] \to \mathbb{R}^{n+1}$$

above $p([0,1]) \subset O$ with $c(0) = (p(0), 0)$. The maps

$$c_l : w \mapsto (p(w), \mathsf{L}\int_{p[0,w]} g(r)\,dr) : [0,1] \to \mathbb{R}^{n+1}$$

$$c_g : w \mapsto (p(w), \mathsf{U}\int_{p[0,w]} g(r)\,dr) : [0,1] \to \mathbb{R}^{n+1}$$

have respectively the least and greatest possible height in their $n+1$ coordinate and thus for $0 \le w \le 1$:

$$(c(w))_{n+1} \in [(c_l(w))_{n+1}, (c_g(w))_{n+1}].$$

We can now define the path integral map

$$\int_{(-)[0,(-)]} (-) : (D^0)^n_s \times P(\mathbb{R}^n) \times [0,1] \to \mathbf{I}\mathbb{R},$$

given by

$$(g, p, w) \mapsto [\mathsf{L}\int_{p[0,w]} g(r)\,dr, \mathsf{U}\int_{p[0,w]} g(r)\,dr]$$

if p is in the closure of a connected component of $\mathrm{dom}(g)$ and bottom otherwise.

Proposition 6.2 *The path integral map is continuous with respect to the Scott topology, the norm topology of paths and the Euclidean topology.*

Definition 6.3 Given a step function $g \in (D^0)^n_s$ and a closed simple path p in the closure of a connected component of $\mathrm{dom}(g)$, we say that g satisfies the *zero-containment loop condition for* p if

$$0 \in \int_{p[0,1]} g(r)\,dr.$$

We say that $g \in (D^0)^n_s$ satisfies the *zero-containment loop condition* if it satisfies the zero-containment loop condition for any closed simple path p in the closure of any connected component of $\mathrm{dom}(g)$.

If g only takes point (maximal) values, then the zero-containment loop condition is simply the standard condition for g to be a gradient i.e., that the path integral of g vanishes on any closed path. Figure 3 gives an example of a step function $g = (g_1, g_2)$, with $\mathrm{dom}(g) = ((0,3) \times (0,3)) \setminus ([1,2] \times [1,2])$ which does not satisfy the zero-containment loop condition. Denote the dashed path by p; it has nodes at $p(0) = p(1) = (1/2, 1/2)$, $p(1/4) = (5/2, 0)$, $p(1/2) = (5/2, 5/2)$ and $p(3/4) = (1/2, 5/2)$. The lower path integral of g over p gives a strictly positve value:

$\mathsf{L}\int_p g(r)dr = \sum_{i=0}^3 \int_{i/4}^{(i+1)/4} -g(p(r)) \odot (-p'(r))dr$
$= -\int_0^{1/4} g(p(r)) \odot (-8,0)dr - \int_{1/4}^{1/2} g(p(r))\odot$
$(0,-8)dr - \int_{1/2}^{3/4} g(p(r)) \odot (8,0)dr - \int_{3/4}^{1} g(p(r))\odot$
$(0,8)dr = 1/4(8\cdot 1 + 8\cdot 1 + 8\cdot 1 + 8\cdot 1) = 8 > 0.$

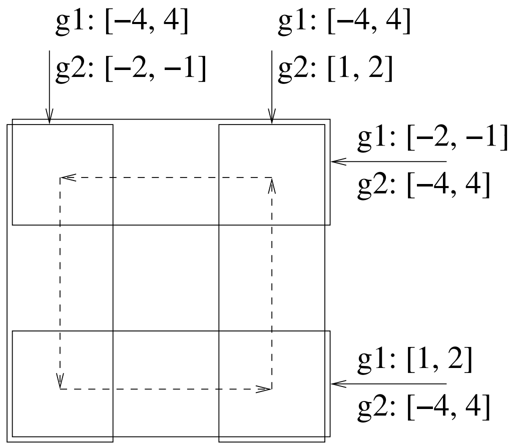

Figure 3. Failure of zero-containment

Recall that $g \in (D^0)^n_s$ is called integrable if $\int g \neq \emptyset$. The following is an extension of Green's Theorem also called the Gradient Theorem in classical differential calculus [7].

Theorem 6.4 *Suppose $g \in (D^0)^n_s$ is an integrable step function. Then g satisfies the zero-containment loop condition.*

Proof Assume $h \in \int g$ and thus, by Corollary 3.3,

$$g \sqsubseteq \frac{dh}{dx}. \tag{9}$$

Take any closed piecewise linear path p_0 in a connected component O of $\mathrm{dom}(g)$ with $p_0(0) = p_0(1) = y_0$ and nodes at $0 = r_0 < r_1 < \cdots\cdots r_{k-1} < r_k = 1$ say. Then h is locally Lipschitz in $\mathrm{cl}(O)$ and thus by Rademacher's theorem [3, p 148] it is differentiable almost everywhere in O. We take small parallel transversal (Poincaré) $(n-1)$-dimensional sections [11] A_i at $p_0(r_i)$, for $i = 0, \cdots, k-1$, to the path p_0 and consider the family of closed piecewise linear paths p_y with $p_y(0) = p_y(1)$, where $y \in A_0$, such that, for each $y \in A_0$ and each i with $0 \leq i \leq k-1$, $p_y(r_i) \in A_i$, for $i = 1, \cdots k-1$, and the line segments $p_y[r_i, r_{i+1}]$ are parallel with $p_{y_0}[r_i, r_{i+1}]$. By Fubini's theorem, it follows that for almost all $y \in A$, with respect to the $(n-1)$-dimensional Lebesgue measure on A, the map h is differentiable on the path p_y almost everywhere with respect to the 1-dimensional Lebesgue measure on p_y. Let $y \in A$ be such a point. Thus $h'(x)$ exists almost everywhere for $x \in p_y[0,1]$ with respect to the 1-dimensional Lebesgue measure on $p_y[0,1]$ and, by Corollary 3.6, at these points we have $h'(x) \in \frac{dh}{dx}$. This together with Equation 9, implies that

$$h'(x) \in g(x), \tag{10}$$

for almost all $x \in p_y([0,1])$. Now the composition $h \circ p : [0,1] \to \mathbb{R}$ of locally Lipschitz maps is locally Lipschitz and thus equal to the integral of its derivative. Thus, the ordinary path integral of the gradient of h along p_y vanishes:

$$\int_0^1 h'(p_y(r)) \cdot p_y'(r)\, dr = \int_0^1 (h \circ p)'(r)\, dr =$$

$$h(p_y(1)) - h(p_y(0)) = h(y) - h(y) = 0. \tag{11}$$

On the other hand, from the definitions of the lower and upper path integrals, using Equation 10, we have:

$$\mathsf{L}\int_{p_y} g(r)\, dr \leq \int_0^1 h'(p_y(r)) \cdot p_y'(r)\, dr,$$

and

$$\int_0^1 h'(p_y(r)) \cdot p_y'(r)\, dr \leq \mathsf{U}\int_{p_y} g(r)\, dr. \tag{12}$$

Thus,

$$\mathsf{L}\int_{p_y} g(r)\, dr \leq 0 \leq \mathsf{U}\int_{p_y} g(r)\, dr.$$

Since $\mathsf{L}\int_{p_y} g(r)\, dr$ and $\mathsf{U}\int_{p_y} g(r)\, dr$ depend continuously on $y \in A$ and since Equations (11) and (12) hold for almost all $y \in A$ with respect to the (n-1)-dimensional Lebesgue measure on A, it follows that these equations hold actually for all $y \in A$ and in particular for $y = y_0$. This establishes that g satisfies the zero-containment loop condition for any closed non-self-intersecting piecewise linear path in O. By continuity, it follows that g also satisfies the zero-containment loop condition for any closed non-self-intersecting piecewise linear path in the closure of O. $\square$

We will now show that if a step function $g \in (D^0)^n_s$ satisfies the zero-containment loop condition, then it is integrable. Let O be a connected component of $\mathrm{dom}(g)$. Note that any step function g can be extended to the boundary of $\mathrm{dom}(g)$ by the lower and upper semi continuity of g^- and g^+ respectively. We will in the following also need to consider paths, which can have a subpath lying on the boundary of O, regarded as the limit of paths lying in O itself. Thus, we adopt the following convention. If two crescents have a common boundary, we consider their common boundary as infinitesimally separated so that they have distinct boundaries. This means that a line segment of a simple path on a common boundary of two different crescents is always regarded as the limit of a sequence of parallel segments contained on one side of this boundary.

We are now ready to introduce a key concept of this paper. For $x, y \in \mathrm{cl}(O))$, we put

$$V_g(x,y) = \sup\{\mathsf{L}\int_{p[0,1]} g(r)\, dr :$$

$$p \text{ a piecewise linear path in } \mathrm{cl}(O) \text{ from } y \text{ to } x\},$$

$$W_g(x,y) = \inf\{\mathsf{U}\int_{p[0,1]} g(r)\, dr :$$

$$p \text{ a piecewise linear path in } \mathrm{cl}(O) \text{ from } y \text{ to } x\}.$$

Proposition 6.5 *Suppose g satisfies the zero-containment loop condition and $x, y \in cl(O)$, then there are locally minimal simple paths p and q from y to x such that:*

$$V_g(x,y) = \mathsf{L}\int_{p[0,1]} g(r)\, dr$$

$$W_g(x,y) = \mathsf{U}\int_{q[0,1]} g(r)\, dr.$$

Moreover, for each $y \in cl(O)$, the two maps given by $V_g(\cdot, y), W_g(\cdot, y) : cl(O) \to \mathbb{R}$ are continuous, piecewise linear and satisfy $V_g(y,y) = W_g(y,y) = 0$,

$$g \sqsubseteq \frac{dV_g(\cdot, y)}{dx} \qquad \textit{and} \qquad g \sqsubseteq \frac{dW_g(\cdot, y)}{dx}.$$

See the Appendix for the proof of Proposition 6.5. Thus, we obtain the following main result:

Theorem 6.6 *A step function $g \in (D^0)^n_s$ is integrable iff it satisfies the zero-containment loop condition.*

Proposition 6.7 *For a step function $g \in (D^0)^n_s$ defined over rational numbers, the zero-containment loop condition is decidable.*

Proof There are a finite number of connected components of $\mathrm{dom}(g)$. In each connected component O of $\mathrm{dom}(g)$, the maximal value of $\mathsf{L}\int_{p[0,1]} g(r)\, dr$, respectively the minimum value of $\mathsf{U}\int_{p[0,1]} g(r)\, dr$, for a closed simple path in $\mathrm{cl}(O)$ is reached for a path p with nodes at the corners of the boundary of O. Since the number of such closed simple paths is finite and since for each such path $\mathsf{L}\int_{p[0,1]} g(r)\, dr$ is a rational number, we can decide in finite time if the zero-containment loop condition holds for g. $\square$

We will now show that for a pair of step functions $(f,g) \in D^1$, with g integrable, the consistency relation $(f,g) \in \mathsf{Cons}$ is decidable. Let x and y be in the same connected component O of dom(g) with $O \cap$ dom$(f) \neq \emptyset$. Let $S_{(f,g)}(x,y) = V_g(x,y) + \underline{\lim} f^-(y)$.

Theorem 6.8 *The maps $V_g(\cdot,y), W_g(\cdot,y) : cl(O) \to \mathbb{R}$ are respectively the least and the greatest continuous maps $L, G : O \to \mathbb{R}$ with $L(y) = 0$ and $G(y) = 0$ such that $g \sqsubseteq \frac{dL}{dx}$ and $g \sqsubseteq \frac{dG}{dx}$.*

Proof We prove that $L = V_g(\cdot,y)$ by showing that the set $A = \{x \in \text{cl}(O) : L(x) = V_g(x,y)\}$ is non-empty, open and closed in cl(O), from which the result follows. Clearly $y \in A$ so $A \neq \emptyset$. Also since L and $V_g(\cdot,y)$ are both continuous, A is closed. It remains to show that A is open. Suppose A is not open. Then cl$(O) \setminus A \neq \emptyset$. We can take $u \in \text{cl}(O) \setminus A$ close to the boundary of A and away from the boundary of O such that the following condition holds: the locally minimal path p from y to u with $V_g(u,y) = \mathsf{L}\int_{p[0,1]} g(r)\, dr$ intersects the boundary of A at a point x such that the subpath q of p from y to x gives: $V_g(x,y) = \mathsf{L}\int_q g(r)\, dr$ and the the subpath of p from x to u is a straight line. Thus, $V_g(u,y) = V_g(x,y) + V_g(u,x)$. On the other hand,

$$L(u) \geq L(x) + V_g(u,x) = V(x,y) + V_g(u,x)$$

and it follows that $L(u) \geq V_g(u,y)$, i.e., $L(u) = V_g(u,y)$, which gives a contradiction. This establishes the proof. The case for G is similar. $\square$

Corollary 6.9 *Let O be a connected component of dom(g) with non-empty intersection with dom(f). For $x \in O$, we have:*

$$s(f,g)(x) = \sup_{y \in O \cap \textit{dom(f)}} S_{(f,g)}(x,y). \tag{13}$$

Proof By Proposition 6.8, the map $h_y = \lambda x.S_{(f,g)}(x,y)$ is the least function with $h_y(y) = \underline{\lim} f^-(y)$ such that $g \sqsubseteq \frac{dh_y}{dx}$. By definition, $s(f,g)$ is precisely the upper envelop of h_y for $y \in O$. $\square$

Proposition 6.10 *There exist a finite number of points $y_0, y_1, \ldots, y_i \in cl(O \cap dom(f))$ with*

$$s(f,g)(x) = \max\{S_{(f,g)}(x,y_j) : j = 0,1,\ldots,i\}$$

for $x \in O$.

Proof For each fixed $x \in O$, the map $\lambda y.S_{(f,g)}(x,y)$ is linear in the coordinates of $y \in$ cl$(O \cap$ dom(f)$)$, thus it will attain its maximum when y is at a corner of cl$(O \cap$ dom(f)$)$. It is thus sufficient to take y_i's to be the corners of the crescents associated with f and g in cl$(O \cap$ dom(f)$)$. $\square$

Results dual to those above are obtained for $t(f,g)$ as follows. We put $T_{(f,g)}(x,y) = W_g(x,y) + \overline{\lim} f^+(y)$. Then, we have

$$t(f,g)(x) = \inf_{y \in O \cap \text{dom(f)}} T_{(f,g)}(x,y),$$

and there exist $y_0, y_1, \ldots, y_i \in$ cl$(O \cap$ dom(f)$)$ with

$$t(f,g)(x) = \min\{T_{(f,g)}(x,y_j) : j = 0,1,\ldots,i\},$$

for $x \in O$.

Corollary 6.11 *The predicate Cons is decidable on basis elements (f,g) consisting of step functions defined over rational numbers.*

Proof Since, by Proposition 6.10 and its dual, $s(f,g)$ and $t(f,g)$ are rational, continuous and piecewise linear, we can test $s(f,g) \leq (t(f,g)$ in each connected component of dom(g) with non-empty intersection with dom(f). $\square$

Theorem 6.12 *The domain D^1 can be given an effective structure using a standard enumeration of its rational basis.*

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Appendix

Proposition 2.2

Proof Suppose $f \in \delta(a,b)$, but $f(y)$ is not maximal for $y \in a^\circ$. Let $\epsilon > 0$ be small enough so that $x \in a^\circ$ with $x_i = [y_i - \epsilon, y_i + \epsilon]$ for $1 \le i \le n$. The diameter of $b(x-x)$ can be made arbitrarily small by choosing ϵ small. However, since $b(x-x) \sqsubseteq f(x)-f(x)$, it follows that the diameter of $b(x-x)$ is larger than the diameter of $f(x)-f(x)$ which is larger than the diameter of $f(y)$. This contradiction establishes the proof. □

Proposition 2.3

Proof (i) $\iff$ (ii). By taking $x \in a^\circ$ with $x_j = z_j$ for $j \neq i$, we have $b_i(z_i - x_i) \sqsubseteq f(z) - f(x)$ iff $b_i \sqsubseteq \frac{\partial f}{\partial x_i}(z)$ iff $f'(z) \in b$.
(iii) $\iff$ (ii). Obvious.□

Proposition 2.4

Proof The "if" part follows immediately from the definition of $\delta(a,b)$. To show the "only if" part, assume $\delta(a,b) \supseteq \delta(c,d)$ holds but not $c \sqsubseteq a$. Then for some i with $1 \le i \le n$, the relation $c_i \sqsubseteq a_i$ does not hold. Let $g : [0,1]^n \to \mathbb{R}$ be the linear partial function defined by

$$g(x) = \begin{cases} \underline{d_i} x & x \in c^\circ \\ (\underline{b_i} - 1)x & x \in (a \setminus c)^\circ \\ \bot & \text{otherwise} \end{cases}$$

Then $g \in \delta(c,d)$ but $g \notin \delta(a,b)$. On the other hand if $b_i \sqsubseteq d_i$ does not hold for some i, consider the function $f : c^\circ \to \mathbb{R}$ with $f(x) = tx_i$ for $t \in d_i \setminus b_i$. Then, $f \in \delta(c,d) \setminus \delta(a,b)$. □

Proposition 2.7

Proof Let $x \in \mathrm{dom}(\Delta)$. Then there exists a step tie $\delta(a,b)$ with $x \in a^\circ$ and $f \in \Delta \subseteq \delta(a,b)$ and the result follows from Proposition 2.2 □.

Proposition 2.8

Proof Suppose there exists $f \in \bigcap_{i\in I} \delta(a_i,b_i)$ but $(a_i \searrow b_i)_{i\in I}$ is not consistent. Then there is a finite subfamily $(a_i \searrow b_i)_{i\in J}$, for $J \subseteq I$, which is not consistent. Therefore there exists a pair, say $\delta(a,b)$ and $\delta(a',b')$, in this finite subfamily such that $a^\circ \cap a'^\circ \neq \emptyset$ with $b_i \cap b'_i = \emptyset$ for some i with $1 \le i \le n$. Take elements $x, y \in a^\circ \cap a'^\circ$ with $x_j = y_j$ for $j \neq i$ and $x_i \neq y_i$ it follows that $b(x-y)$ and $b'(x-y))$ are disjoint.

Since by assumption $f \in \delta(a,b) \cap \delta(a',b')$, we have

$$b(x-y) \sqsubseteq f(x) - f(y)$$

$$b'(x-y) \sqsubseteq f(x) - f(y)$$

which gives a contradiction, since the two intervals on the left hand side are disjoint. □

Proposition 2.10

Proof Let $b \sqsubseteq \bigsqcup_{a_i \sqsubseteq a} b_i$ and assume $f \in \bigcap_{i\in I} \delta(a_i,b_i)$. Let $x, y \in a^\circ$. For each $i \in I$ with $a_i \sqsubseteq a$ we have:

$$b_i(x-y) \sqsubseteq f(x) - f(y).$$

By the Scott continuity of product of two intervals and our assumption, we get:

$$b(x-y) \sqsubseteq \bigsqcup_{a_i \sqsubseteq a} b_i(x-y) \sqsubseteq f(x) - f(y),$$

as required. □

Proposition 2.13

Proof By Corollary 2.11, $\int$ is well-defined and monotone. Let $(g_i)_{i\in I}$ be a directed set in D^0 with $g_i = \bigsqcup_{j \in I_i} a_j \searrow b_j$. Then, $\int \bigsqcup_{i\in I} g_i = \int \bigsqcup_{i\in I} \bigsqcup_{j\in I_i} a_j \searrow b_j = \bigcap_{i\in I} \bigcap_{j\in I_i} \delta(a_j,b_j) = \bigcap_{i\in I} \int g_i = \bigsqcup_{i\in I} \int g_i$. □

Theorem 3.2

Proof (i) Let the indexing set I be defined by $i \in I \iff f \in \delta(a_i,b_i)$. Then $\bigcap_{i\in I} \delta(a_i,b_i) \neq \emptyset$. Hence, $(a_i \searrow b_i)_{i\in I}$ is consistent. Therefore, $\frac{df}{dx} = \bigsqcup_{i\in I} a_i \searrow b_i$ exists and is Scott continuous.

(ii) By Proposition 2.3, $f \in \delta(a,b) \iff a \searrow b \sqsubseteq f'$. Hence, $f' = \bigsqcup_{f \in \delta(a,b)} a \searrow b$.

(iii) Obvious. □

Corollary 3.3

Proof Let $g = \bigsqcup_{i\in I} a_i \searrow b_i$, then by Theorem 3.2(iii):

$$f \in \int g \iff f \in \bigcap_{i\in I} \delta(a_i,b_i) \iff$$

$$\forall i \in I.\ a_i \searrow b_i \sqsubseteq \frac{df}{dx} \iff \bigsqcup_{i\in I} a_i \searrow b_i \sqsubseteq \frac{df}{dx}. \qquad \square$$

Proposition 3.4

Proof (i) By Proposition 2.2, for $b \neq \bot$, we have $f \in \delta(a,b) \Rightarrow z \notin a^\circ$. Hence,

$$\frac{df}{dx}(z) = \bigsqcup\{a \searrow b \mid f \in \delta(a,b) \,\&\, z \in a^\circ\} = \bot.$$

(ii) We have $c = \frac{df}{dx}(y) = \bigsqcup\{b \mid y \in a^\circ \;\&\; f \in \delta(a,b)\}$. Let $\epsilon > 0$ be given. Now take $a \searrow b \sqsubseteq \frac{df}{dx}$ with $y \in a^\circ$ and $w(b_i) < \epsilon/n$ for $1 \le i \le n$. By Proposition 2.2, this implies that $f(x)$ is maximal for each $x \in U$ with $U = a^\circ$. Furthermore, $b \sqsubseteq c$ and there exists $\delta > 0$ such that $\|x -$

$y\| < \delta$ implies $x \in a^\circ$. We have $b(x-y) \sqsubseteq f(x) - f(y)$ for $x \in a^\circ$, and

$$w(b(x-y)) \leq \sum_{i=1}^{n} w(b_i)|x_i - y_i| \leq \frac{\epsilon}{n}\sum_{i=1}^{n}|x_i - y_i| \leq \epsilon\|x-y\|.$$

We have $b(x-y) \sqsubseteq c(x-y)$ and therefore $|f(x) - f(y) - c(x-y)| \leq \epsilon\|x-y\|$ and the result follows. $\Box$

Proposition 5.1

Proof Suppose h_1 is real-valued. Let $a \searrow b \sqsubseteq g$. We have $f_i(x) - f_i(y) \in b(x-y)$ for all $i \in I$. Thus, $\underline{b(x-y)} \leq f_i(x) - f_i(y) \leq \overline{b(x-y)}$. Thus, $\inf_{i\in I} f_i(x) \leq f_i(y) + \overline{b(x-y)}$. Taking infimum again, we obtain $\inf_{i\in I} f_i(x) \leq \inf_{i\in I} f_i(y) + \overline{b(x-y)}$ and hence: $h_1(x) - h_1(y) \leq \overline{b(x-y)}$. Similarly, $\underline{b(x-y)} \leq h_1(x) - h_1(y)$ and the result follows. The case of h_2 is similar. $\Box$

Proposition 5.2

Proof If $\mathrm{dom}(f) \cap \mathrm{dom}(g) = \emptyset$, then the three statements hold trivially. So assume in the following proof that $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$.
(ii) $\Rightarrow$ (i). Suppose $s(f,g) \leq t(f,g)$. Then, $s(f,g) \in \uparrow\! f \cap \int g$ and hence $(f,g) \in \mathsf{Cons}$.
(i) $\Rightarrow$ (ii). Suppose $(f,g) \in \mathsf{Cons}$. Assume $h \in \uparrow\! f \cap \int g$. Then, the induced map $h : \mathrm{dom}(g) \to \mathbb{R}$ satisfies $h \in \int g$. Hence, $f^- \leq h \leq f^+$ and thus $s(f,g) \leq t(f,g)$.
(ii) $\Rightarrow$ (iii). Suppose $s(f,g) \leq t(f,g)$. Put $h = s(f,g)$.
(iii) $\Rightarrow$ (ii). We have $s(f,g) \leq h \leq t(f,g)$. $\Box$

Proposition 5.3

Proof Consider the map s. If $f_1 \sqsubseteq f_2$ and $g_1 \sqsubseteq g_2$, then we have $\int g_1 \supseteq \int g_2$ and $f_1^- \leq f_2^-$ and it follows that $s(f_1,g_1) \leq s(f_2,g_2)$. Let $\{(f_i,g_i)\}_{i\in\omega}$ be an increasing chain and put $f = \bigsqcup_{i\in\omega} f_i$ and $g = \bigsqcup_{i\in\omega} g_i$. To show the continuity of s we need to show that $\sup_i s(f_i,g_i) \geq s(f,g)$ on any connected component of $\mathrm{dom}(g) = \bigcup_{i\in\omega}\mathrm{dom}(g_i)$. Take any such connected component $A \subseteq \mathrm{dom}(g)$. If $A \cap \mathrm{dom}(f) = \emptyset$ then $s(f,g) = -\infty$ on A and the result follows. Assume that $A \cap \mathrm{dom}(f) \neq \emptyset$, i.e., $\mathrm{dom} f_{i_0} \cap \mathrm{dom} g_{i_0} \neq \emptyset$ for some $i_0 \in \omega$. If $s(f_i,g_i) = \infty$ on $A \cap \mathrm{dom}(g_i)$ for some $i \geq i_0$, then $\sup_{i\in\omega} s(f_i,g_i) = \infty$ on A and the result follows again. Otherwise, assume without loss of generality that $-\infty < s(f_i,g_i) < \infty$ on $A \cap \mathrm{dom}(g_i)$ for all $i \in \omega$. Then from $(s(f_i,g_i))\restriction_A \in \int g_i$ it follows that $\forall i \geq j.\ (s(f_i,g_i))\restriction_A \in \int g_j$, and hence, by Proposition 5.1, $(\sup_{i\in\omega}(s(f_i,g_i)\restriction_A)) \in \int g_j$. Thus $(\sup_{i\in\omega} s(f_i,g_i))\restriction_A \in \bigsqcup_j \int g_j = \int \bigsqcup g_j$. On the other hand, $s(f_i,g_i) \geq f_i^-$ on A implies $\sup_{i\in\omega} s(f_i,g_i) \geq f_i^-$ on A and hence $\sup_{i\in\omega} s(f_i,g_i) \geq f^-$ on A. This shows that s is continuous. Similarly t is continuous. $\Box$

Corollary 5.4

Proof Let $(f_i,g_i)_{i\geq 0} \subset D^0 \times (D^0)^n$ be an increasing chain with $(f_i,g_i) \in \mathsf{Cons}$ for all $i \geq 0$. Then, by Proposition 5.2, $s(f_i,g_i) \leq t(f_i,g_i)$ for all $i \in I$. Hence, $s(f,g) = \sup_{i\geq 0} s(f_i,g_i) \leq \inf_{i\geq 0} t(f_i,g_i) = t(f,g)$. $\Box$

Theorem 5.8

Proof Note that Γ^1 is well-defined by Corollary 3.3: we have $f \in \int \frac{df}{dx}$ and thus, $(f, \frac{df}{dx}) \in D^1$. We now show that $(f, \frac{df}{dx})$ is maximal. If $\frac{df}{dx} \sqsubseteq g$ and $(f,g) \in D^1$ then for any $a \searrow b \sqsubseteq g$ we have $f \in \int g \subseteq \delta(a,b)$, which implies $a \searrow b \sqsubseteq \frac{df}{dx}$, thus $g \sqsubseteq \frac{df}{dx}$, i.e., $g = \frac{df}{dx}$ and $(f, \frac{df}{dx})$ is maximal. $\Box$

Proposition 6.5

Proof We prove the statements for the lower integral; those for the upper integral are dual to these. Consider $V_g(x_0,y)$ for some $x_0 \in \mathrm{cl}(O)$ and a piecewise linear path p^{x_0} in $\mathrm{cl}(O)$ with $p^{x_0}(0) = y$ and $p^{x_0}(1) = x_0$ and nodes $p^{x_0}(r_i)$ for $0 = r_0 < r_1 < \cdots < r_{k-1} < r_k$. The lower integral $\mathsf{L}\int_{p[0,1]} g(r)\,dr$ depends linearly on the coordinates $(p^{x_0}(r_i))_j$, $j = 1, \cdots, n$, of the nodes of p^{x_0}. Thus, the extremal values of $\mathsf{L}\int_{p^{x_0}} g(r)\,dr$ are reached when the inner nodes of the path are at the corners of the boundary of O. It then follows from the zero-containment loop condition that the maximum value of $\mathsf{L}\int_{p^{x_0}} g(r)\,dr$ is reached for a simple path, with inner nodes at the corners of the boundary of O, which also has to be locally minimal in order to induce an extremal value for the lower integral. Since there are a finite number of corners at the boundary of O, it follows that there are a finite number of simple paths, say $p_1, \cdots, p_N$ for some some $N \geq 1$, from y to x with inner nodes at the corners of the boundary of O.

For a fixed $y \in \mathrm{cl}(O)$, as x varies locally near x_0, each locally minimal path p_m^x, for $1 \leq m \leq N$, depends continuously on x, with respect to the norm given in Formula (8), and thus $\mathsf{L}\int_{p_m^{x_0}} g(r)\,dr$ depends piecewise linearly on the coordinates of x, with the linearity coefficients changing only at the boundary of the associated crescents of g. Thus, for each $m = 1, \cdots, N$, we obtain for x near x_0 a family of N locally minimal paths p_m^x $(1 \leq m \leq M)$ with $p_m^x(0) = y$ and $p_m^x(1) = x$ such that the mapping $\lambda x.\mathsf{L}\int_{p_m^x} g(r)\,dr$ is a continuous piecewise linear surface. Therefore the map $V_g(\cdot, y)$, which is locally the maximum of N continuous piecewise linear local surfaces is itself a continuous piecewise linear surface.

Globally we have an N valued multi-surface and we can move continuously on this multi-surface from the point $(x, \mathsf{L}\int_{p_m^x} g(r)\,dr)$ on the graph of $\lambda x.\mathsf{L}\int_{p_m^x} g(r)\,dr$ to the point $(x, \mathsf{L}\int_{p_\ell^x} g(r)\,dr)$ on the graph of $\lambda x.\mathsf{L}\int_{p_\ell^x} g(r)\,dr$ (for $m \neq \ell$) by moving x continuously, around the holes of O, along some closed path back to its original position.

In order to show that $V_g(y,y)=0$, we note that the trivial constant p path with constant value y is a locally minimal simple path from y to y with $\mathsf{L}\int_{p[0,1]} g(r)\,dr = 0$. By the zero-containment loop condition, any other closed simple path q from y to y satisfies $\mathsf{L}\int_q g(r)\,dr \leq 0$ and thus $V_g(y,y) = \mathsf{L}\int_{p[0,1]} g(r)\,dr = 0$.

Finally, we prove that $g \sqsubseteq \frac{dV_g(\cdot,y)}{dx}$. First we show that there exists small $a>0$ such that this relation holds for all y and x with in any closed n-dimensional disc of radius a with center in cl(O). For any $w \in$ cl(O) there exists some $b_w > 0$ such for all y and x in the closed n-dimensional disc $\mathsf{D}_{b_w}(w)$ of radius b_w and center w, the the locally minimal simple path p from y to x with $V_g(x,y) = \mathsf{L}\int_{p[0,1]} g(r)\,dr$ will either be,

(i) the straight path from y to x, or,

(ii) the simple path from y to x with two line segments and inner node w.

In fact if w is not a corner, we can choose b_w small enough so that $\mathsf{D}_{b_w}(w)$ does not contain any corners and that it satisfies (i) above, whereas if w is a corner we can choose b_w such that (ii) above holds. Now, we consider the covering of the compact set cl(O) by the collection

$$\{\mathsf{D}^\circ_{b_w/2}(w) : w \in \text{cl}(O)\}$$

of the open discs $\mathsf{D}^\circ_{b_w/2}(w)$. Let

$$\{\mathsf{D}^\circ_{b_{w_i/2}}(w_i) : i = 1,\cdots,M\}$$

be a finite subcovering and let

$$a = \min\{b_{w_i}/2 : i = 1,\cdots,M\}.$$

Then, for any $w \in$ cl(O), we have $w \in \mathsf{D}^\circ_{b_{w_i/2}}(w_i)$ for some $i = 1,\cdots,M$. We have $\mathsf{D}_a(w) \subset \mathsf{D}^\circ_{b_{w_i}}(w_i)$ and thus for any $x,y \in \mathsf{D}_a(w)$, the locally minimal path p from y to x will be either of type (i) or (ii) above, and $g \sqsubseteq \frac{dV_g(\cdot,y)}{dx}$ will hold as it is easily seen from the definition of lower path integral.

Now consider any $x,y \in$ cl(O) and consider the n-dimensional sphere $\mathsf{S}_a(x)$ and the disc $\mathsf{D}_a(x)$ of radius a with center x. For any $z \in \mathsf{S}_a(x)$ and $u \in \mathsf{D}_a(x)$, we have: $V_g(u,y) \geq V_g(z,y) + V_g(u,z)$ since any pair of piecewise linear paths p_1 (from y to z) and p_2 (from z to u) gives rise, by concatenation, to a piecewise linear path (p_1 followed by p_2) from y to u. Moreover,

$$V_g(u,y) = \sup\{V_g(z,y) + V_g(u,z)) : z \in \mathsf{S}_a(x)\},$$

since $V_g(u,y)$ is the maximum value of the lower path integral over all piecewise linear paths from y to u and any path p_0 as such will intersect $\mathsf{S}_a(x)$ at some point z and thus gives rise to a piecewise linear path p_1 from y to z and a piecewise linear path p_2 from z to u. Now for fixed $y, x \in$ cl(O) and fixed $z \in \mathsf{S}_a(x)$, the map $\lambda u.V_g(z,y) + V_g(u,z)$ satisfies

$$g \sqsubseteq \frac{d(V_g(z,y) + V_g(\cdot,z))}{du}$$

since $V_g(z,y)$ is a constant and $u \in \mathsf{D}_a(x)$. Thus by Proposition 5.1, we have $g \sqsubseteq \frac{dV_g(\cdot,y)}{du}$. Since $u \in \mathsf{D}_a(x)$ is an arbitrary point, in particular we have $g \sqsubseteq \frac{dV_g(\cdot,y)}{dx}$ and the proof is complete. □