A computable approach to measure and integration theory

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Abstract

We introduce a computable framework for Lebesgue's measure and integration theory in the spirit of domain theory. For an effectively given locally compact second countable Hausdorff space and an effectively given locally finite Borel measure on the space, we define the notion of a computable measurable set with respect to the given measure, which is stronger than Šanin's recursive measurable set. The set of computable measurable subsets is closed under complementation, finite unions and finite intersections. We then introduce interval-valued measurable functions and develop the notion of computable measurable functions using interval-valued simple functions. This leads us to the interval versions of the main results of the theory of Lebesgue integration which provide a computable framework for measure and integration theory. The Lebesgue integral of a computable integrable function with respect to an effectively given (σ-)finite Borel measure on an effectively given (locally) compact second countable Hausdorff space can be computed up to any required accuracy.

Key Words: Domain theory, data type, interval-valued measurable function, interval-valued Lebesgue integral.

Dedication: This paper is dedicated to the historical memory of Jamshid Kashani (d. 1429), the Iranian mathematician who was the first to use the recursive fixed point method in analysis with which he computed $\sin 1^\circ$ correct to 9 sexagesimal places; he also computed π to 16 decimal places [3, pages 7 and 151].

1. Introduction

In the past decades, there have been a wide range of applications of measure and integration theory in different branches of computer science including in probabilistic semantics [20, 15], stochastic hybrid systems [2] and labelled Markov processes [9, 5]. Nevertheless, a systematic general framework for computability in measure and integration theory still remains in its infancy.

Computability of continuous functions and their integrals has been addressed by different schools in computable analysis (for example [18, p 37] and [23, p 182]). In early 1990's, the author developed a domain-theoretic framework for measure and integration theory which gave rise to a generalized Riemann integral [7, 6, 8, 10, 16, 1, 17]. It has provided a computable framework for measure theory and the Riemann theory of integration. However, this only deals with almost everywhere continuous functions. Computability of measures on the unit interval has also been developed in type two theory [22] and, in addition, by using the Prokhorov distance in the metric space of measures [12].

Computability of measurable subsets has a different story. In 1950's, based on the Russian approach to computability in analysis, N. A. Šanin [21] initiated research into computability of measurable sets in Euclidean spaces with respect to the Lebesgue measure. According to his definition, a bounded measurable set is *recursive* if there exists a recursive sequence of "simple" open sets, namely finite unions of bounded rational open intervals, such that the Lebesgue measure of the symmetric difference of the set and the elements of the sequence tends to zero effectively.

The notion of a recursive measurable set is equivalent to that of a *recursively approximable set*, defined by Ker-I Ko in terms of a function-oracle [13]. The measure of the symmetric difference of two sets provides a pseudo-metric on the space of measurable subsets. Thus, Šanin's notion is also at the basis of the approach adopted by researchers in type two theory of computability [25, 24], where an abstract computable measure space is defined as one which is generated by a countable ring of subsets and which is endowed with the pseudo-metric of the measure of symmetric difference.

We aim to develop here a new approach to computability of measurable sets and functions that is motivated by interval analysis and domain theory, where data types for mathematical objects are produced by providing lower and upper bounds for them. As a directly relevant example, in [11] a computable framework for geometric objects is constructed in which a subset of a topological space is approximated from inside and outside by open subsets.

In this paper, a combined measure-theoretic and set-theoretic approach is designed in which a computable measurable set of a countably based locally compact Hausdorff space is given by the intersection of a recursive sequence of open sets containing the set and the union of a recursive sequence of closed sets contained in the set. Our notion of a computable measurable set, which gives approximations to a measurable set both from within and from outside, is stronger than Šanin's recursive measurable sets.

A computable measurable set in our framework is characterized for each positive integer n by a recursively given closed set contained in the set and a recursively given open set containing the set, whose measure differ by less that $1/2^n$. This provides a data type for measurable sets and naturally leads us to consider interval-valued measurable functions, which in turn give us a date type for measurable functions. We derive interval versions of all major results in Lebesgue's theory of integration which also furnish an effective method to compute the Lebesgue integral of a computable measurable function.

It is assumed that the reader is familiar with the basic concepts of recursion theory as in [4], the elements of real number computability, in particular the notions of a computable real number and computable sequences of real numbers together with their main properties, as in [18, Chapter 0] and finally a basic knowledge of measure and integration theory as in [19, 14].

2. Computable measurable sets

In this section we introduce our notion of a computable measurable set for Borel measures on locally compact second countable Hausdorff spaces, which includes the case of the n-dimensional Lebesgue measure on $\mathbb{R}^n$. We first recall a number of definitions.

A Hausdorff topological space is said to be *σ-compact* if there exists an increasing sequence of compact sets $(X_i)_{i\in\omega}$ with $X = \bigcup_{i\in\omega} X_i$.

A measure μ on a topological space is said to be *regular* if for any μ-measurable set A we have:

$$\mu(A) = \inf\{\mu(O) : A \subset O,\ O \text{ open}\}$$

$$= \sup\{\mu(C) : C \subset A,\ C \text{ compact}\}.$$

A Borel measure μ on a Hausdorff space X is said to be *Locally finite* if for any compact subset $K \subset X$ we have $\mu(K) < \infty$.

We start by providing a simple characterization of a measurable set on locally compact second countable Hausdorff spaces.

Theorem 2.1 *Let X be a locally compact second countable Hausdorff space and μ a locally finite Borel measure on X. Then a subset $A \subset X$ is μ-measurable iff for each $\epsilon > 0$ there exist an open set O and a closed set C such that $C \subset A \subset O$ with $\mu(O \setminus C) < \epsilon$.*

Proof The "only if" part follows from Theorem 2.14 (Riesz's Representation Theorem) and Theorems 2.17 and 2.18 in [19] as follows. Since X is second countable, every open set in X is σ-compact, it follows from Theorem 2.18 above that μ is regular. By Theorems 2.14 and 2.17, for the μ-measurable subset A and any $\epsilon > 0$, there exist an open set O and a closed set C such that $C \subset A \subset O$ with $\mu(O \setminus C) < \epsilon$ as required. For the "if part", we put $\epsilon = 1/n$ for any positive integer n. Then there are open and closed sets O_n and C_n such that $C_n \subset A \subset O_n$ with $\mu(O_n \setminus C_n) < 1/n$. Let $F = \bigcup_{n\geq 1} C_n$ and $G = \bigcap_{n\geq 1} O_n$. Then F and G are F_σ and G_δ subsets respectively and we have $F \subset A \subset G$ with $\mu(G \setminus F) = 0$. Hence, A is μ-measurable.□

We will use the following consequence of the above theorem to develop our computability theory for measurable subsets.

Corollary 2.2 *Let X be a locally compact second countable Hausdorff space, with $X = \bigcup_{i\in\omega} X_i$ where each X_i is compact, and μ a locally finite Borel measure on X. Then $A \subset X$ is μ-measurable iff for each $i \geq 0$ and $\epsilon > 0$, there are open sets $U \subset X$ and $V \subset X$ such that*

$$X_i \setminus U \subset A \cap X_i \subset V \cap X_i$$

and

$$\mu(X_i \cap U) + \mu(X_i \cap V) - \mu(X_i) < \epsilon.$$

Proof Suppose A is μ-measurable and $\epsilon > 0$. Then $X_i \cap A$ is μ-measurable for each $i \geq 0$ and by Theorem 2.1, there exist a closed set C and an open set O such that $C \subset A \subset O$ and $\mu(O \setminus C) < \epsilon$. Put $U = X \setminus C$ and $V = O$. Conversely, if for each $i \geq 0$ and $\epsilon > 0$ two open sets U and V with the above properties exist, then by Theorem 2.1, $X_i \cap A$ is μ-measurable, and thus $A = \bigcup_{i\in\omega}(X_i \cap A)$ is also μ-measurable.□

We develop the notion of a computable measurable set by using the effective version of the statement in Corollary 2.2. We need the following basic results about computable sequences of real numbers.

Proposition 2.3 (i) *If $(a_{ijn})_{i,j,n\in\omega}$ is a computable triple sequence of real numbers converging effectively in i, j and n as $j \to \infty$ with $\lim_{j\to\infty} a_{ij,n} = b_{in}$ then $(b_{in})_{i,n\in\omega}$ is a computable double sequence of real numbers.*

(ii) *If $(a_{ij})_{i,j\in\omega}$ is a computable double sequence of real numbers which converges monotonically as $j \to \infty$ to a computable sequence of real numbers $(b_i)_{i\in\omega}$, i.e., $a_{ij} \le a_{i(j+1)}$ for all $i, j \in \omega$ and $b_i = \lim_{j\to\infty} a_{ij}$ then the convergence of $(a_{ij})_{i,j\in\omega}$ is effective in both i and j as $j \to \infty$.*

Proof (i) This is a straightforward extension of the result in [18, page 20] for computable double sequences of real numbers to computable triple sequence of real numbers.

(ii) [18, page 20].□

We first develop the notion of computable measurable sets for a finite Borel measure on a compact second countable Hausdorff space. This will allow us to see the main ideas of the construction before extending the notions to locally finite Borel measures on a locally compact second countable Hausdorff space.

Let X be a compact second countable Hausdorff space.

Definition 2.4 We say that X is effectively given with respect to an effective enumeration $(O_i)_{i\in\omega}$ of a countable basis of open sets closed under finite union and intersection if the following holds:

- $O_0 = \emptyset$ and $O_1 = X$.
- The predicates $O_i \subseteq O_j$ and $\overline{O_i} \subseteq O_j$ are decidable for $i, j \in \omega$.
- There are total recursive functions ϕ and ψ such that $O_i \cup O_j = O_{\phi(i,j)}$ and $O_i \cap O_j = O_{\psi(i,j)}$.
- For each $i \in \omega$, the predicate $X \setminus O_i \subset \bigcup_{1\le m\le n} O_{i_m}$ is decidable for any finite set of integers $i_m \in \omega$ with $1 \le m \le n$, i.e., from any effective covering of the compact subset $X \setminus O_i$ by basic open subsets, one can effectively obtain a finite subcovering.

Since $O_i = \emptyset$ iff $\overline{O_i} \subseteq O_0$, it follows that the equality relation $O_i = \emptyset$ is decidable and we can assume, by redefining the enumeration O, that $O_i = \emptyset$ iff $i = 0$. We note here that it would be possible to drop the requirement for the decidability of $O_i \subseteq O_j$ in (ii) at the expense of some more work. For simplicity though, we choose to keep this condition in our framework.

From our assumptions, it follows that there exists a total recursive function $\alpha : \mathbb{N} \to \mathbb{N}^2$ such that $\vec{O}_{\alpha(j)} := (O_{\alpha_1(j)}, O_{\alpha_2(j)})$ gives an enumeration of covers of X by pairs of basic open sets, i.e., $O_{\alpha_1(j)} \cup O_{\alpha_2(j)} = X$.

Now assume that μ is a finite Borel measure on the effectively given compact second countable space X with its effective enumeration $(O_i)_{i\in\omega}$ of basic open sets. We say that μ is effectively given on X if $(\mu(O_i))_{i\in\omega}$ is a computable sequence of real numbers.

Definition 2.5 We say $A \subset X$ is a *μ-computable measurable* set if there exists a total recursive function $\beta : \mathbb{N}^2 \to \mathbb{N}$ such that the following holds:

(i) The two sequences $(O_{\alpha_1(\beta(j,n))})_{j,n\in\omega}$ and $(O_{\alpha_2(\beta(j,n))})_{j,n\in\omega}$ of open sets are increasing in $j \in \omega$ for fixed $n \in \omega$ and decreasing in n for fixed j.

(ii) For all $n \in \omega$, we have: $(\bigcup_{j\in\omega} O_{\alpha_1(\beta(j,n))})^c \subseteq A \subseteq (\bigcup_{j\in\omega} O_{\alpha_2(\beta(j,n))})$.

(iii) The two sequences $(\mu(O_{\alpha_1(\beta(j,n))}))_{j,n\in\omega}$ and $(\mu(O_{\alpha_2(\beta(j,n))}))_{j,n\in\omega}$ are computable double sequences of real numbers converging effective in j and n as $j \to \infty$.

(iv) For all $n \in \omega$, we have: $\mu(\bigcup_{j\in\omega} O_{\alpha_1(\beta(j,n))}) + \mu(\bigcup_{j\in\omega} O_{\alpha_2(\beta(j,n))}) - \mu(X) < 1/2^n$.

We know by Theorem 2.1 that the conditions in Definition 2.5 above imply that A is μ-measurable. Moreover, we have:

Proposition 2.6 *If A is a computable μ-measurable set then $\mu(A)$ is a computable real number.*

Proof Since the convergence in (iii) above is effective in j and n as $j \to \infty$, it follows from Proposition 2.3 that $\mu(X\setminus\bigcup_{j\in\omega} O_{\alpha_1(\beta(j,n))})$ and $\mu(\bigcup_{j\in\omega} O_{\alpha_2(\beta(j,n))})$ are computable sequences of real numbers, with the first one increasing and the second one decreasing in n. From (ii) and (iv), it follows that the common limit of these two sequences, i.e., $\mu(A)$, is a computable real number.□

Proposition 2.7 *Let $O = \bigcup_{j\in\omega} O_{\gamma(j)}$ where $\gamma : \mathbb{N} \to \mathbb{N}$ is a total recursive function. Then O is a μ-computable open set iff $\mu(O)$ is a computable real number.*

Proof If O is μ-computable then by Proposition 2.6, $\mu(O)$ is a computable real number. Now, for the converse, assume $\mu(O)$ is a computable real number. It suffices to show that there is an effective decreasing sequence of basic open sets whose complements are contained within O and whose μ-measure tend to $\mu(O)$. From γ we can effectively obtain a total recursive function $\delta : \mathbb{N} \to \mathbb{N}$ such that $O = \bigcup_{j\in\omega} O_{\delta(j)}$ and $\overline{O_{\delta j}} \subseteq O_{\delta(j+1)}$ for all $j \in \omega$. Since $(\mu(O_{\delta(j)}))_{j\in\omega}$ increases monotonically to the computable real number $\mu(O)$, the convergence is effective in j. Let n be any positive integer. Since X is effectively compact with respect to the basis $(O_i)_{i\in\omega}$, we can effectively find a finite open covering $X \setminus O_{\delta(j+1)} \subset \bigcup_{1\le m\le n} O_{i_m}$ with $O_{i_k} \cap O_{\delta(j)} = \emptyset$ for $1 \le k \le m$. Using the total recursive function ϕ for binary union (Definition 2.4), we obtain a total recursive function σ such that $X \setminus O_{\delta(j+1)} \subset O_{\sigma(j)} := \bigcup_{1\le m\le n} O_{i_m}$. By construction, we have $\mu(X) - \mu(O_{\sigma(j)}) > \mu(O) - 1/2^n$. Moreover, by putting $\theta(0) := \sigma(0)$ and $O_{\theta(j+1)} := O_{\sigma(j)} \cap O_{\theta(j)} = O_{\psi(\sigma(j),\theta(j))}$, we obtain a total recursive function θ which induces an effective decreasing sequence of basic open sets that shares the above properties of the sequence induced by σ. This completes the proof.□

The above characterization of a μ-computable open set coincides with that in [11].

2.1 Locally compact spaces

We now extend the notions of the previous section to a locally compact second countable Hausdorff space X. Then X is σ-compact and there exists an increasing sequence of compact sets $(X_i)_{i\in\omega}$ with $X = \bigcup_{i\in\omega} X_i$. Let $(O_j)_{j\in\omega}$ be an effective enumeration of a basis of relatively compact open sets, which is closed under non-empty finite intersection and finite union. Then, for each $i \in \omega$, the collection $(O_j \cap X_i)_{j\in\omega}$ is a countable basis of the relative topology for X_i.

Definition 2.8 We say that X is effectively given with respect to $(X_i)_{i\in\omega}$ and $(O_j)_{j\in\omega}$ if the following holds:

(i) $O_0 = \emptyset$.

(ii) The predicates $O_i \subseteq O_j$ and $\overline{O_i} \subseteq O_j$ are decidable for $i, j \in \omega$.

- There are total recursive functions ϕ and ψ such that $O_i \cup O_j = O_{\phi(i,j)}$ and $O_i \cap O_j = O_{\psi(i,j)}$.
- For each $i, j \in \omega$, the predicate $X_i \setminus O_j \subset \bigcup_{1\leq m\leq n} O_{i_m} \cap X_i$ is decidable, i.e., from any effective covering of the compact subset $X_i \setminus O_j$ one can effectively obtain a finite subcovering.

As in the compact case, we can and will assume that $O_i = \emptyset$ iff $i = 0$. From our assumptions, it follows that there exists a total recursive function $\alpha : \mathbb{N}^2 \to \mathbb{N}^2$ such that $\vec{O}_{\alpha(i,j)} := (O_{\alpha_1(i,j)}, O_{\alpha_2(i,j)})$ gives an enumeration of covers of X_i by pairs of basic open sets, i.e., $X_i \subset O_{\alpha_1(i,j)} \cup O_{\alpha_2(i,j)}$.

Assume that μ is a locally finite Borel measure on the effectively given locally compact second countable space X with its effective enumeration $(O_i)_{i\in\omega}$ of basis. We say that μ is effectively given on X if $(\mu(O_j \cap X_i))_{i,j\in\omega}$ is a computable double sequence of real numbers.

Definition 2.9 We say $A \subset X$ is a *μ-computable measurable* set if there exists a total recursive function $\beta : \mathbb{N}^3 \to \mathbb{N}$ such that

(i) The two triple sequences $(O_{\alpha_1(\beta(i,j,n))})_{i,j,n\in\omega}$ and $(O_{\alpha_2(\beta(i,j,n))})_{i,j,n\in\omega}$ of open sets are both increasing in i for fixed j and n, increasing in j for fixed i and n and decreasing in n for fixed i and j.

(ii) For all $i, n \in \omega$, we have: $(X_i \setminus \bigcup_{j\in\omega} O_{\alpha_1(\beta(i,j,n))}) \subseteq X_i \cap A \subseteq (X_i \cap \bigcup_{j\in\omega} O_{\alpha_2(\beta(i,j,n))})$.

(iii) The two sequences $(\mu(X_i \cap O_{\alpha_1(\beta(i,j,n))}))_{i,j,n\in\omega}$ and $(\mu(X_i \cap O_{\alpha_2(\beta(i,j,n))}))_{i,j,n\in\omega}$ are computable triple sequences of real numbers converging effectively in i, j and n as $j \to \infty$.

(iv) For all $i, n \in \omega$, we have:
$\mu(X_i \cap (\bigcup_{j\in\omega} O_{\alpha_1(\beta(i,j,n))})) +$
$\mu(X_i \cap (\bigcup_{j\in\omega} O_{\alpha_2(\beta(i,j,n))})) - \mu(X_i) < 1/2^n$.

We know by Theorem 2.1 that the conditions in Definition 2.9 above imply that A is μ-measurable. Moreover, we have:

Proposition 2.10 *If A is a computable μ-measurable set then $(\mu(A\cap X_i))_{i\in\omega}$ is a computable sequence of real numbers.*

Proof Since the convergence in Definition 2.9(iii) above is effective in i, j and n as $j \to \infty$, it follows from Proposition 2.3 that $\mu(X_i \setminus \bigcup_{j\in\omega} O_{\alpha_1(\beta(i,j,n))})$ and $\mu(X_i \cap \bigcup_{j\in\omega} O_{\alpha_2(\beta(i,j,n))})$ are computable double sequences of real numbers, with the first one increasing and the second one decreasing in n for fixed i. From (iv) it follows that these two double sequences of real numbers converge effectively in n and i to $\mu(A \cap X_i)$ as $n \to \infty$. Thus, by Proposition 2.3, $(\mu(A \cap X_i))_{i\in\omega}$ is a computable sequence of real numbers. □

Proposition 2.11 *Let O be a recursive union of basic open sets. Then O is a μ-computable open set iff $(\mu(X_i\cap O))_{i\in\omega}$ is a computable sequence of real numbers.*

Proof If O is μ-computable then by Proposition 2.10, $(\mu(X_i \cap O))_{i\in\omega}$ is a computable sequence of real numbers. Now, for the converse, let $O = \bigcup_{j\in\omega} O_{\gamma(j)}$ where the sequence of open sets is increasing and $\gamma : \mathbb{N} \to \mathbb{N}$ is a total recursive function and assume $(\mu(X_i \cap O))_{i\in\omega}$ is a computable sequence of real numbers. Since $\bigcup_{j\in\omega} O_{\gamma(j)}$ can be used as the first sequence of open sets in Definition 2.9, it suffices to construct the second sequence. From γ we can effectively obtain a total recursive function $\delta : \mathbb{N} \to \mathbb{N}$ such that $O = \bigcup_{j\in\omega} O_{\delta(j)}$ and $\overline{O_{\delta(j)}} \subseteq O_{\delta(j+1)}$ for all $j \in \omega$. Consider the computable double sequence of real numbers $(\mu(X_i\cap O_{\delta(j)})_{i,j\in\omega}$. It monotonically converges to the computable sequence of real numbers $(\mu(X_i \cap O))_{i\in\omega}$. Thus, by Proposition 2.3, the convergence is effective in i and j. Let n be any positive integer. Since X is effectively locally compact with respect to the basis $(O_j)_{j\in\omega}$ and the sequence of compact subsets $(X_i)_{i\in\omega}$, we can effectively find a finite open covering $X_i \setminus O_{\delta(j+1)} \subset \bigcup_{1\leq m\leq N} O_{t_m}$ with $O_{t_m} \cap O_{\delta(j)} = \emptyset$ for $1 \leq m \leq N$. Using the total recursive function ϕ for binary union (Definition 2.8), we obtain a total recursive function σ such that

$X_i \setminus O_{\delta(j+1)} \subset O_{\sigma(j)} := \bigcup_{1\le m\le N} O_{t_m}$. By construction, we have $\mu(X_i) - \mu(O_{\sigma(j)} \cap X_i) > \mu(O \cap X_i) - 1/2^n$. Moreover, by putting $\theta(0) := \sigma(0)$ and $O_{\theta(j+1)} := O_{\sigma(j)} \cap O_{\theta(j)} = O_{\psi(\sigma(j),\theta(j))}$, for $j \ge 0$, we obtain a total recursive function θ which induces an effective decreasing sequence of basic open sets which shares the above properties of the sequence induced by σ. Since the construction is effective in i, this completes the proof. □

It follows from Definition 2.9 that any total recursive function $\beta : \mathbb{N}^3 \to \mathbb{N}$ which satisfies the following three conditions:

- the two sequences $(O_{\alpha_1(\beta(i,j,n))})_{i,j,n\in\omega}$ and $(O_{\alpha_2(\beta(i,j,n))})_{i,j,n\in\omega}$ of open sets are increasing in i for fixed j and n, increasing in j for fixed i and n and decreasing in n for fixed i and j,
- the two sequences $(\mu(X_i \cap O_{\alpha_1(\beta(i,j,n))}))_{i,j,n\in\omega}$ and $(\mu(X_i \cap O_{\alpha_2(\beta(i,j,n))}))_{i,j,n\in\omega}$ are computable triple sequences of real numbers converging effectively in i, j and n as $j \to \infty$,
- for all $i, n \in \omega$, we have the relation:
  $\mu(X_i \cap (\bigcup_{j\in\omega} O_{\alpha_1(\beta(i,j,n))})) +$
  $\mu(X_i \cap (\bigcup_{j\in\omega} O_{\alpha_2(\beta(i,j,n))})) - \mu(X_i) < 1/2^n$,

defines an equivalence class of μ-computable measurable sets which differ by a null set. Two canonical representatives of this class are given by the G_δ set $\bigcap_{n\in\omega}\bigcup_{i,j\in\omega} O_{\alpha_2(\beta(i,j,n))}$ and the F_σ set $(\bigcap_{n\in\omega}\bigcup_{i,j\in\omega} O_{\alpha_1(\beta(i,j,n))})^c$.

Moreover, the two parameter family of pairs of closed and relatively open sets in Definition 2.9(ii), for $i, n \in \omega$, represent a data-type for any member A of this equivalence class where $(X_i \setminus \bigcup_{j\in\omega} O_{\alpha_1(\beta(i,j,n))}) \subseteq X_i \cap A \subseteq (X_i \cap \bigcup_{j\in\omega} O_{\alpha_2(\beta(i,j,n))})$ and the measure of the relatively open subset and the closed subset in each pair differ by at most $1/2^n$, as it follows from 2.9(iv).

Proposition 2.12 (i) *The complement of a computable measurable set is another computable measurable set.*

(ii) *A finite union or intersection of computable measurable subsets is a computable measurable subset.*

Proof (i) Interchange 1 and 2 in the indices of α and β in Definition 2.9

(ii) This follows easily using the total recursive functions ϕ and ψ for binary union and binary intersection of basic open sets. intersection. □

However, computable measurable subsets are not closed under countable union or intersection as the following example shows.

Example 2.13 Consider the Lebesgue measure λ on the real line and let $(r_k)_{k\in\omega}$ be an effective increasing sequence of positive rational numbers converging to a left-computable but non-computable real number $r \in \mathbb{R}$. (Such a sequence can be constructed from a recursively enumerable but non-recursive subset of natural numbers.) Then by Proposition 2.7, the open interval $(0, r) = \bigcup_{k\in\omega}(0, r_k)$ is not μ-computable though for each $k \in \omega$ the open interval $(0, r_k)$ is λ-computable.

Finally, for later use, we define the notion of a computable sequence $(A_k)_{k\in\omega}$ of computable measurable functions by requiring that the properties required in Definition 2.9 for the computable measurable sets A_k for each $k \in \omega$ hold effectively in $k \in \omega$:

Definition 2.14 We say $(A_k)_{k\in\omega}$ is a *computable sequence* of μ-computable measurable sets if there exists a total recursive function $\beta : \mathbb{N}^4 \to \mathbb{N}$ such that the following four conditions hold:

(i) The two sequences $(O_{\alpha_1(\beta(i,j,n,k))})_{i,j,n,k\in\omega}$ and $(O_{\alpha_2(\beta(i,j,n,k))})_{i,j,n,k\in\omega}$ of open sets are increasing in i for fixed j, n and k, increasing in j for fixed i, n and k, increasing in i for fixed j, n and k and decreasing in n for fixed i, j and k.

(ii) For all $i, n, k \in \omega$, we have:
$(X_i \setminus \bigcup_{j\in\omega} O_{\alpha_1(\beta(i,j,n,k))}) \subseteq X_i \cap A_k \subseteq$
$(X_i \cap \bigcup_{j\in\omega} O_{\alpha_2(\beta(i,j,n,k))})$.

(iii) The two sequences $(\mu(X_i \cap O_{\alpha_1(\beta(i,j,n,k))}))_{i,j,n,k\in\omega}$ and $(\mu(X_i \cap O_{\alpha_2(\beta(i,j,n,k))}))_{i,j,n,k\in\omega}$ are computable quartic sequences of real numbers converging effectively in i, j, n and k as $j \to \infty$.

(iv) For all $i, n, k \in \omega$, we have:
$\mu(X_i \cap (\bigcup_{j\in\omega} O_{\alpha_1(\beta(i,j,n,k))})) +$
$\mu(X_i \cap (\bigcup_{j\in\omega} O_{\alpha_2(\beta(i,j,n,k))})) - \mu(X_i) < 1/2^n$.

3 Measurable functions

Let $(X, \mathcal{M})$ be a measure space with the underlying set X and a σ-algebra $\mathcal{M}$ of subsets of X. We work with such a general space first to develop the notions of interval-valued measurable functions. Later, in order to develop a computability theory for measurable functions, we assume that X is a locally compact second countable Hausdorff space, equipped with its σ-algebra of measurable subsets induced by a Borel measure μ on X, i.e. $\mathcal{M}$ will be the set of all μ-measurable subsets of X.

Given any topological space Y, we say that a function $f : X \to Y$ is measurable if $f^{-1}(B) \in \mathcal{M}$ for any Borel subset $B \subset Y$. Let $\mathbf{I}\mathbb{R}$ be the domain of the non-empty compact intervals of the real line ordered by reverse inclusion, equipped with its σ-algebra of Borel subsets induced from the Scott topology.

Consider the set $X \to_m \mathbf{I}\overline{\mathbb{R}}$ of measurable functions $f : X \to \mathbf{I}\overline{R}$, where $\overline{R}$ is the extended real line, i.e., the two point compactification $[-\infty, \infty]$ of $\mathbb{R}$, where the basic open sets are of the form (a, b), $[-\infty, b)$ and $(a, \infty]$, with $a, b \in \mathbb{R}$. Each such function is determined by the extended real-valued lower and upper parts f^- and f^+ of f defined such that for each $x \in X$ we have $f(x) = [f^-(x), f^+(x)]$.

Proposition 3.1 *We have* $f \in (X \to_m \mathbf{I}\overline{\mathbb{R}})$ *iff* f^-, f^+ *are measurable as extended real valued functions.*

Since the supremum, respectively infimum, of an increasing, respectively decreasing, sequence of real-valued measurable functions is measurable, the poset $X \to_m \mathbf{I}\overline{\mathbb{R}}$ is ω-bi-complete, i.e. the supremum (respectively infimum) of any increasing (respectively decreasing) sequence of interval valued measurable functions is an interval-valued measurable function. Similarly, since the supremum (respectively infimum) of any (finite or) countable set of measurable functions is measurable, it follows that $X \to_m \mathbf{I}\overline{\mathbb{R}}$ is ω-inf complete and bounded ω-sup complete.

Given a sequence of intervals $x_i \in \mathbf{I}\overline{\mathbb{R}}$, $i \in \omega$ and $x \in \mathbf{I}\overline{\mathbb{R}}$, we write $\lim_{i\to\infty} x_i = x$ if $x^- = \lim_{i\to\infty} x_i^-$ and $x^+ = \lim_{i\to\infty} x_i^+$ both exist in $\overline{\mathbb{R}}$ with respect to its compact topology.

Furthermore, we introduce the $\liminf - \limsup$ operation on sequences in $\mathbf{I}\overline{\mathbb{R}}$ which we denote by $\lim^*$:

$$\begin{array}{rcl} \lim^* : (\mathbf{I}\overline{\mathbb{R}})^\omega & \to & \mathbf{I}\overline{\mathbb{R}} \\ (x_i)_{i\in\omega} & \mapsto & [\liminf_{i\to\infty} x_i^-, \limsup_{i\to\infty} x_i^+] \end{array}$$

Note that $\lim^*_{i\to\infty} x_i$ for $x_i \in \mathbf{I}\overline{\mathbb{R}}$ is precisely the set of all limits of convergent sequences $(a_i)_{i\in\omega}$ with $a_i \in x_i$. Note that $\lim^*$ is monotone but not continuous.

This induces a $\liminf - \limsup$ operation on $(X \to_m \mathbf{I}\overline{\mathbb{R}})$ as follows:

$$\begin{array}{rcl} \lim^* : (X \to_m \mathbf{I}\overline{\mathbb{R}})^\omega & \to & (X \to_m \mathbf{I}\overline{\mathbb{R}}) \\ (f_i)_{i\in\omega} & \mapsto & [\liminf_{i\to\infty} f_i^-, \limsup_{i\to\infty} f_i^+] \end{array}$$

Thus, $(X \to_m \mathbf{I}\overline{\mathbb{R}})$ is also $\lim^*$-*complete* since it has limits of all countable sequences of intervals. If indeed $f^- = \lim_{i\to\infty} f_i^-$ and $f^+ = \lim_{i\to\infty} f_i^+$ both exist then we write $\lim_{i\to\infty} f_i = f = [f^-, f^+]$. Clearly in this case $\lim_{i\to\infty} f_i = \lim^*_{i\to\infty} f_i$.

For a subset $A \subset X$, let $\chi_A : X \to \{0, 1\}$ be the characteristic function of the set A, i.e., $\chi_A(x) = 1$ iff $x \in A$. In analogy with simple functions in classical measure theory on the one hand and step functions in domain theory on the other hand, we define:

Definition 3.2 Let $A_i \subset X$ be measurable subsets for $1 \le i \le n$ and let $\alpha_i \in \mathbf{I}\mathbb{R}$ be real intervals for $1 \le i \le n$. Then

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i} : X \to \mathbf{I}\mathbb{R}$$

is called an *interval-valued simple function*.

Note that $s = [s^-, s^+]$ where $s^\pm : X \to \mathbb{R}$ with $s^\pm = \sum_{i=1}^n \alpha^\pm \chi_{A_i}$ are both measurable functions. It follows that s is an interval-valued measurable function. Note also that we exclude extended real intervals from the definition of a simple function and that, as in the classical case, s takes only a finite number of values and does not depend on the particular representation in terms of measurable sets A_i's and intervals α_i's. There is indeed a canonical representation of s for which the α_i's are precisely the distinct non-zero values of s and A_i is precisely the set where s takes value α_i. We define the *order* $o(s)$ of s to be the number of distinct non-zero values of s. Using the canonical representation of s we also define the *width* $w(s)$ of s as the maximum length of the intervals α_i, i.e., $w(s) = \max\{\alpha_i^+ - \alpha_i^- : 1 \le i \le n\}$. Finally, for the canonical representation, we define the *maximum absolute value* of s by $m(s) = \max\{|\alpha_i^-|, |\alpha_i^+| : 1 \le i \le n\}$.

We say $f : X \to \mathbf{I}\overline{\mathbb{R}}$ is *bounded* by a compact interval $K \in \mathbf{I}\mathbb{R}$, if for all $x \in X$ we have: $K \sqsubseteq f(x)$; we denote this by $K \sqsubseteq f$. We first deal with bounded measurable functions; unbounded functions are addressed in the end of this section.

Proposition 3.3 *Every bounded real-valued measurable function is the supremum of an increasing sequence of interval-valued simple functions* s_n *with* $o(s_n) \le c\,2^n$, *where* c *is a positive constant independent of* n, *and* $w(s_n) \le 1/2^n$.

Proof Let the real-valued measurable $f : X \to \mathbb{R}$ be bounded so that $|f| \le M$ for some $M \ge 0$. Let m be the least non-negative integer such that $M \le 2^m$. For a positive integer n and $-2^{m+n} + 1 \le k \le 2^{m+n}$ let

$$A_{nk} =$$

$$\{x \in X : \frac{(k-1)}{2^n} \le f(x)\} \cap \{x \in X : f(x) < \frac{k}{2^n}\},$$

which is measurable as f is measurable. Let

$$s_n = \sum_{k=-2^{(m+n)}+1}^{2^{(n+m)}} \left[\frac{(k-1)}{2^n}, \frac{k}{2^n}\right] \chi_{A_{nk}}.$$

Then we have $f = \bigsqcup_{n\ge 0} s_n$ with $w(s_n) \le 1/2^n$ with $o(s_n) = 2^{n+m+1}$ as required. □

The above proposition can easily be extended to bounded interval-valued measurable functions of type $X \to \mathbf{I}\mathbb{R}$.

3.1 Computable measurable functions

An effective version of Proposition 3.3 provides us with the notion of a computable real-valued measurable function and our date type for such functions. Let X be an effectively given locally compact second countable Hausdorff space as in Section 2.

Definition 3.4 (i) A simple function

$$s = \sum_{i=1}^{k} \alpha_i \chi_{A_i} : X \to \mathbf{I}\mathbb{R}$$

is *computable* if for $1 \leq i \leq k$, the subset $A_i \subset X$ is a computable measurable set and the real numbers α_i^- and α_i^+ are computable real numbers.

(ii) We say that $f : X \to \mathbb{R}$ is a computable bounded measurable function if there is an effective increasing sequence of computable simple functions $s_n : X \to \mathbf{I}\mathbb{R}$ with $f = \bigsqcup_{n\in\omega} s_n$ such that

- there is an effectively given non-negative integer $M \geq 0$ with $m(s_n) < M$ for all $n \in \omega$,
- $w(s_n) \leq 1/2^n$ for all $n \in \omega$, and,
- $o(s_n) \leq c\,2^n$ for some effectively given positive constant c independent of $n \in \omega$.

Three remarks regarding the notion of a computable simple function are in order.

(i) Firstly, note that Definition 3.4(i) of a computable simple function is independent of the choice of the representative of s as the complement and the finite union of computable measurable sets are both computable measurable sets and also the sum of two computable real numbers is another computable real number.

(ii) For a computable simple function s the real numbers $w(s)$ and $m(s)$, defined with respect to the canonical representation, are computable real numbers.

(iii) We note that for a classical simple function $s = \sum_{i=1}^{k} a_i \chi_{A_i} : X \to \mathbb{R}$ with $a_i \in \mathbb{R}$ the two definitions of computability in parts (i) and (ii) of Definition 3.4 are consistent. Indeed, if A_i's are computable measurable sets and a_i's are computable real numbers, so that s is a computable simple function according to Definition 3.4(i), then putting $s_n = s$ for all $n \in \omega$ we see that s is computable as a measurable function in the sense of Definition 3.4(ii). On the other hand, suppose the simple function s with canonical representation $s = \sum_{i=1}^{k} a_i \chi_{A_i} : X \to \mathbb{R}$ is a computable measurable function in the sense of Definition 3.4(ii). Let $V = \{0\} \cup \{a_i : 1 \leq i \leq k\}$ and put $r = \min\{|v - w| : v, w \in V \text{ with } v \neq w\}$. By assumption, there is an increasing sequence of computable simple functions s_n with $s = \bigsqcup_n s_n$ and $w(s_n) \leq 1/2^n$. Fix i with $1 \leq i \leq k$, and let n be such that $1/2^n < r/2$. Assume $s_n = \sum_{t=1}^{m_n} \beta_{nt} \chi_{B_{nt}}$. Then we have

$$A_i = \bigcup_{a_i \in \alpha_{nt}} B_{nt}$$

and it follows that A_i is the finite union of computable measurable subsets and is thus a computable measurable subset by Proposition 2.12. Moreover, for any t such that $B_{nt} \subset A_i$ we have $\beta_{nt}^- \leq a_i \leq \beta_{nt}^+$ with $\beta_{nt}^+ - \beta_{nt}^- < 1/2^n$. Since n can be chosen arbitrarily large and since β_{nt}^+ and β_{nt}^- are computable real numbers, it follows that a_i is a computable real number.

Proposition 3.5 *The maximum and minimum of a finite number of bounded real-valued computable measurable functions are bounded computable measurable functions.*

Consider the increasing sequence of bounded computable measurable functions $f_n = (\chi_{(0,r_k)})_{k\in\omega}$, where $(r_k)_{k\in\omega}$ is the sequence of rational numbers in Example 2.13. Since $\sup_n f_n = \chi_{(0,r)}$, we see that the supremum of a countable set of bounded computable measurable functions is not necessarily computable.

3.2 Unbounded measurable functions

The definitions and results above for bounded measurable functions can be extended to unbounded functions as follows. A *countably valued simple function* is a measurable function $f : X \to \mathbb{R}$ which takes countably many distinct values; this class of functions has been used in classical measure and integration theory [14]. We now introduce a subclass of countably valued simple functions as follows. A countably valued simple function is *locally finite* if in any bounded region in $\mathbb{R}$ it only takes a finite number of distinct values. These definitions extend to interval-valued functions of type $X \to \mathbf{I}\mathbb{R}$. A locally finite interval-valued simple function is a countably valued simple function such that any bounded region in $\mathbb{R}$ intersects only a finite number of its distinct values. For a locally finite simple function s and for a compact interval K of the real line, we denote by $o(s, K)$ the number of distinct non-zero values of s which intersect K. For any measurable function $f : X \to \mathbb{R}$, there is an increasing sequence of locally finite interval-valued simple functions with supremum f. Indeed, for any $n \in \omega$, let

$$s_n = \sum_{k\in\mathbb{Z}} \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \chi_{A_k},$$

where $A_k = \{x \in X : f(x) \in [k/2^n, (k+1)/2^k)$. Then A_k's are measurable sets and we have $f = \bigsqcup_{n\in\omega} s_n$ where

s_n is locally finite. The definitions of the previous section can then be extended to unbounded measurable functions as follows.

We say that a locally finite simple function $s = \sum_{k\in\omega} \alpha_k \chi_{A_k}$ is *computable* if $(A_k)_{k\in\omega}$ is a computable sequence of computable measurable subsets (see Definition 2.14), and $(\alpha_k^-)_{k\in\omega}$ and $(\alpha_k^+)_{k\in\omega}$ are both computable sequences of real numbers. Finally, we say a measurable map is computable if there is an effective increasing sequence of computable locally finite simple functions $s_n : X \to \mathbf{I}\mathbb{R}$ with $f = \bigsqcup_{n\in\omega} s_n$ and $w(s_n) \leq 1/2^n$ such that $o(s_n, [N, N+1]) \leq c\,2^n$ for some effectively given positive constant c independent of n and N. All of the results of the previous section can be extended to the unbounded case.

4 Interval Lebesgue Integral

We are now in a position to define the notion of interval Lebesgue integral as a map $\int : (X \to_m \mathbf{I}\overline{\mathbb{R}}) \to \overline{\mathbb{R}}$ with respect to a measure μ on the measure space $(X, \mathcal{M})$. Later in this section, in order to develop a computability theory, we work with a Borel measure μ on a compact or a locally compact second countable Hausdorff space X.

For a simple function $s \in (X \to_m \mathbf{I}\overline{\mathbb{R}})$ with a representative $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i} : X \to \mathbf{I}\mathbb{R}$, which vanishes outside a set of finite measure, we define the μ-*integral* of s as:

$$\int_X s\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i).$$

It follows that $\int_X s\, d\mu = [\int_X s^-\, d\mu, \int_X s^+\, d\mu]$. Thus, as in the classical case, the integral of a simple function is independent of its representative. If $E \subset X$ is measurable, then $s \cdot \chi_E = \sum_{i=1}^{n} \alpha_i \chi_{A_i \cap E}$ is also a simple function and, as in the classical case, when $\mu(E) < \infty$, we define:

$$\int_E s\, d\mu = \int_X s \cdot \chi_E\, d\mu.$$

We also immediately deduce the following. If $s$and t are simple interval-valued functions which are zero outside a set of measure zero then, for compact intervals $a, b \in \mathbf{I}\mathbb{R}$:

(i) $\int (as + bt)\, d\mu = a \int s\, d\mu + b \int t\, d\mu$.

(ii) If $s \sqsubseteq t$ holds a.e., then $\int s\, d\mu \sqsubseteq \int t\, d\mu$.

Now we deal with measurable functions. We first consider a measurable function $f \in (X \to_m \mathbf{I}\overline{\mathbb{R}})$ bounded on a set E with finite measure and define:

Definition 4.1 The Lebesgue integral of any bounded interval-valued measurable function f on a measurable subset E with respect to a measure μ on X, such that $\mu(E) < \infty$, is defined as:

$$\int_E f\, d\mu = \bigsqcup \{ \int_E s\, d\mu : \text{ simple } s \sqsubseteq f \}.$$

We have the equality: $\int_E f\, d\mu = [\int_E f^-\, d\mu, \int_E f^+\, d\mu]$. We usually write $\int f\, d\mu$ for $\int_X f\, d\mu$. The following results easily follows as in the classical case.

Proposition 4.2 *If f and g are bounded measurable interval-valued functions which vanish outside a set of finite measure then:*

(i) $\int (af + bg)\, d\mu = a \int f\, d\mu + b \int g\, d\mu$.

(ii) *If $f \sqsubseteq g$ holds a.e., then $\int f\, d\mu \sqsubseteq \int g\, d\mu$.*

(iii) *If A and B are disjoint measurable subsets then*

$$\int_{A\cup B} f\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu.$$

We can now obtain in a straightforward way the interval version of some of the classical results in measure theory. Recall the definitions of $\lim$ and $\lim^*$ in $(X \to_m \mathbf{I}\overline{\mathbb{R}})$.

Proposition 4.3 Bounded Convergence Theorem *Let $f_n \in (X \to_m \mathbf{I}\overline{\mathbb{R}})$ be a sequence of uniformly bounded measurable functions, which vanish outside a set of finite measure, such that $\lim_{n\to\infty} f_n = f$ exists. Then $\int \lim_{n\to\infty} f_n\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu$.*

Corollary 4.4 Monotone Convergence Theorem *Let $f_n \in (X \to_m \mathbf{I}\overline{\mathbb{R}})$ be an increasing sequence of measurable functions, which vanish outside a set of finite measure, with f_0 bounded. Then $\bigsqcup_{n\in\omega} \int f_n\, d\mu = \int (\bigsqcup_{n\in\omega} f_n)\, d\mu$.*

Finally, from the Bounded Convergence Theorem above we obtain the interval version of Fatou's lemma.

Lemma 4.5 Fatou's Lemma *Let $f_n \in (X \to_m \mathbf{I}\overline{\mathbb{R}})$ be a sequence of uniformly bounded measurable functions, which vanish outside a set of finite measure. Then $\int \lim^* f_n\, d\mu \sqsubseteq \lim^* \int f_n\, d\mu$.*

4.1 Computability of Lebesgue integral

We now assume X is an effectively given second countable compact Hausdorff space and μ is an effectively given finite Borel measure μ on it as described in Section 2. Recall that we have an effective enumeration $(O_j)_{j\in\omega}$ of a countable basis of X with $X = O_1$ such that $(\mu(O_j))_{j\in\omega}$ is a computable sequence of real numbers. The following theorem, which is our main result, brings together and uses all the results in the previous sections, on computable measurable sets and functions and on the interval-valued Lebesgue integral, for the case of a compact second countable Hausdorff space.

Theorem 4.6 *Suppose f is a bounded computable real-valued measurable function on X. Then the Lebesgue integral of f with respect to μ is computable, i.e., given any positive integer k we can effectively compute the Lebesgue integral of f up to $1/2^k$ accuracy.*

Proof Let M be an effectively given bound for f and $(s_n)_{n\in\omega}$ be the increasing sequence of computable simple functions in $(X \to_m \mathbf{I}\overline{\mathbb{R}})$ which witnesses the computability of f according to Definition 3.4 with effectively given constant $c > 0$. By the interval version of the Monotone Convergence Theorem (Corollary 4.4), we know that $\int f\,d\mu = \bigsqcup_{n\in\omega}\int s_n\,d\mu$, which means that the required integral lies in each compact interval of the shrinking sequence of compact intervals given by the integrals of the simple functions. Our task is to effectively find n such that $\int s_n\,d\mu$ provides the required estimate. In fact, using the canonical representation:

$$s_n = \sum_{i=1}^{o(s_n)} \alpha_i \chi_{A_i}$$

we obtain $\int s_n\,d\mu = \sum_{i=1}^{o(s_n)} \alpha_i \mu(A_i)$. Since each A_i is a computable measurable subset, we can effectively obtain for each nonnegative integer p an open set O_{ip} and a closed set C_{ip} such that $C_{ip} \subset A_i \subset O_{ip}$ with $\mu(O_{ip})$ and $\mu(C_{ip})$ computable real numbers satisfying $\mu(O_{ip}) - \mu(C_{ip}) < 1/2^p$. We have

$$\sum_{i=1}^{o(s_n)} \alpha_i^- \mu(C_{ip}) \leq \int s_n\,d\mu \leq \sum_{i=1}^{o(s_n)} \alpha_i^+ \mu(O_{ip}).$$

Estimating the difference between the two sums above we have:

$$\sum_{i=1}^{o(s_n)} \alpha_i^+ \mu(O_{ip}) - \alpha_i^- \mu(C_{ip})$$

$$= \sum_{i=1}^{o(s_n)} \alpha_i^+ \mu(O_{ip}) - \alpha_i^+ \mu(C_{ip}) + \alpha_i^+ \mu(C_{ip}) - \alpha_i^- \mu(C_{ip})$$

$$= \sum_{i=1}^{o(s_n)} \alpha_i^+ (\mu(O_{ip}) - \mu(C_{ip})) + (\alpha_i^+ - \alpha_i^-)\mu(C_{ip})$$

$$\leq \sum_{i=1}^{o(s_n)} \frac{m(s_n)}{2^p} + w(s_n)\mu(C_{ip}) \leq \frac{m(s_n)o(s_n)}{2^p} + \frac{\mu(X)}{2^n},$$

since $w(s_n)$ is bounded by $1/2^n$ and C_{ip}'s, being contained in the disjoint sets A_i, are disjoint for fixed p and $1 \leq i \leq o(s_n)$ and their total μ-measure is therefore bounded by $\mu(X)$. Thus, for our estimate, we conclude that

$$\sum_{i=1}^{o(s_n)} \alpha_i^+ \mu(O_{ip}) - \alpha_i^- \mu(C_{ip}) \leq \frac{m(s_n)c2^n}{2^p} + \frac{\mu(X)}{2^n},$$

since $o(s_n)$ is bounded by $c2^n$.

Note that $\mu(X)$ is a computable number as $X = O_1$; we can thus effectively obtain a nonnegative integer t such that $\mu(X) < 2^t$. Let the positive integer k be given and put $n = k + t + 1$ and put

$$p = \lceil \log m(s_{k+t+1}) \rceil + \lceil \log c \rceil + 2k + t + 2.$$

Using the above effectively obtained n and p, we get

$$\sum_{i=1}^{o(s_n)} \alpha_i^+ \mu(O_{ip}) - \alpha_i^- \mu(C_{ip})$$

$$\leq 1/2^{k+1} + 1/2^{k+1} = 1/2^k.$$

It follows that for the above values of n and p the computable real number

$$\sum_{i=1}^{o(s_n)} \alpha_i^+ \mu(O_{ip})$$

is within $1/2^k$ of the value of the integral $\int f\,d\mu$. $\square$

4.2 Integral of unbounded functions

The interval valued Lebesgue integral can be defined for measurable functions on a general measure space $(X, \mathcal{M})$ as in Section 3. We have shown there that any interval valued measurable function is the supremum of an increasing sequence of locally finite interval valued simple functions. Here, we first assume that our measurable functions are defined on a set $E \in \mathcal{M}$ of finite measure $\mu(E) < \infty$. We say that an infinite sum of compact intervals $\sum_{i\in\omega}\beta_i$ *converges absolutely* if the two infinite sums $\sum_{i\in\omega}\beta_i^-$ and $\sum_{i\in\omega}\beta_i^+$ of real numbers converge absolutely. In this case we write $\sum_{i\in\omega}\beta_i = [\sum_{i\in\omega}\beta_i^-, \sum_{i\in\omega}\beta_i^+]$. The Lebesgue integral of a countably valued, interval valued simple function

$$s = \sum_{i\in\omega} \alpha_i \chi_{A_i} : X \to \mathbf{I}\mathbb{R}$$

which vanishes outside A is defined as

$$\int s\,d\mu = \sum_{i\in\omega} \alpha_i \mu(A_i)$$

provided the infinite sum of compact intervals above converges absolutely. In this case we say s is *integrable*. The Lebesgue integral of any interval-valued measurable function f on E is defined as:

$$\int_E f\,d\mu$$

$$= \bigsqcup \{ \int_E s\,d\mu : \text{ integrable countably valued simple } s \sqsubseteq f \}.$$

One can then extend the basic results of Lebesgue integration to the interval case. In particular, we have:

Theorem 4.7 Lebesgue's bounded convergence theorem *Let $(f_n)_{n\in\omega}$ be a convergent sequence of interval-valued measurable functions defined on E with $f = \lim_{n\to\infty} f_n$ and suppose that there is an integrable interval-valued function g also defined on E with $\mu(E) < \infty$ and with $g \sqsubseteq f_n$ for all $n \in \omega$. Then, f is integrable on E and $\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$.*

Using the notion of a computable measurable function as in Section 3 on an effectively given second countable compact Hausdorff space, which employs the set of locally finite simple function, we also have the following generalization of Theorem 4.6 to unbounded functions.

Theorem 4.8 *Suppose f is a computable real-valued measurable function on an effectively given second countable compact Hausdorff space X then the Lebesgue integral of f with respect to any effectively given finite measure on X is computable, i.e., given any positive integer k we can effectively compute the Lebesgue integral of f up to $1/2^k$ accuracy.*

Finally, we consider the interval version of the Lebesgue integral on a measure space with respect to a measure which is σ-finite, i.e., $X = \bigcup_{i\in\omega} E_i$ with $\mu(E_i) < \infty$ for each $i \in \omega$. Such a sequence of measurable subsets in called *exhausting* [14]. An interval-valued measurable function would then be integrable if it is integrable on each measurable set of finite measure and if $\lim_{i\to\infty} \int_{E_i} f \, d\mu$ exists and is a compact interval for any exhausting sequence $(E_i)_{i\in\omega}$. As in the classical theory, the limit will be independent of the exhausting sequence and we define it as the Lebesgue integral of f.

This framework will then enable us to develop a computability theory, as in Theorem 4.8, for an effectively given σ-finite measure on an effectively given second countable locally compact Hausdorff space.

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