THE GENERALIZED RIEMANN INTEGRAL ON LOCALLY COMPACT SPACES

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Abstract

We extend the basic results on the theory of the generalized Riemann integral to the setting of bounded or locally finite measures on locally compact second countable Hausdorff spaces. The correspondence between Borel measures on X and continuous valuations on the upper space UX gives rise to a topological embedding between the space of locally finite measures and locally finite continuous valuations, both endowed with the Scott topology. We construct an approximating chain of simple valuations on the upper space of a locally compact space, whose least upper bound is the given locally finite measure. The generalized Riemann integral is defined for bounded functions with respect to both bounded and locally finite measures. Also in this setting, generalized R-integrability for a bounded function is proved to be equivalent to the condition that the set of its discontinuities has measure zero. Furthermore, if a bounded function is R-integrable then it is also Lebesgue integrable and the two integrals coincide. Finally, we extend R-integration to an open set and provide a sufficient condition for the computability of the integral of a bounded almost everywhere continuous function.

1 Introduction

Domain theory was introduced by Dana Scott in 1970 ([27]) as a foundational basis for the semantics of computation.

In [6], a basic connection between domain theory and some main branches of mathematics has been established, giving rise, in particular, to a novel computational approach to measure theory and integration. A domain-theoretic framework for measure and integration has been provided in [7], by proving that any bounded Borel measure on a compact metric space X can be obtained as the least upper bound of simple valuations (measures) on the upper space UX, the set of non-empty compact subsets of X ordered by reverse inclusion.

Simple valuations approximating the given measure play the role of partitions in Riemann integration and are used to obtain increasingly better approximations of the integral of a bounded real-valued function on a compact metric space. Thus, instead of approximating the *function* with simple functions as is done in Lebesgue theory, the *measure* is approximated with simple measures. This idea leads to a new notion of integration, called generalized Riemann integration, R-integration for short, similar in spirit but more general than Riemann integration.

All the basic results of the theory of Riemann integration can be extended to this setting. For instance, it is proved in [7] that a bounded real-valued function on a compact metric space is R-integrable with respect to a bounded Borel measure if and only if its set of discontinuities has measure zero and that if the function is R-integrable then it is also Lebesgue integrable and the two integrals coincide.

This theory has had applications in exact computation of integrals [8], in the semantics of programming languages [12], in the 1-dimensional random fields Ising model in statistical physics [9], in forgetful neural networks [10], in stochastic processes [5] and in chaos theory [8].

Apart from the domain-theoretic integral, there are two other notions of generalized Riemann integrals in the literature, namely, the McShane and the Henstock integrals [25, 15]. These are basically integrals for real valued functions on $\mathbb{R}$. Their generalizations to $\mathbb{R}^n$ also exist but they are much more involved. The basic McShane integral is equivalent to the Lebesgue integral with respect to to the Lebesgue measure. The Henstock integral, sometimes called the Henstock-Kurzweil integral, is a generalization of the McShane integral in the sense that any McShane integrable function is Henstock integrable but not conversely. The Henstock integral has the property that every continuous, nearly everywhere differentiable function can be recovered by integration from its derivative. This property, which does not hold for the Riemann or the Lebesgue integral, was historically the motivation behind the definition of this integral.

The reason why the McShane and the Henstock integrals are called generalized Riemann integrals is the following. In order to define the ordinary Riemann integral of a (bounded) function one partitions the *domain* of the function, whereas to define the Lebesgue integral of a function one partitions its *range* to obtain an increasing sequence of simple functions converging pointwise to the given function. In the McShane integral, as well as in the Henstock integral, one returns to the idea of partitioning the domain of the function with a more sophisticated notion of "a tagged partition of an interval subordinate to a given positive valued function on the interval". Using such partitions one can obtain generalizations of the Riemann integral and in fact that of the Lebesgue integral. The theory however is, like the Lebesgue theory, non-constructive and without any effective framework.

The domain-theoretic generalization of the Riemann integral works generally for integration of functions with respect to Borel measures on Polish spaces (topologically complete separable metrizable spaces) [13, 22, 17] which include locally compact second countable spaces. Here, one also deals with the domain of the function rather than its range. But now one goes beyond the notion of partitions and uses finite covers by open subsets to provide approximations to the measure. These approximations give generalized upper and lower sums with which we define the integral. The theory, like the Riemann theory, has a constructive and effective framework. The generalized Riemann integral of a Hölder function with respect to an effectively given measure can be approximated up to any desired accuracy by upper and lower sums [8].

In order to apply the generalized Riemann integration to a wider range of problems, we look for an extension of the results in [7] and [8] to the more general setting of locally compact second countable Hausdorff spaces with bounded or locally finite measures, the latter being measures which are finite on compact subsets of the space. A number of these extensions do require further work; others are straightforward generalizations of the compact case.

These more general hypotheses, however, cover some central fields of application. For instance, probability distributions on the real line are examples of bounded measures on a locally compact space, whereas the Lebesgue measure on the real line is an example of a locally finite measure on a locally compact space.

In this paper, after reviewing the domain-theoretic notions needed here, we show in section 3 that the set of locally finite measures on a locally compact space are in one to one correspondence with the set of locally finite continuous valuations on the upper space which are supported on its maximal elements. Indeed, this gives rise to a topological embedding when both spaces are endowed with the Scott topologies induced by the pointwise order.

In section 4, we show how to approximate a locally finite valuation on a locally compact space by means of an increasing sequence of simple valuations on the upper space, which is built up from any given presentation of the locally compact space as an increasing countable union of relatively compact open subsets.

We proceed in section 5 with the definition of the generalized Riemann integral of a bounded real-valued function on a locally compact space with respect to a bounded Borel measure. As in the case of compact spaces [7], generalized R-integrability for a function f is seen to be equivalent to the condition that the set of discontinuities of f has measure zero. Moreover, if the R-integral exists then the Lebesgue integral exists and the two integrals are equal. The notion of an effective approximation to a bounded measure by simple valuations on the upper space, which is used to compute integrals of Hölder continuous functions up to any precision, has a straightforward extension to locally compact spaces.

We then define, in section 6, the R-integral with respect to a locally finite measure. In the case of functions with compact support, the R-integral reduces to the R-integral in compact spaces with respect to the restriction of the measure to the support of the function. In the case of functions with non-compact support, the R-integral is defined in a way similar to the improper Riemann integral. It is shown, in this more general case, that if a function is R-integrable then it is also Lebesgue integrable and the two integrals coincide.

Finally in section 7, following the work in [4], we extend the definition of R- integration to an open set, and prove that the lower integral coincides with the Lebesgue integral on the open set, whereas the upper integral coincides with the Lebesgue integral on its closure. As a consequence, when the boundary of the open set has measure zero, we obtain two sequences that converge from below and from above to the value of the integral of a bounded almost everywhere continuous function. This allows a computation of the integral on the open set up to any desired degree of accuracy.

2 Valuations on continuous posets

We first recall some basic notions from domain theory which we need in this paper. A non-empty subset $A \subseteq P$ of a partially ordered set (poset) $(P, \sqsubseteq)$ is *directed* if for any pair of elements $x, y \in A$ there is $z \in A$ with $x, y \sqsubseteq z$. A *directed complete partial order (dcpo)* is a partial order in which every directed subset A has a least upper bound (lub), denoted by $\bigsqcup A$. An open set $O \subseteq P$ of the Scott topology of P is a set which is upward closed (i.e. $x \in O$ & $x \sqsubseteq y \ \Rightarrow \ y \in O$) and is inaccessible by lubs of directed sets (i.e. if A is directed with a lub, then $\bigsqcup A \in O \ \Rightarrow \ \exists x \in A. x \in O$). The Scott topology of any poset is T_0. Given two elements x, y in a poset P, we say x is *way-below* y or x *approximates* y, denoted by $x \ll y$, if whenever $y \sqsubseteq \bigsqcup A$ for a directed set A with lub, then there is $a \in A$ with $x \sqsubseteq a$. We sometimes write $\ll_P$ to emphasise that the way-below relation is with respect to P. We say that a subset $B \subseteq D$ is a *basis* for D if for each $d \in D$ the set A of elements of B way-below d is directed and $d = \bigsqcup A$. We say D is *continuous* if it has a basis; it is ω-*continuous* if it has a countable basis. In any continuous poset, subsets of the form $\twoheaduparrow b = \{x \mid b \ll x\}$ where b belongs to a basis give a basis of the Scott topology.

A *valuation* on a topological space Y is a measure-like function defined on the lattice $\Omega(Y)$ of open sets. Here is the precise definition.

Definition 2.1 *A* valuation *on a topological space Y is a map*

$$\nu : \Omega(Y) \to [0, \infty)$$

which, for all $U, V \in \Omega(Y)$, satisfies

1. $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$;
2. $\nu(\emptyset) = 0$;
3. $U \subseteq V \Rightarrow \nu(U) \leq \nu(V)$.

A continuous valuation *is a valuation such that whenever $\mathcal{A}$ is a directed family in $\Omega(Y)$ then*

4. $\nu(\bigcup_{O \in \mathcal{A}} O) = \sup_{O \in \mathcal{A}} \nu(O)$.

A locally finite continuous valuation *is a map*

$$\nu : \Omega(Y) \to [0, \infty]$$

satisfying the same properties of a continuous valuation and

5. $\nu(V) < \infty$ *for* $V \ll Y$.

The set of locally finite continuous valuations on Y is denoted by $P^{\ell}(Y)$; it is pointwise ordered by:

$$\mu_1 \sqsubseteq \mu_2 \equiv (\forall O \in \Omega(X))(\mu_1(O) \leq \mu_2(O)) \ .$$

The subposet of continuous valuations bounded by one, i.e. with $\mu(Y) \leq 1$, is denoted by $P(Y)$ and the set of normalized valuations, i.e. those with $\mu(Y) = 1$ is denoted by $P^1(Y)$.

The *point valuation* based at $b \in Y$ is the valuation

$$\delta_b : \Omega(Y) \to [0, \infty)$$

defined by

$$\delta_b(O) \quad = \quad \begin{cases} 1 & \text{if } b \in O \\ 0 & \text{otherwise.} \end{cases}$$

Any finite linear combination

$$\sum_{i=1}^{n} r_i \delta_{b_i} \quad (r_i \in [0, \infty), \ i = 1, \ldots, n)$$

of point valuations is called a *simple valuation*.

If Y is an (ω)-continuous dcpo then $P(Y)$ is also an (ω)-continuous dcpo with a basis of simple valuations [19]. We have the following property:

Proposition 2.2 *[20, page 46] Let $\nu = \sum_{b \in B} r_b \delta_b$ be a simple valuation and μ a continuous valuation on a continuous dcpo Y. Then $\nu \ll \mu$ iff for all $A \subseteq B$ we have*

$$\sum_{b \in A} r_b < \mu(\Uparrow A), \qquad \Box$$

where $\Uparrow A = \{y \in Y \mid \exists a \in A.\, a \ll y\}$.

If Y is an (ω)-continuous dcpo with bottom then $P^1(Y)$ is also an (ω)-continuous dcpo with a basis of simple valuations and we have the following results [7]:

Lemma 2.3 *Let Y be a dcpo with bottom and let*

$$\nu_1 = \sum_{a \in A} r_a \delta_a \qquad \nu_2 = \sum_{b \in B} s_b \delta_b$$

be simple valuations in $P^1(Y)$. Then we have $\nu_1 \sqsubseteq \nu_2$ iff, for all $a \in A$ and for all $b \in B$, there exists $t_{a,b} \geq 0$ such that $\sum_{b \in B} t_{a,b} = r_a$, $\sum_{a \in A} t_{a,b} = s_b$ and $t_{a,b} > 0$ implies $a \sqsubseteq b$.

Define the maps

$$\begin{array}{rccc} m^+ : & PY & \to & P^1(Y) \\ & \mu & \mapsto & \mu^+ \end{array} \qquad \begin{array}{rccc} m^- : & PY & \to & PY \\ & \mu & \mapsto & \mu^- \end{array}$$

where

$$\mu^+(O) = \begin{cases} \mu(O) & \text{if } O \neq Y \\ 1 & \text{otherwise} \end{cases}$$

and

$$\mu^-(O) = \begin{cases} \mu(O) & \text{if } O \neq Y \\ \mu(Y - \{\perp\}) & \text{otherwise.} \end{cases}$$

Proposition 2.4 *(i)* $\nu \ll_{PY} \mu \;\&\; \mu \in P^1Y \;\Rightarrow\; \nu^+ \ll_{P^1Y} \mu$.

(ii) $\nu \ll_{P^1Y} \mu \;\Rightarrow\; \nu^- \ll_{PY} \mu^- \sqsubseteq \mu$.

Lemma 2.5 *Suppose* $\mu, \nu \in P^1Y$. *Then* $\nu \ll_{P^1Y} \mu$ *implies* $\nu(Y - \{\perp\}) < 1$.

Proposition 2.6 *For two simple valuations*

$$\mu_1 = \sum_{b \in B} r_b \delta_b \qquad \mu_2 = \sum_{c \in C} s_c \delta_c$$

in P^1Y, *we have* $\mu_1 \ll_{P^1Y} \mu_2$ *iff* $\perp \in B$ *with* $r_\perp \neq 0$, *and, for all* $b \in B$ *and all* $c \in C$, *there exists a nonnegative number* $t_{b,c}$ *with* $t_{\perp,c} \neq 0$ *such that*

$$\sum_{c \in C} t_{b,c} = r_b \qquad \sum_{b \in B} t_{b,c} = s_c$$

and $t_{b,c} \neq 0$ *implies* $b \ll c$.

When considering locally finite valuations closure under directed joins does not hold any more, i.e. $P^\ell(Y)$ is not a dcpo. However, it is still true that locally finite valuations can be approximated by simple valuations, since this results holds in general for (unbounded) valuations (cf. theorem 5.2, page 46 in [20]), and simple valuations are in particular locally finite. So we have:

Proposition 2.7 *For any continuous dcpo* Y, *any* $\mu \in P^\ell(Y)$ *is the supremum of the (directed) set of simple valuations way-below it and, therefore,* $P^\ell(Y)$ *is a continuous poset.*

3 Locally finite measures on a locally compact space

Throughout the paper, X will denote a second countable locally compact Hausdorff space. We will use the decomposition $X = \bigcup_{i \in \mathbb{N}} X_i$, where $\langle X_i \rangle_{i \in \mathbb{N}}$ is an increasing sequence of relatively compact open subsets of X such that $\overline{X_i} \subseteq X_{i+1}$. We start with some definitions:

Definition 3.1 *A Borel measure* μ *on a locally compact Hausdorff space is* locally finite *if* $\mu(C) < \infty$ *for all compact* $C \subseteq X$. $M^\ell(X)$ *will denote the set of locally finite measures on* X. *The set of measures bounded by one and the set of normalized measures are denoted respectively by* $M(X)$ *and* $M^1(X)$.

We recall from [6] that the upper space UX of a topological space is the set of all non-empty compact subsets of X, with the base of the *upper topology* given by the sets

$$\Box a = \{C \in UX \mid C \subseteq a\},$$

where $a \in \Omega(X)$. When X is a second countable locally compact Hausdorff space, then the upper space UX of X is an ω-continuous dcpo and the Scott topology of $(UX, \supseteq)$ coincides with the upper topology. The lub of a directed subset is the intersection and $A \ll B$ iff B is contained the interior of A. The singleton map $s : X \to UX$ with $s(x) = \{x\}$ is a topological embedding onto the set of maximal elements of UX.

In [6], it was shown that the map

$$\begin{array}{rcl} M(X) & \to & P(UX) \\ \mu & \mapsto & \mu \circ s^{-1} \end{array}$$

is an injection into the set of maximal elements of $P(UX)$ and it was conjectured that its image is the set of maximal elements. This conjecture was later proved by Lawson in a more general setting [23]. The continuous dcpo UX does not necessarily have a bottom element. Therefore, in order to consider normalized valuations, we will adjoin a bottom element $\bot = X$ and denote the dcpo with bottom thus obtained with $(UX)_\bot$. Then the injective map $\mu \mapsto \mu \circ s^{-1} : M^1(X) \to P^1(UX)_\bot$ is onto the set of maximal elements of $P^1(UX)_\bot$. Here, we will show a one-to-one correspondence between locally finite Borel measures on X and locally finite continuous valuations on the upper space supported in $s(X)$.

Proposition 3.2 *Let $s : X \to UX$ be the singleton map. Then the map*

$$\begin{array}{rcl} e : M^\ell(X) & \to & P^\ell(UX) \\ \mu & \mapsto & \mu \circ s^{-1} \end{array}$$

is well defined.

Before proving the above proposition we need the following lemmas, connecting the way-below relation on the upper space UX of X with the one on X.

Lemma 3.3 *Let $\{O_i : i \in I\}$ be a directed family in $\Omega(X)$. Then*

$$\Box(\bigcup_{i\in I} O_i) = \bigcup_{i\in I} \Box O_i.$$

Proof: The inclusion from right to left trivially holds. For the converse, assume that C is a non-empty compact subset of $\bigcup_{i\in I} O_i$. By compactness, C has a finite subcover, and therefore, since the family of opens is directed, there exists $i \in I$ such that C is a subset of O_i. $\Box$

Lemma 3.4 *Let V be an open set in the Scott topology of UX. Then*

$$V \ll UX \text{ in } \Omega(UX) \text{ if and only if } s^{-1}(V) \ll X \text{ in } \Omega(X).$$

Proof:

$\Rightarrow$: Suppose $X \subseteq \bigcup_{i\in I} O_i$ where the right-hand side is a directed union and $O_i \in \Omega(X)$. Then, by lemma 3.3, we have $UX = \Box X \subseteq \bigcup_{i\in I} \Box O_i$. Since $V \ll UX$ in $\Omega(UX)$, there exists $i \in I$ such that $V \subseteq \Box O_i$, and therefore $s^{-1}(V) \subseteq O_i$, thus proving the claim.

$\Leftarrow$: Suppose $UX \subseteq \bigcup_{i\in I} O_i$, where the right-hand side is the union of a directed family of opens of the upper space. Since by hypothesis $s^{-1}(V) \ll X$ and X is a locally compact space, there exists a compact set $C \subseteq X$ such that $s^{-1}(V) \subseteq C \subseteq X$. Since $C \in UX$, there exists $i \in I$ such that $C \in O_i$ and therefore $\uparrow C \subseteq O_i$ as O_i is open in the Scott topology of UX and thus an upper set with respect to reverse inclusion. This implies $V \subseteq \uparrow C$. For, if $K \in V$ and $x \in K$, then $K \sqsubseteq \{x\}$ and, hence, $\{x\} \in V$ since V is upward closed. Thus $x \in s^{-1}(V) \subseteq C$, i.e. $K \subseteq C$. Therefore $V \subseteq \uparrow C \subseteq O_i$, i.e., $V \ll UX$. □

Proof of proposition 3.2: Let μ be a locally finite Borel measure on X. Then it is immediate to verify that $\mu \circ s^{-1}$ satisfies conditions 1-4 of definition 2.1 since μ is a measure and s^{-1} preserves (directed) unions and intersections. The continuous valuation $\mu \circ s^{-1}$ is locally finite since, if $V \ll UX$, then, by lemma 3.4, $s^{-1}(V) \ll X$. Since X is locally compact, there exists a compact subset K of X such that $s^{-1}(V) \subseteq K$. Therefore, by monotonicity of μ, $\mu \circ s^{-1}(V) \leq \mu(K)$, and by the assumption on local finiteness of μ the conclusion follows. □

By a result from [24] (theorem 3.9), if P is a continuous dcpo equipped with its Scott topology, then every continuous valuation ν on P extends uniquely to a measure, denoted by ν^*. Thus, in particular, $\mu \circ s^{-1}$ extends to a unique measure $(\mu \circ s^{-1})^*$ on UX, which, by abuse of notation, we will denote by μ^* too. Such a measure satisfies $\mu^*(UX - s(X)) = 0$, that is we have:

Proposition 3.5 *The valuation $\mu \circ s^{-1}$ is supported in $s(X)$.*

Proof: We have $UX - s(X) = \bigcup_{i \in \mathbb{N}}(UX_i - s(X_i))$. By proposition 5.9 and corollary 5.10 in [6], $s(X) and s(X_i)$ are G_δ subsets of UX and there exists a countable decreasing sequence of opens $\langle O_{i,j}\rangle_{j \in \mathbb{N}}$ such that $s(X_i) = \cap_{j \in \mathbb{N}} O_{i,j}$ and $s^{-1}(O_{i,j}) = X_i$. Therefore

$$\mu^*(UX - s(X)) = \sup_{i \in \mathbb{N}} \mu^*(UX_i - s(X_i)),$$

so it is enough to prove that, for all $i \in \mathbb{N}$, $\mu^*(UX_i - s(X_i)) = 0$. We have $\mu^*(s(X_i)) = \inf_{j \in \mathbb{N}} \mu^*(O_{i,j}) = \inf_{j \in \mathbb{N}} \mu \circ s^{-1}(O_{i,j}) = \mu(X_i) = \mu \circ s^{-1}(UX_i) = \mu^*(UX_i)$. It follows that $\mu^*(UX_i - s(X_i)) = 0$. □

Adapting the notation from [6], we will denote with $S^\ell(X)$, resp. $S^1(X)$, the locally finite valuations, resp. the normalized valuations, on UX which are supported in $s(X)$. We have:

Theorem 3.6 *The map*

$$\begin{array}{rcl} e : M^\ell(X) & \to & S^\ell(X) \\ \mu & \mapsto & \mu \circ s^{-1} \end{array}$$

is a bijection with inverse given by

$$\begin{array}{rcl} j : S^\ell(X) & \to & M^\ell(X) \\ \nu & \mapsto & \nu^* \circ s\ . \end{array}$$

The proof of the theorem is based on the following lemma.

Lemma 3.7 *If μ and ν are locally finite Borel measures which have the same restriction to $\Omega(X)$ then $\mu = \nu$.*

Proof: We have $\mu(X_i) = \nu(X_i) < \infty$. Furthermore, for all Borel sets B, $B \cap X_i$ is a Borel set in the induced topology of X_i. By proposition 5.2 in [6], any finite continuous valuation on a locally compact Hausdorff space has a unique extension to a measure, so that we have $\mu(X_i \cap B) = \nu(X_i \cap B)$ and therefore $\mu(B) = \mu(X \cap B) = \mu((\bigcup_{i \in \mathbb{N}} X_i) \cap B) = \mu(\bigcup_{i \in \mathbb{N}}(X_i \cap B)) = \sup_{i \in \mathbb{N}} \mu(X_i \cap B) = \sup_{i \in \mathbb{N}} \nu(X_i \cap B) = \nu(B)$. □

Proof of theorem 3.6: By the foregoing discussion, e is well defined. To prove that j is well defined we only have to prove $j(\nu)$ is a locally finite measure when ν is a locally finite continuous valuation. Let C be a compact subset of X. There exists a relatively compact open subset O, i.e. $O \ll X$, with $C \subseteq O$. Then $(j(\nu))(C) \leq (j(\nu))(O) = \nu^*(s(O)) \leq \nu(\Box O) < \infty$, where the last inequality holds since $\Box O \ll UX$ by lemma 3.4.

The proof that j is the inverse of ν proceeds as in [6], theorem 5.21. $\Box$

In [8] it has been proved that the space of probability measures of a compact metric space equipped with the weak topology is topologically embedded, via the map $\mu \mapsto \mu \circ s^{-1}$, onto the subspace of the maximal elements of the probabilistic power domain of the upper space.

Here a topological embedding can be obtained by considering the Scott topologies on $M^\ell(X)$ and $P^\ell(UX)$: We define a partial order on $M^\ell(X)$, as in $P^\ell(UX)$, by

$$\mu_1 \sqsubseteq \mu_2 \equiv (\forall O \in \Omega(X))(\mu_1(O) \leq \mu_2(O)) .$$

The relation $\sqsubseteq$ is clearly reflexive and transitive, and it is antisymmetric by lemma 3.7. Observe that the space $(M^\ell(X), \sqsubseteq)$ is not a dcpo. In fact, a directed subset D of $M^\ell(X)$ has least upper bound in $M^\ell(X)$ if and only if for all open sets O such that $O \ll X$ in $\Omega(X)$ the set $\{\mu(O) : \mu \in D\}$ is bounded. It is immediate to check that the maps e and j are continuous with respect to the Scott topologies on $M^\ell(X)$ and $P^\ell(UX)$, i.e., e is a topological embedding.

4 Approximation of locally finite measures

Given a measure on a compact metric space it is possible to construct a chain of simple valuations on the upper space with the measure as its least upper bound (cf. [8]). In this section we will generalize this construction to the case of locally finite measures on a locally compact space.

The relatively compact subsets X_i are used to construct a sequence of simple valuations on UX approximating any given locally finite measure on X, thus generalizing the procedure which is worked out in [8]. As in that work, we assume that the measure is given by its values on a given countable basis closed under finite unions and finite intersections.

Since the subsets X_i are relatively compact, for all $i \in \mathbb{N}$ there exists an ordered finite covering $\mathcal{C}_i$ of $\overline{X_i}$ made of relatively compact open sets of the basis with diameter $\leq \frac{1}{i}$. Moreover such coverings can be chosen satisfying the additional requirement $\cup\mathcal{C}_i \subseteq \cup\mathcal{C}_{i+1}$. These hypotheses are needed in order to obtain an increasing sequence of simple valuations. We assume in this section, and the rest of the paper, that open covers are always constructed from the given countable base satisfying the above property.

For each $i \in \mathbb{N}$ we define inductively a finite ordered open cover $\mathcal{D}_i$ for $\overline{X_i}$ as a list of opens sets in the following way, where the symbol $*$ denotes the concatenation operation for lists:

$\mathcal{D}_1 \equiv \mathcal{C}_1$;
$\mathcal{D}_2 \equiv \mathcal{D}_1 \wedge \mathcal{C}_2 * \langle O \in \mathcal{C}_2 : O \not\subseteq \ \cup \mathcal{C}_1 \rangle$;
$D_3 \equiv \mathcal{D}_2 \wedge \mathcal{C}_3 * \langle O \in \mathcal{C}_3 : O \not\subseteq \ \cup \mathcal{C}_2 \rangle$;
...
$\mathcal{D}_{i+1} \equiv \mathcal{D}_i \wedge \mathcal{C}_{i+1} * \langle O \in \mathcal{C}_{i+1} : O \not\subseteq \ \cup \mathcal{C}_i \rangle$;
...

where, for given ordered covers $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \wedge \mathcal{B}$ denotes the cover

$$\langle O_1 \cap O_2 : O_1 \in \mathcal{A}, O_2 \in \mathcal{B} \rangle$$

ordered lexicographically. We recall from [8] that, for an ordered cover $\mathcal{A} = \langle O_1, \ldots, O_n \rangle$, the simple valuation $\mu_{\mathcal{A}}$ associated with $\mathcal{A}$ is given by

$$\mu_{\mathcal{A}} \equiv \sum_{i=1}^{n} r_i \delta_{\overline{O_i}} , \text{ where } r_i = \mu(O_i - \bigcup_{j<i} O_j).$$

We will denote with μ_i the simple valuation $\mu_{\mathcal{D}_i}$ associated with $\mathcal{D}_i$.

We will now prove that the μ_i's constitute a chain of simple valuations with least upper bound μ. The following lemma is proved as proposition 3.1 in [8].

Lemma 4.1 *Let $\mathcal{A} = \langle O_1, \ldots, O_n \rangle$ be an ordered cover of X_i with $O_j \ll X$ for all $j \leq n$, μ a locally finite measure on X and the r_i's as above. Then for all open subsets O of X_i we have*

$$\sum_{O_i \subseteq O} r_i \leq \mu(O) \leq \sum_{\overline{O_i} \cap O \neq \emptyset} r_i \ .$$

Corollary 4.2 *For all $i \in \mathbb{N}$, $\mu_i \sqsubseteq e(\mu)$ in $P^\ell(UX)$.*

Proof: Let $V = \bigcup_{\lambda \in \Lambda} \Box O_\lambda$ be an open set of the upper space. Then

$$\begin{aligned} \mu_i(V) &= \textstyle\sum_{\overline{O_i} \in \bigcup_{\lambda \in \Lambda} \Box O_\lambda} r_i \leq \sum_{O_i \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda} r_i \\ &\leq \mu(\textstyle\bigcup_{\lambda \in \Lambda} O_\lambda) = e(\mu)(V) \ . \end{aligned}$$

where the last inequality holds by lemma 4.1. $\Box$

Proposition 4.3 *For all $i \in \mathbb{N}$, $\mu_i \sqsubseteq \mu_{i+1}$ in $P^\ell(UX)$.*

Proof: Let $O \in \Omega(UX)$, $\mathcal{D}_i = \langle O_1, \ldots, O_n \rangle$, $\mathcal{C}_{i+1} = \langle V_1, \ldots, V_m \rangle$ and let, for $1 \leq i \leq n$, $1 \leq j \leq m$, $r_i = \mu(O_i - \bigcup_{i' < i} O_{i'})$, $r_{(i,j)} = \mu(O_i \cap V_j - \bigcup_{(i',j') < (i,j)} O_{i'} \cap V_{j'})$. Then $\mu_{i+1}(O) \geq \mu_{D_i \wedge C_{i+1}}(O) = \sum_{\overline{O_i \cap V_j} \in O} r_{(i,j)}$. Furthermore,

$$\sum_{\overline{O_i \cap V_j} \in O} r_{(i,j)} \geq \sum_{j \leq m} \sum_{\overline{O_i} \in O} r_{(i,j)} = \sum_{\overline{O_i} \in O} \sum_{j \leq m} r_{(i,j)} \ .$$

As in the proof of proposition 3.4 in [8], it follows that $\sum_{j \leq m} r_{(i,j)} = r_i$, and therefore $\sum_{\overline{O_i} \in O} \sum_{j \leq m} r_{(i,j)} \geq \sum_{\overline{O_i} \in O} r_i = \mu_i(O)$, and the result follows. $\Box$

Proposition 4.4 $\mu = \bigsqcup_{i \in \mathbb{N}} \mu_i$.

Proof: Since, for all i, $\mu_i \sqsubseteq \mu$, we have $\bigsqcup_{i \in \mathbb{N}} \mu_i \sqsubseteq \mu$.

For the converse inequality, it is sufficient by lemma 3.3 in [8] to check that, for all opens $O \in \Omega(X)$, $\sup_i \mu_i(\Box O) \geq \mu(O)$. We distinguish two cases, i.e. $\mu(O) < \infty$ and $\mu(O) = \infty$. If $\mu(O) < \infty$, let $O_j \equiv O \cap X_j$, for $j \in \mathbb{N}$. Then, for all j, $O_j \ll X$ and $O = \bigcup_{j \in \mathbb{N}} O_j$. By theorem 3.5 in [8], we have, for all $j \in \mathbb{N}$, $\mu(O_j) = \bigsqcup_{i \in \mathbb{N}} \mu_i(O_j)$ and therefore $\mu(O) = \bigsqcup_{j \in \mathbb{N}} \mu(O_j) = \bigsqcup_{j \in \mathbb{N}} \bigsqcup_{i \in \mathbb{N}} \mu_i(O_j) = \bigsqcup_{i \in \mathbb{N}} \bigsqcup_{j \in \mathbb{N}} \mu_i(O_j) = \bigsqcup_{i \in \mathbb{N}} \mu_i(O)$. If $\mu(O) = \infty$, then for all $M > 0$ there exists $V \ll O$ such that $\mu(V) > M$. Since $V \ll O$ and μ is a locally finite measure, $\mu(V)$ is finite and therefore, by the previous case, $\mu(V) = \bigsqcup \mu_i(\Box V)$. Thus, since $\Box O \supseteq \Box V$, we have $\bigsqcup \mu_i(\Box O) \geq \bigsqcup \mu_i(\Box V) = \mu(V) > M$, and the conclusion follows since M is arbitrary. $\Box$

5 Integration with respect to bounded measures

Let μ be a Borel measure on X such that $0 < \mu(X) < \infty$. By rescaling, we can suppose that μ is normalized, i.e. that $\mu(X) = 1$. By corollary 3.3 in [7], $P^1(UX)_\perp$, the set of normalized valuations of the probabilistic power domain of $(UX)_\perp$, is an ω-continuous dcpo with a basis of normalized simple valuations.

We also know from the previous section that there exists an increasing chain of simple valuations $\langle\mu_i\rangle_{i\geq 0}$ in $P(UX)_\perp$ such that $\mu\circ s^{-1} = \sup_{i\geq 0}\mu_i$. For convenience, in the rest of this section, we identify $\mu\circ s^{-1}$ with μ and, therefore, write $\mu = \sup_{i\geq 0}\mu_i$.

The valuations

$$\mu_i^+ \equiv \mu_i + (1-\mu_i((UX)_\perp))\delta_\perp$$

are in $P^1(UX)_\perp$. Moreover, for any proper open subset O of $(UX)_\perp$, we have $\mu_i^+(O) = \mu_i(O)$ and $\mu\circ s^{-1} = \sup_{i\geq 0}\mu_i^+$, where $\langle\mu_i^+\rangle_{i\geq 0}$ is again an increasing chain of simple valuations.

Let f be a bounded real-valued function defined on X and let μ be a bounded Borel measure on X. For a simple valuation $\nu \equiv \sum_{b\in B} r_b\delta_b$, define, as in [7], the *lower* and *upper sums* by

$$S^\ell_X(f,\nu) \equiv \sum_{b\in B} r_b \inf f[b]$$

$$S^u_X(f,\nu) \equiv \sum_{b\in B} r_b \sup f[b],$$

where $f[b] = \{f(x)|x\in b\}$. Observe that, since f is bounded, $S^\ell_X(f,\nu)$ and $S^u_X(f,\nu)$ are well defined real numbers. For a choice of $x_b\in b$ for all $b\in B$, we also have a *Riemann sum* $S_x(f,\nu) = \sum_{b\in B} r_b f(x_b)$. By proposition 4.2 in [7] we have:

Proposition 5.1 *Let $\nu_1,\nu_2 \in P^1(UX)_\perp$ be simple valuations with $\nu_1 \sqsubseteq \nu_2$. Then $S^\ell_X(f,\nu_1) \leq S^\ell_X(f,\nu_2)$ and $S^u_X(f,\nu_2)\leq S^u_X(f,\nu_1)$.*

Corollary 5.2 *If $\nu_1,\nu_2\in P^1(UX)_\perp$ are simple valuations with $\nu_1,\nu_2 \ll \mu$, then $S^\ell_X(f,\nu_1)\leq S^u_X(f,\nu_2)$*

Using the notation introduced above we define the lower and upper R-integrals as follows:

Definition 5.3 $\mathbf{R}\underline{\int} f d\mu \equiv \sup_{\nu\ll\mu} S^\ell_X(f,\nu)$ *(the lower R-integral);*

$\mathbf{R}\overline{\int} f d\mu \equiv \inf_{\nu\ll\mu} S^u_X(f,\nu)$ *(the upper R-integral).*

By corollary 5.2 we have

$$\mathbf{R}\underline{\int} f d\mu \leq \mathbf{R}\overline{\int} f d\mu\ .$$

Definition 5.4 *We say that f is R-integrable with respect to μ and write $f\in R_X(\mu)$ if $\mathbf{R}\underline{\int} f = \mathbf{R}\overline{\int} f$.*

As a consequence of the definition we have:

Proposition 5.5 (The R-condition). *We have $f\in R(\mu)$ iff for all $\epsilon>0$ there exists a simple valuation $\nu\in P^1(UX)_\perp$ with $\nu\ll\mu$ such that*

$S^u_X(f,\nu) - S^\ell_X(f,\nu) < \epsilon.$

If f is R-integrable then the integral of f can be calculated by using the increasing sequence of simple valuations $\langle\mu_i^+\rangle_{i\geq 0}$:

Proposition 5.6 *If f is R-integrable with respect to μ, and $\langle\mu_i\rangle_{i\geq 0}$ is an increasing sequence of simple valuations on UX with least upper bound μ, then*

$$\int_X f d\mu = \sup_{i\geq 0} S^\ell_X(f,\mu_i^+) = \inf_{i\geq 0} S^u_X(f,\mu_i^+).$$

Proof: Cf. proposition 4.9 in [7]. □

The following properties are easily shown as in [7]:

Proposition 5.7 *1. If f and g are R-integrable with respect to μ then $f+g$ is also R-integrable with respect to μ and $\int (f+g)d\mu = \int f d\mu + \int g d\mu$.*

2. If f is R-integrable with respect to μ and $c \in \mathbb{R}$ then cf is R-integrable with respect to μ and $\int cf d\mu = c \int f d\mu$.

3. If f and g are R-integrable with respect to μ then so is their product fg.

5.1 Lebesgue criterion for R-integrability

We will extend the Lebesgue criterion for Riemann integration of a bounded function on a compact real interval to R-integration of a bounded real-valued function on a locally compact space, by generalizing the result in [7], that is the analogous criterion for R-integration on compact spaces.

We first recall some definitions.

Definition 5.8 *Let $T \subseteq X$ and $r > 0$ and define*

$\Omega_f(T) \equiv \sup\{f(x) - f(y) \mid x, y \in T\}$, *called the* oscillation of f on T*;*

$\omega_f(x) \equiv \lim_{h \to 0^+} \Omega_f(B(x,h))$, *where $B(x,h)$ is the open ball of radius h centred at x;*

$D_r \equiv \{x \in X \mid \omega_f(x) \geq \frac{1}{r}\}$.

Then we have (cf. [2], p. 170, and [7]):

Proposition 5.9 *The following statements hold:*

(i) f is continuous at x iff $\omega_f(x) = 0$.

(ii) If X is compact and $\omega_f(x) < \epsilon$ for all $x \in X$, then there exists $\delta > 0$ such that, for all compact subsets $b \subseteq X$ with $|b| < \delta$, we have $\Omega_f(b) < \epsilon$.

(iii) For any $r > 0$, the set D_r is closed.

If D is the set of discontinuities of f, we have $D = \bigcup_{n \geq 1} D_n$ where $D_1 \subseteq D_2 \subseteq D_3 \subseteq \ldots$ is an increasing chain of closed sets. Hence D is an F_δ and therefore a Borel set. In the following, $\mu \in M^1(X)$. We have:

Lemma 5.10 *Let d be a compact subset of X and let $\nu = \sum_{b \in B} r_b \delta_b \in P^1(UX)$. Then:*

1. If $\nu \sqsubseteq \mu$ then $\sum_{b \cap d \neq \emptyset} r_b \geq \mu(d)$.

2. If $\nu \ll \mu$ then $\sum_{b^\circ \cap d \neq \emptyset} r_b \geq \mu(d)$, where b° denotes the interior of b.

Proof: Cf. lemma 6.4 in [7]. □

Proposition 5.11 *Let $\langle \mu_i \rangle_{i \in I}$ be a directed set of simple valuations*

$$\mu_i = \sum_{b \in B_i} r_{i,b} \delta_b$$

in $P^1(UX)$. Then $\bigsqcup_{i \in I} \mu_i \in im(e)$ iff all $\epsilon > 0$ and all $\delta > 0$, there exists $i \in I$ with

$$\sum_{b \in B_i, |b| \geq \delta} r_{i,b} < \epsilon$$

where $|b|$ is the diameter of the compact set $b \subseteq X$.

Proof: The two directions in the proof were shown for a compact X in proposition 4.14 in [7] and in proposition 5.1 in [8], but the proofs there hold for a locally compact X as well. □

We recall from [26] that a bounded measure on a second countable locally compact Hausdorff space is regular, i.e., for any Borel set B, it satisfies

$$\mu(B) = \inf\{\mu(O) \mid B \subseteq O,\ O \text{ open}\} = \sup\{\mu(K) \mid K \subseteq B,\ K \text{ compact}\}\ .$$

Then the Lebesgue criterion follows:

Theorem 5.12 *A bounded real-valued function on a locally compact second countable Hausdorff space is R-integrable with respect to a bounded Borel measure μ on X iff its set of discontinuities has measure zero.*

Proof: Suppose $\mu(D) > 0$. Since $D = \bigcup_{n\geq 1} D_n$, there exists $n \geq 1$ such that $\mu(D_n) > 0$. As $D_n = \bigcup_{i\geq 0} D_n \cap \overline{X_i}$, there exists i such that $\mu(D_n \cap \overline{X_i}) > 0$. We put $D'_n \equiv D_n \cap \overline{X_i}$. Let $\nu = \sum_{b\in B} r_b\delta_b$ be a simple valuation with $\nu \ll \mu$. Then

$$\begin{aligned} S^u(f,\nu) - S^\ell(f,\nu) &= \textstyle\sum_{b\in B} r_b(\sup f[b] - \inf f[b]) \\ &\geq \textstyle\sum_{b^\circ \cap D'_n \neq \emptyset} r_b(\sup f[b] - \inf f[b]) \\ &\geq \textstyle\sum_{b^\circ \cap D'_n \neq \emptyset} r_b/n \geq \mu(D'_n)/n > 0 \end{aligned}$$

by definition of D'_n and the above lemma. Thus f does not satisfy the R-condition and therefore it is not R-integrable.

Conversely, assume $\mu(D) = 0$. For all $n \geq 1$, there exists a compact subset Y of X such that $\mu(X - Y) < \frac{1}{n}$. Let T and Z be regular compact subsets of X with $Y \subseteq Z^\circ$ and $Z \subseteq T^\circ$. Let $D_n = \{x \in T \mid w_f(x) \geq \frac{1}{n}\}$. We know that D_n is closed and $\mu(D_n) = 0$. By regularity of μ, there exists an open $V \subseteq X$ such that $D_n \cap Z \subseteq V$ and $\mu(V) < \frac{1}{n}$. By taking the intersection of V with T°, if necessary, we can assume $V \subseteq T^\circ$. Since $D_n \cap Z$ is compact, there exists an open set W with $D_n \cap Z \subseteq W$ and $\overline{W} \subseteq V$. Let $d(\overline{W}, V^c) = \delta_1$ and $d(Y, \partial Z) = \delta_2$, where ∂Z denotes the boundary of Z. Observe that $Z - W$ is compact and for all $x \in Z - W$, $w_f(x) < \frac{1}{n}$. By proposition 5.9(ii), there exists δ_3 such that for all compact $b \subseteq Z - W$ with $|b| < \delta_3$ we have $\Omega_f(b) < \frac{1}{n}$. Let $0 < \delta < min(\delta_1, \delta_2, \delta_3)$. By proposition 5.11, there exists $\gamma \in P^1(UX)_\perp$, $\gamma = \sum_{b\in B} r_b\delta_b$, such that $\sum_{|b|\geq\delta} r_b < \frac{1}{n}$ and $\gamma \sqsubseteq \mu$. Observe that, by the choice of δ, $|b| < \delta$ implies $b \subseteq Z$ or $b \subseteq X - Y$. Moreover, $|b| < \delta$ and $b \subseteq Z$ imply $b \subseteq V \cap Z$ or $b \subseteq Z - W$. Then we have

$$\begin{aligned} &S^u(f,\gamma) - S^\ell(f,\gamma) = \sum_{b\in B} r_b(\sup f[b] - \inf f[b]) \\ &\leq \sum_{|b|\geq\delta} r_b(\sup f[b] - \inf f[b]) + \sum_{|b|<\delta, b\subseteq Z} r_b(\sup f[b] - \inf f[b]) \\ &\quad + \sum_{|b|<\delta, b\subseteq X-Y} r_b(\sup f[b] - \inf f[b]) \\ &\leq \sum_{|b|\geq\delta} r_b(\sup f[b] - \inf f[b]) + \sum_{|b|<\delta, b\subseteq V\cap Z} r_b(\sup f[b] - \inf f[b]) \\ &\quad + \sum_{|b|<\delta, b\subseteq Z-W} r_b(\sup f[b] - \inf f[b]) + \sum_{|b|<\delta, b\subseteq X-Y} r_b(\sup f[b] - \inf f[b]) \\ &\leq (M-m)/n + (M-m)/n + 1/n + (M-m)/n = [3(M-m)+1]/n \end{aligned}$$

and we conclude that f is R-integrable. □

5.2 R-integration and Lebesgue integration

In this section we will prove that the result in [7] connecting R-integration and Lebesgue integration for bounded real-valued functions defined on a compact space extends to the case of bounded functions defined on a locally compact second countable Hausdorff space. The proof for the compact case uses the domain-theoretic result for the existence of a directed set of deflations whose lub is the identity function on the domain; it can be extended to the locally compact case using the one-point compactification of the space. However, the following alternative proof which uses the construction of the simple valuations in section 4 approximating a given measure is elementary and conceptually simpler than the proof in [7]. Furthermore, the new proof can be used to generalize the result to other spaces such as complete separable metric spaces where the domain-theoretic property on the existence of deflations does not exist.

Theorem 5.13 *If a bounded real-valued function f is R-integrable with respect to a bounded Borel measure μ on a locally compact second countable Hausdorff space, then it is also Lebesgue integrable and the two integrals coincide.*

Proof: We will use the increasing chain of simple valuations $\langle \mu_i \rangle_{i\in\mathbb{N}}$, constructed in section 4, such that $\mu = \bigsqcup_{i\in\mathbb{N}} \mu_i$. We recall that the simple valuation μ_i associated with the ordered cover $\mathcal{D}_i = \langle O_{i,1}, \dots, O_{i,n_i} \rangle$ of X_i is given by

$$\mu_i = \sum_{j=1}^{n_i} r_{i,j} \delta_{\overline{O_{i,j}}} \text{ , where } r_{i,j} = \mu(O_{i,j} - \bigcup_{k<j} O_{i,k}) \ .$$

Let $V_{i,j} \equiv O_{i,j} - \bigcup_{k<j} O_{i,k}$. Observe that for all $x \in \cup\mathcal{D}_i$ there exists a unique j such that $x \in V_{i,j}$. Moreover, let m and M be a lower bound and an upper bound of f on X, respectively. For all $i \in N$ we define two functions

$$\begin{array}{rccl} f_i^- : & \cup\mathcal{D}_i & \to & \mathbb{R} \\ & x & \mapsto & \inf f[V_{i,j}], \ for\ x \in V_{i,j} \end{array} \qquad \begin{array}{rccl} f_i^+ : & \cup\mathcal{D}_i & \to & \mathbb{R} \\ & x & \mapsto & \sup f[V_{i,j}], \ for\ x \in V_{i,j} \end{array}$$

By putting $f_i^-(x) = m$ and $f_i^+(x) = M$ for $x \in X - \cup\mathcal{D}_i$ we obtain two functions

$$f_i^- : X \to \mathbb{R} \qquad f_i^+ : X \to \mathbb{R}$$

Observe that f_i^- and f_i^+ are simple measurable functions. Moreover, since the cover $\mathcal{D}_{i+1}$ is a refinement of the cover $\mathcal{D}_i$, for all $x \in X$ we have

$$m \leq \ldots \leq f_i^-(x) \leq f_{i+1}^-(x) \ldots \leq f(x) \leq \ldots \leq f_{i+1}^+(x) \leq f_i^+(x) \leq M \ .$$

Let

$$\begin{array}{rccl} f^- : & X & \to & \mathbb{R} \\ & x & \mapsto & \lim_{i\to\infty} f_i^-(x) \end{array} \qquad \begin{array}{rccl} f^+ : & X & \to & \mathbb{R} \\ & x & \mapsto & \lim_{i\to\infty} f_i^+(x). \end{array}$$

Then $f^-(x) \leq f(x) \leq f^+(x)$ for all $x \in X$, and, by the monotone convergence theorem, f^- and f^+ are Lebesgue integrable.

We will now estimate their Lebesgue integrals. We have

$$\begin{array}{rcl} S^\ell(f, \mu_i) & = & \sum_{j=1}^{n_i} \mu(V_{i,j}) \inf f[\overline{O_{i,j}}] \\ & \leq & \sum_{j=1}^{n_i} \mu(V_{i,j}) \inf f[V_{i,j}] \quad = \quad \mathbf{L} \int f_i^- \, d\mu - m\,\mu(X - \cup\mathcal{D}_i) \end{array}$$

and

$$\begin{array}{rcl} S^u(f, \mu_i) & = & \sum_{j=1}^{n_i} \mu(V_{i,j}) \sup f[\overline{O_{i,j}}] \\ & \geq & \sum_{j=1}^{n_i} \mu(V_{i,j}) \sup f[V_{i,j}] \quad = \quad \mathbf{L} \int f_i^+ \, d\mu - M\,\mu(X - \cup\mathcal{D}_i) \end{array}$$

Since $f_i^- \leq f_i^+$ implies $\mathbf{L}\int f_i^- d\mu \leq \mathbf{L}\int f_i^+ d\mu$, we obtain

$$m\,\mu(X - \cup\mathcal{D}_i) + S^\ell(f,\mu_i) \leq \mathbf{L}\int f_i^- d\mu \leq \mathbf{L}\int f_i^+ d\mu \leq S^u(f,\mu_i) + M\,\mu(X - \cup\mathcal{D}_i).$$

Since f is assumed to be R-integrable, we know by proposition 5.6 that $S^\ell(f,\mu_i)$ increases to $\mathbf{R}\int f d\mu$ and $S^u(f,\mu_i)$ decreases to $\mathbf{R}\int f d\mu$. Moreover $\mu(X - \cup\mathcal{D}_i) \to 0$ as $i \to \infty$ since $\bigcup_{i>0} \cup\mathcal{D}_i = X$.

Therefore, both $\mathbf{L}\int f_i^- d\mu$ and $\mathbf{L}\int f_i^+ d\mu$ converge to $\mathbf{R}\int f d\mu$, and thus, by the monotone convergence theorem, $\mathbf{L}\int f^- d\mu = \mathbf{L}\int f^+ d\mu = \mathbf{R}\int f d\mu$.

We thus obtain $\mathbf{L}\int (f^+ - f^-) d\mu = 0$, which implies that $f^+ = f^-$ almost everywhere. Therefore $f = f^+ = f^-$ almost everywhere and we can conclude that f is Lebesgue integrable and that

$$\mathbf{L}\int f d\mu = \mathbf{L}\int f^- d\mu = \mathbf{L}\int f^+ d\mu = \mathbf{R}\int f d\mu$$

as required. $\square$

5.3 Computation of integrals

Following [8], we can develop an effective framework for computing integrals of bounded Hölder continuous functions with respect to a normalized measure on a locally compact second countable metric space X. This is a straightforward generalization of the compact case and is presented here for completeness. Given a measure $\mu \in M^1(X)$, a chain $\langle \mu_i \rangle_{i\in\mathbb{N}}$ of simple valuations in $P^1(UX)$ is an *effective approximation* to μ if $\bigsqcup_{i\in I} \mu_i = \mu$ and for all positive integers m and n there exists $i \geq 0$, recursively given in terms of m and n, such that $\mu_i = \sum_{c\in C} r_c \delta_c$ satisfies $\sum_{|c|\geq 1/m} r_c < 1/n$. Suppose such an effective approximation exists for μ. Let $f : X \to \mathbb{R}$ be a bounded Hölder continuous function with constants $k \geq 0$ and $h > 0$ such that $|f(x) - f(y)| \leq k(d(x,y))^h$ for all $x, y \in X$, and let $|f(x)| \leq K$ for all $x \in X$. In this setting one can compute the integral of f with respect to μ up to any desired accuracy as follows. Let $\epsilon > 0$ be given. To compute $\int f\, d\mu$ to within ϵ accuracy, we choose positive integers m and n with $1/m < (\epsilon/2k)^{1/h}$ and $1/n < \epsilon/(4K)$, and let the integer i be such that $\nu_i = \sum_{c\in C} r_c\delta_c$ satisfies $\sum_{|c|\geq 1/m} r_c < 1/n$. We have

$$S^\ell(f,\mu_i) \leq \int f\, d\mu \leq S^u(f,\mu_i), \qquad S^\ell(f,\mu_i) \leq S_\xi(f,\mu_i) \leq S^u(f,\mu_i),$$

where $S_\xi(f,\mu_i)$ is any generalised Riemann sum for μ_i. For any $c \in C$ we have $\sup f[c] - \inf f[c] \leq 2K$; whereas for $c \in C$ with $|c| < 1/m$ we have $\sup f[c] - \inf f[c] < \epsilon/2$. Hence,

$$\begin{aligned}
|\textstyle\int f\, d\mu - S_\xi(f,\mu_i)| &\leq S^u(f,\mu_i) - S^\ell(f,\mu_i) = \sum_{c\in C}^{N} r_c(\sup f[c] - \inf f[c]) \\
&= \sum_{|c|\geq 1/m} r_c(\sup f[c] - \inf f[c]) + \sum_{|c|<1/m} r_c(\sup f[c] - \inf f[c]) \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Therefore any Riemann sum for μ_i gives the value of the integral up to ϵ accuracy. We have then shown:

Theorem 5.14 *The expected value of any Hölder continuous function with respect to any normalised measure on a second countable locally compact metric space can be obtained up to any given accuracy with an effective approximation of the measure by an increasing chain of normalised valuations on the upper space of the metric space.*

6 Integration with respect to locally finite measures

6.1 Functions with compact support

In this section we define the R-integral for a real-valued function with compact support defined on a locally compact space with respect to a locally finite measure.

We first need the following:

Definition 6.1 *For any Borel measure μ and any Borel set B, let $\mu|_B$ be the unique extension to a measure of the continuous valuation given by*

$$\mu|_B(O) \equiv \mu(B \cap O) \ .$$

Let f be a real-valued function on X with compact support C and let μ be a locally finite measure on X. Then the integral of f with respect to μ is given by

$$\int f d\mu = \int f d\mu|_C \ .$$

The computation of the above integral can be obtained by cutting down, to the compact support C of f, the construction of the chain of simple valuations approximating the given locally finite measure μ.

Recall from section 4 that $\mu = \bigsqcup \mu_i$, where μ_i is the simple valuation, associated with the ordered cover $\mathcal{D}_i$ of X_i, given by $\sum_{j=1}^{n_i} r_{i,j}\delta_{\overline{O_{i,j}}}$, with $r_{i,j} = \mu(O_{i,j} - \bigcup_{k<j} O_{i,k})$. Since C is compact, there exists $n \in \mathbb{N}$ such that $C \subseteq X_i$ for all $i > n$. For $i > n$, define

$$\mu_i^C = \sum_{j=1}^{n_i} r_{i,j}^C \delta_{\overline{O_{i,j}} \cap C}, \text{ where } r_{i,j}^C = \mu(O_{i,j} \cap C - \bigcup_{k<j} O_{i,k} \cap C) \ .$$

The following properties are then easily derived as in section 4 (observe that it is enough to know the value of μ on the induced topology of C):

Lemma 6.2 $\sum_{O_{i,j} \cap C \subseteq O} r_{i,j}^C \leq \mu(O \cap C) \leq \sum_{\overline{O_{i,j}} \cap C \cap O \neq \emptyset} r_{i,j}^C$.

Then we have:

Proposition 6.3 *Let $\mu|_C$ and μ_i^C be defined as above. Then:*

1. *For all $i > n$, $\mu_i^C \sqsubseteq \mu|_C$;*
2. *For all $i > n$, $\mu_i^C \sqsubseteq \mu_{i+1}^C$;*
3. $\mu|_C = \bigsqcup_{i>n} \mu_i^C$.

The above proposition shows that the integral of f can be computed by means of the chain of simple valuations $\langle \mu_i^C \rangle_i$. Therefore R-integrability of f with respect to μ reduces to the R-integrability of $f|_C$ with respect to $\mu|_C$.

6.2 Functions with non-compact support

Let $f : X \to \mathbb{R}$ be a positive function which is bounded on compact sets. Given a subset A of X let χ_A denote the characteristic function of A. In order to extend the theory of generalized Riemann integration to this setting we need the following.

Definition 6.4 *A positive function f, bounded on compact sets, is R-integrable on X with respect to μ if, for all $i \in \mathbb{N}$, the function with compact support $f \cdot \chi_{X_i}$ is R-integrable and the limit*

$$\lim_{i \to \infty} \mathbf{R} \int_X f \cdot \chi_{X_i} d\mu$$

is finite. In this case we put

$$\mathbf{R}\int_X f d\mu = \lim_{i\to\infty} \mathbf{R}\int_X f \cdot \chi_{X_i} d\mu \ .$$

The above definition of R-integrability for positive functions bounded on compact sets does not depend on the choice of the increasing sequence of relatively compact subsets X_i's such that $X = \bigcup_i X_i$. Indeed, if $\langle Y_i\rangle_{i\in\mathbb{N}}$ is another such sequence then, for all i, $f \cdot \chi_{X_i}$ is R-integrable if and only if, for all i, $f \cdot \chi_{Y_i}$ is R-integrable and

$$\lim_{i\to\infty} \mathbf{R}\int_X f \cdot \chi_{X_i} d\mu = \lim_{i\to\infty} \mathbf{R}\int_X f \cdot \chi_{Y_i} d\mu.$$

In fact, for all i, there exists j such that $X_i \subseteq Y_j$, hence

$$\mathbf{R}\int_X f \cdot \chi_{X_i} d\mu \leq \mathbf{R}\int_X f \cdot \chi_{Y_i} d\mu$$

and conversely.

For a general function f we can use the decomposition

$$f = f^+ - f^-$$

where $f^+ = max(f, 0)$ and $f^- = max(-f, 0)$, and say that f is R-integrable if both f^+ and f^- are R-integrable and we put

$$\mathbf{R}\int_X f d\mu = \mathbf{R}\int_X f^+ d\mu - \mathbf{R}\int_X f^- d\mu \ .$$

The following result connects R-integration with respect to locally finite measures to Lebesgue integration, generalizing theorem 5.13.

Theorem 6.5 *If a real-valued function f bounded on compact sets is R- integrable with respect to a locally finite Borel measure μ on a locally compact second countable Hausdorff space, then it is also Lebesgue integrable and the two integrals coincide.*

Proof: By using the decomposition $f = f^+ - f^-$ and the fact that Lebesgue integrable functions are closed under sum, we can suppose $f \geq 0$.

By hypothesis, for all i, $f \cdot \chi_{X_i}$ is R-integrable. Since it is a function with compact support, by the results of the previous section, its R-integral can be computed with respect to the bounded measure $\mu|_{X_i}$. By theorem 5.13, it is Lebesgue integrable and

$$\mathbf{L}\int f \cdot \chi_{X_i} d\mu = \mathbf{R}\int f \cdot \chi_{X_i} d\mu \ .$$

Moreover $\langle f \cdot \chi_{X_i}\rangle_{i\in\mathbb{N}}$ is an increasing monotonic sequence of functions that converges pointwise to f. Therefore, by the monotone convergence theorem, f is Lebesgue integrable and

$$\mathbf{L}\int f d\mu = \lim_{i\to\infty} \mathbf{L}\int f \cdot \chi_{X_i} d\mu \ .$$

Since $\lim_{i\to\infty} \mathbf{L}\int f \cdot \chi_{X_i} d\mu = \lim_{i\to\infty} \mathbf{R}\int f \cdot \chi_{X_i} d\mu = \mathbf{R}\int_X f d\mu$, we have $\mathbf{L}\int f d\mu = \mathbf{R}\int_X f d\mu$. $\square$

7 The generalized Riemann integral on an open Set

In this section we generalize the definition of R-integration to an open set, in such a way that it gives the usual one when the open set is the whole space. We remark that for this purpose we could take O itself as a locally compact space and then use the results already established in the theory of integration developed so far. Nevertheless this extension is necessary if we want to compute the integral of f on O by using the original chain of simple valuations $\langle \mu_i \rangle_{i\in\mathbb{N}}$ approximating μ on X. Once this chain has been obtained, it will not be necessary to construct a new chain for each subspace O.

In what follows, f is a bounded non-negative real valued function on X and μ a normalized Borel measure on X.

Definition 7.1 *Let $\nu = \sum_{a\in A} r_a \delta_a$ be a simple valuation. The generalized lower and upper Darboux sums relative to the open O are, respectively:*

$$\begin{aligned} S^{\ell}_O(f,\nu) &\equiv \textstyle\sum_{a\in A, a\subseteq O} r_a \inf f[a] \ , \\ S^{u}_O(f,\nu) &\equiv \textstyle\sum_{a\in A, a\cap O\neq\emptyset} r_a \sup f[a] \ . \end{aligned}$$

Note that for $O = X$, the above sums reduce to the earlier definitions.

As in section 5, we will consider the ω-continuous dcpo with bottom $(UX)_\perp$ and obtain the generalization of proposition 5.1 for the lower and upper sums relative to an open set.

Proposition 7.2 *Let $\nu_1, \nu_2 \in P^1(UX)_\perp$ be simple valuations with $\nu_1 \sqsubseteq \nu_2$. Then $S^{\ell}_O(f,\nu_1) \leq S^{\ell}_O(f,\nu_2)$ and $S^{u}_O(f,\nu_2) \leq S^{u}_O(f,\nu_1)$.*

Proof: The first inequality holds as before. For the second, by the above lemma, we have:

$$\begin{aligned} S^u_O(f,\nu_2) &\equiv \sum_{b\in B, b\cap O\neq\emptyset} s_b \sup f[b] = \sum_{b\in B, b\cap O\neq\emptyset} \sum_{a\in A} t_{a,b} \sup f[b] \\ &= \sum_{b\in B, b\cap O\neq\emptyset} \sum_{a\in A, a\cap O\neq\emptyset} t_{a,b} \sup f[b] \leq \sum_{b\in B, b\cap O\neq\emptyset} \sum_{a\in A, a\cap O\neq\emptyset} t_{a,b} \sup f[a] \\ &= \sum_{a\in A, a\cap O\neq\emptyset} r_a \sup f[a] \equiv S^u_O(f,\nu_1) \ . \ \square \end{aligned}$$

Definition 7.3 *The* lower R-integral *of f on O with respect to μ is*

$$\mathbf{R}\underline{\int}_O f d\mu = \sup\nolimits_{\nu\ll\mu} S^{\ell}_O(f,\nu) \ .$$

The upper R-integral *of f on O with respect to μ is*

$$\mathbf{R}\overline{\int}_O f d\mu = \inf\nolimits_{\nu\ll\mu} S^{u}_O(f,\nu) \ .$$

We clearly have $\mathbf{R}\underline{\int}_O f d\mu \leq \mathbf{R}\underline{\int}_O f d\mu = \sup_{\nu\ll\mu} S^{\ell}_O(f,\nu)$.

Proposition 7.4 *If f is a non-negative, real-valued function on X which is continuous almost everywhere with respect to μ, then for all $\epsilon > 0$ there exists $\nu \ll \mu$, $\nu = \sum_{b\in B} r_b \delta_b$, such that*

$$\sum_{b\in B, b\subseteq O} r_b \sup f[b] - S^{\ell}_O(f,\nu) < \epsilon \ .$$

Proof: The proof proceeds as the proof of the 'if' part of theorem 5.12 by showing that for all $\epsilon > 0$ there exists a simple valuation $\nu = \sum_{b\in B} r_b\delta_b \ll \mu$, such that $\sum_{b\in B, b\subseteq O} r_b(\sup f[b] - \inf f[b]) < \epsilon$. $\square$

7.1 R-integration on an open set and Lebesgue integration

We will now investigate the connection between R-integration on an open set and Lebesgue integration. We will see that when the function f is continuous almost everywhere and $f \geq 0$ then the lower integral coincides with the Lebesgue integral of f on O whereas the upper integral coincides with the Lebesgue integral of f on $\overline{O}$. An important consequence of this fact is that the lower and the upper integral are equal when the boundary of O has measure zero.

In what follows, μ is a normalized Borel measure on X, f is a bounded, non-negative, real-valued function on X whose set of discontinuities has measure zero and O is an open subset of X.

Lemma 7.5 *Let $\mu = \bigsqcup_{\nu \in D} \nu$ be the lub of a directed set of simple valuations on UX and O an open set. For $\nu = \sum_{b \in B} r_b \delta_b$, define $\nu_O = \sum_{b \in B, b \subseteq O} r_b \delta_b$. Then the set $\{\nu_O : \nu \in D\}$ is directed and $\bigsqcup_{\nu \in D} \nu_O = \mu|_O$.*

Proof: First we prove that for two simple valuations ν and α, $\nu \sqsubseteq \alpha$ implies $\nu_O \sqsubseteq \alpha_O$ and therefore $\{\nu_O : \nu \in D\}$ is directed: For any open $A \subseteq UX$ we have $\nu_O(A) = \nu(A \cap \Box O) \leq \nu'(A \cap \Box O) = \nu'_O(A)$.

Then we have

$$\begin{aligned} \mu|_O(A) &\equiv \mu|_O \circ s^{-1}(A) \equiv \mu(O \cap s^{-1}(A)) = \mu|_O \circ s^{-1}(\Box O \cap A) \\ &= \mu(\Box O \cap A) = (\textstyle\bigsqcup_{\nu \in D})\nu(\Box O \cap A) \\ &= \textstyle\bigsqcup_{\nu \in D} \nu_O(A) \ .\Box \end{aligned}$$

Lemma 7.6 *If f is a non-negative real valued function on X which is continuous almost everywhere with respect to μ then*

$$\mathbf{R}\underline{\int_O} f d\mu = \sup\{\int_C f d\mu \mid C \subseteq O, C \textit{ compact}\} \ .$$

Proof: For $O = X$, the result is clear. Assume that $O \neq X$. We first prove

$$\sup_{\nu \ll \mu} S^{\ell}_O(f, \nu) \leq \sup\{\int_C f d\mu \mid C \subseteq O, C \text{ compact}\}.$$

Let $\nu \equiv \sum_{b \in B} r_b \delta_b \ll_{P^1(UX)_\perp} \mu$, $\nu_O \equiv \sum_{b \in B, b \subseteq O} r_b \delta_b$ and $C \equiv \bigcup_{b \in B, b \subseteq O} b$. Observe that C is compact since it is a finite union of compact sets and $\nu_O \in P(UC)$. By proposition 2.4(ii), $\nu^- \ll_{P(UX)_\perp} \mu$. It follows by proposition 2.2, applied first to $P(UX)_\perp$ and then to $P(UC)$, that $(\nu^-)_O \ll_{P(UC)} \mu|_C$. We also have $\nu_O = (\nu^-)_O$ since $O \neq X$ by assumption. Hence, $\nu_O \ll_{P(UC)} \mu|_C$, and consequently, $S^{\ell}_O(f, \nu) = S^{\ell}_C(f, \nu_O) \leq R \int_C f d\mu$.

For the converse inequality, let C be a compact subset of O and let $\mu|_C$ and $\mu|_O$ be the restrictions of μ to C and to O respectively. Since f is non-negative,

$$\int_C f d\mu \leq \int_X f d\mu|_O \ .$$

By lemma 7.5, $\mu|_O = \bigsqcup_{\nu \ll \mu} \nu_O$ and hence $\int_X f d\mu|_O = \sup_{\nu \ll \mu} S^{\ell}(f, \nu_O) = \sup_{\nu \ll \mu} S^{\ell}_O(f, \nu)$. $\Box$

We observe that by the above lemma the generalization to open sets that we gave for R-integration extends the usual for ordinary Riemann integration: indeed, if $]a, b[$ is an open interval of the real line the Riemann integral of f on it can be defined in the following way:

$$\int_{]a,b[} f(x)dx \equiv \lim_{\epsilon \to 0} \int_{a+\epsilon}^{b-\epsilon} f(x)dx = \sup\{\int_I f(x)dx \mid I \subseteq]a,b[,\ I \text{ compact}\}$$

Therefore R-integration on open sets extends usual Riemann integration on open intervals.

As announced, we have:

Proposition 7.7 *If f is a non-negative real valued function on X which is continuous almost everywhere with respect to μ then*

$$\mathbf{R}\underline{\int}_O f d\mu = \mathbf{L}\int_O f d\mu \ .$$

Proof: Since X is locally compact and second countable, there exists a chain $\langle C_i \rangle_{i \in \mathbb{N}}$ of compact sets such that $O = \bigcup_{i \in \mathbb{N}} C_i$. By applying Lebesgue's monotone convergence theorem to the sequence of functions $\langle f\chi_{C_i} \rangle_{i \in \mathbb{N}}$, we obtain

$$\mathbf{L}\int_O f d\mu = \sup_{C \subseteq O,\, C\ compact} \mathbf{L}\int_C f d\mu \ .$$

The conclusion then follows by lemma 7.6 and the fact that we already know that on compact spaces the Lebesgue and the generalized Riemann integral coincide. $\square$

Recall that for any subset $A \subseteq X$ of a metric space (X,d) and any $r \geq 0$, the r-parallel body A_r of A is given by $A_r = \{x \in X \mid \exists y \in A.\, d(x,y) \leq r\}$.

Proposition 7.8 *If f is a non-negative real valued function on X which is continuous almost everywhere with respect to μ then*

$$\mathbf{R}\overline{\int}_O f d\mu = \mathbf{L}\int_{\overline{O}} f d\mu \ .$$

Proof: We will start with proving that $\mathbf{R}\overline{\int}_O f d\mu \leq \mathbf{L}\int_{\overline{O}} f d\mu$. Let $O_{1/n}$ be the $1/n$-parallel body of O. Since

$$\mathbf{L}\int_{\overline{O}} f d\mu = \inf_{n \geq 0} \mathbf{L}\int_{O_{1/n}} f d\mu \ ,$$

it is enough to prove that, for all positive integer n, $\mathbf{R}\overline{\int}_O f d\mu \leq \mathbf{L}\int_{O_{1/n}} f d\mu$. Fix $n \geq 1$ and let $\epsilon > 0$ be given.

Let $M \equiv \sup f[X]$. By proposition 5.11 there exists $\nu_1 \ll \mu$ such that, for all $\nu \equiv \sum_{a \in A} r_a \delta_a \sqsupseteq \nu_1$, we have

$$\sum_{a \in A, |a| \geq 1/n} r_a < \frac{\epsilon}{2M} \ .$$

By applying propositions 7.4 and 7.7 to $O_{1/n}$, there exists $\nu_2 \ll \mu$ such that, for all $\nu \equiv \sum_{a \in A} r_a \delta_a \sqsupseteq \nu_2$, we have:

$$\sum_{a \in A, a \subseteq O_{1/n}} r_b \sup f[b] - \mathbf{L}\int_{O_{1/n}} f d\mu < \epsilon/2 \ .$$

Take $\nu \equiv \sum_{a \in A} r_a \delta_a \ll \mu$ with $\nu_1, \nu_2 \sqsubseteq \nu$. Then,

$$
\begin{aligned}
\mathbf{R}\overline{\int}_O f d\mu &\leq S^u_O(f,\nu) \\
&= \textstyle\sum_{a \in A, a \cap O \neq \emptyset, |a| < 1/n} r_a \sup f[a] + \sum_{a \in A, a \cap O \neq \emptyset, |a| \geq 1/n} r_a \sup f[a] \\
&\leq \textstyle\sum_{a \in A, a \subseteq O_{1/n}} r_a \sup f[a] + M \sum_{a \in A, |a| \geq 1/n} r_a \\
&\leq \textstyle\sum_{a \in A, a \subseteq O_{1/n}} r_a \sup f[a] + \epsilon/2 \\
&\leq \mathbf{L} \textstyle\int_{O_{1/n}} f d\mu + \epsilon.
\end{aligned}
$$

The conclusion follows since ϵ is arbitrary.

As for the converse inequality, we have to prove that, for all $\nu \ll \mu$, $S^u_O(f,\nu) \geq \mathbf{L} \int_{\overline{O}} f d\mu$. For this purpose, we will use the chain of simple valuations $\langle \nu_i \rangle_{i \in \mathbb{N}}$ constructed in section 4 with $\mu = \bigsqcup_{i \in \mathbb{N}} \nu_i$.

If $O \ll X$, there exists $i \in \mathbb{N}$ such that $O \subseteq X_i$. Moreover, since $\nu \ll \mu$, there exists j such that $\nu \sqsubseteq \nu_j$. Let $k = \max\{i,j\}$. Then $O \subseteq X_k$ and and $\nu \sqsubseteq \nu_k$. To fix the notation, let ν_k be the simple valuation associated to the ordered cover $\langle O_1, \ldots, O_n \rangle$ of $\overline{X}_k$. Since the upper sums decrease, we have:

$$
\begin{aligned}
S^u(f,\nu) &\geq S^u_O(f,\nu_k) \\
&\equiv \textstyle\sum_{\overline{O_i} \cap O \neq \emptyset} \mu(O_i - \cup_{j<i} O_j) \sup f[\overline{O_i}] \\
&\geq \textstyle\sum_{\overline{O_i} \cap O \neq \emptyset} \mu((O_i - \cup_{j<i} O_j) \cap \overline{O}) \sup f[(O_i - \cup_{j<i} O_j) \cap \overline{O}] \\
&\geq \textstyle\sum_{\overline{O_i} \cap O \neq \emptyset} \mathbf{L} \int_{(O_i - \cup_{j<i} O_j) \cap \overline{O}} f d\mu = \mathbf{L} \int_{\overline{O}} f d\mu
\end{aligned}
$$

If O is not way-below X we can use the decomposition $O = \bigcup_{i \in \mathbb{N}} O \cap X_i$, where, for all i, $O \cap X_i \ll X$. By the above argument, for all i we have

$$\mathbf{R}\overline{\int}_{O \cap X_i} f d\mu \geq \mathbf{L} \int_{\overline{O \cap X_i}} f d\mu$$

and therefore, for all i,

$$\mathbf{R}\overline{\int}_{O} f d\mu \geq \mathbf{L} \int_{\overline{O \cap X_i}} f d\mu$$

that gives

$$\mathbf{R}\overline{\int}_{O} f d\mu \geq \sup_{i \in \mathbb{N}} \mathbf{L} \int_{\overline{O \cap X_i}} f d\mu$$

and the conclusion follows since $\sup_{i \in \mathbb{N}} \mathbf{L} \int_{\overline{O \cap X_i}} f d\mu = \mathbf{L} \int_{\overline{O}} f d\mu$. $\square$

The following linearity properties are easily derived from the corresponding properties of the Lebesgue integral:

Proposition 7.9

1. *$\mathbf{R}\underline{\int}_O (f+g) d\mu = \mathbf{R}\underline{\int}_O f d\mu + \mathbf{R}\underline{\int}_O g d\mu$.*
2. *If c is any positive real number, then $\mathbf{R}\underline{\int}_O c f d\mu = c\ \mathbf{R}\underline{\int}_O f d\mu$.*

Similarly we have:

Proposition 7.10 *If O_1, O_2 are disjoint open sets then*

$$\mathbf{R}\underline{\int}_{O_1\cup O_2} f d\mu = \mathbf{R}\underline{\int}_{O_2} f d\mu + \mathbf{R}\underline{\int}_{O_2} f d\mu \ .$$

Proposition 7.11 *If a sequence $\langle f_j\rangle_{j\in\mathbb{N}}$ of R-integrable, real-valued functions defined on X is uniformly convergent to the function f and O is an open subset of X, then f is R-integrable on O and $\mathbf{R}\underline{\int}_O f d\mu = \lim_{j\to\infty} \mathbf{R}\underline{\int}_O f_j d\mu$.*

By the next proposition, in the following we can dispense with the assumption on f being non-negative.

Proposition 7.12 *If f is a bounded real-valued function which is continuous almost everywhere then $\mathbf{L}\int_O f d\mu = \mathbf{R}\int_O f^+ d\mu - \mathbf{R}\int_O f^- d\mu$.*

Proof: We have $\mathbf{L}\int_O f d\mu = \mathbf{L}\int_O f^+ d\mu - \mathbf{L}\int_O f^- d\mu = \mathbf{R}\int_O f^+ d\mu - \mathbf{R}\int_O f^- d\mu$. □

7.2 Computation of the R-integral on an open set

In order to have a computation of the integral $\int_O f d\mu$ of a bounded function continuous almost everywhere with respect to μ, we require to have two sequences which converge to the expected value of the integral from below and from above, so that at each stage of the computation a lower bound and an upper bound for the value of the integral is obtained.

First, by definition, for every simple valuation ν,

$$S^{\ell}_O(f,\nu) \leq S^{u}_O(f,\nu) \ .$$

Then it follows from proposition 7.2 that for any two simple valuations $\nu_1, \nu_2 \ll \mu$ we have

$$S^{\ell}_O(f,\nu_1) \leq S^{u}_O(f,\nu_2)$$

and therefore

$$\sup_{\nu\ll\mu} S^{\ell}_O(f,\nu) \leq \inf_{\nu\ll\mu} S^{u}_O(f,\nu) \ .$$

The lower sums and the upper sums of a continuous function relative to a given a chain of valuations and a given open may not converge to the same limit, as the following example of an iterated function system (IFS) with probabilities [18, 3] will show.

Example. Consider the IFS with probabilities on the space $X = [0,1]$:

$$f_1 : x \mapsto x/2 \qquad p_1 = 1/3$$

$$f_2 : x \mapsto 1/2 \qquad p_2 = 1/3$$

$$f_3 : x \mapsto x/2 + 1/2 \qquad p_1 = 1/3$$

This IFS gives rise to the following chain $\langle \nu_n\rangle_{n\geq 1}$ of simple valuations on the upper space of X, where

$$\nu_n = 1/3^n \sum_{i_1,\ldots,i_n=1}^{3} \delta_{f_{i_1},\ldots,f_{i_n}[0,1]}$$

It follows from [6, 7] that $\bigsqcup_{n\geq 1}\nu_n$ is maximal in $P^1(UX)$ and gives the unique measure μ satisfying

$$\mu = 1/3(\mu\circ f_1^{-1} + \mu\circ f_2^{-1} + \mu\circ f_3^{-1}) \ .$$

Observe that there is a non-zero mass on all points $1/2^n$.

If we consider as f the identity function, we have, for all n,

$$S^u_{[0,1/2)}(f,\nu_n) - S^\ell_{[0,1/2)}(f,\nu_n) \geq 1/6$$

Therefore the upper sums and the lower sums cannot converge to the same limit.

The lower sums and the upper sums of f with respect to O are proved to converge to the same limit under the auxiliary assumption that the boundary of O has measure zero.

Theorem 7.13 *Let X be a compact metric space, μ a normalized measure on X, O an open subset of X, $f : X \to \mathbb{R}$ a bounded function continuous almost everywhere with respect to μ. If the boundary of O has measure zero, then the lower sums and the upper sums of f on O converge to the same limit which is the Lebesgue integral of f on O.*

Proof: We already know that the lower sums of f on O converge from below to $\mathbf{L}\int_O f d\mu$ and that the upper sums converge from above to $\mathbf{L}\int_{\overline{O}} f d\mu$. If the boundary of O has measure zero then $\mathbf{L}\int_O f d\mu = \mathbf{L}\int_{\overline{O}} f d\mu$ and the conclusion follows. □

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