A Computational Model for Multi-Variable Differential Calculus

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Abstract

We develop a domain-theoretic computational model for multi-variable differential calculus, which for the first time gives rise to data types for differentiable functions, by constructing an effectively given continuous Scott domain for real-valued Lipschitz functions on finite dimensional Euclidean spaces. The model for real-valued Lipschitz functions of n variables, is built as a sub-domain of the product of $n+1$ copies of the function space on the domain of intervals by tupling together consistent information about locally Lipschitz (piecewise differentiable) functions and their differential properties (partial derivatives). The main result of the paper is to show, in two stages, that consistency is decidable on basis elements, which implies that the domain can be given an effective structure. First, a domain-theoretic notion of line integral is used to extend Green's theorem to interval-valued vector fields and show that integrability of the derivative information is decidable. Then, we use techniques from the theory of minimal surfaces to construct the least and the greatest piecewise linear functions that can be obtained from a tuple of $n+1$ rational step functions, assuming the integrability of the n-tuple of the derivative part. This provides an algorithm to check consistency on the rational basis elements of the domain, giving an effective framework for multi-variable differential calculus.

1 Introduction

We develop a domain-theoretic computational model for multi-variable differential calculus, which for the first time gives rise to data types for real-valued Lipschitz or differentiable functions on finite dimensional Euclidean spaces. This extends the corresponding result in [11] for dimension $n = 1$ to higher dimensions $n > 1$. While many of the properties of the domain of Lipschitz functions on $\mathbb{R}$ extend easily, as shown in [8], even to infinite dimensional Banach spaces, constructing an effective structure for the domain in the finite dimensional case $n > 1$ has been a challenge.

The model is a continuous Scott domain for Lipschitz functions of n variables. It allows us to deal with Lipschitz or differentiable functions in a recursion theoretic setting, and is thus fundamental for applications in computational geometry, geometric modelling, ordinary and partial differential equations and other fields of computational mathematics. The overall aim of the framework is to synthesize differential calculus and computer science, which are two major pillars of modern science and technology.

As in dimension one, the basic idea of the model, for a finite dimensional Euclidean space $\mathbb{R}^n$ or for an infinite dimensional Banach space X, is to collect together the local differential properties of the function by developing a generalization of the concept of Lipschitz constant to a non-empty, convex and compact set-valued Lipschitz constant in $\mathbb{R}^n$ in the finite dimensional case and a non-empty, convex and weak* compact set-valued Lipschitz constant in the dual X^* for the infinite dimensional case. The collection of these local differentiable properties are then used to define the Lipschitz or *L-derivative* of a function and the primitives of a Scott continuous, non-empty, convex and compact (respectively, weak* compact) set-valued vector field in $\mathbb{R}^n$ (respectively, in X^*), which leads to a fundamental theorem of calculus for set-valued derivatives, which was shown first for dimension one [10] and then for infinite dimensions [8].

The L-derivative is in fact closely related to Clarke's gradient, which is a key tool in nonsmooth analysis, control theory and optimization theory [5, 6] and is defined by using the generalized directional derivative based on taking the limsup of the rate of change of the function along a given direction. It has been shown in [8] that the L-derivative and the Clarke's gradient coincide in finite dimensions.

In a series of papers, Borwein and his collaborators have studied various properties of the Clarke gradient and developed new related notions [2, 3, 4]. In particular, given a weak* upper semi-continuous map g that is non-empty, convex and weak* compact set-valued from a Banach space to the space of subsets of its dual, a *g-Lipschitz map* is defined as one whose Clarke gradient at every point is contained in the set value of g at that point. In finite dimensions, the set of g-Lipschitz maps is precisely the set of primitives of g, a result which is a direct consequence of the equivalence of the L-derivative and Clarke gradient. Whereas in the domain-theoretic setting the generalized differential properties are used to develop the notion of primitives and the fundamental theorem of calculus is then deduced as a proposition, in the work of Borwein et al the g-Lipschitz maps are defined precisely by using the relation that expresses the fundamental theorem of calculus.

The fundamental theorem is used here, in the finite dimensional case, to construct the domain of differentiable functions as a sub-domain of the product of the space of interval-valued function of n variables and the space of functions of n variables that take nonempty compact sets as values. Geometrically, the first component of such a pair serves as an approximation of the function value, and the second component simultaneously approximates all n partial derivatives. We call such a pair consistent if there exists a piecewise differentiable function, equivalently a piecewise linear function, which is approximated by the first component, and whose n partial derivatives, wherever defined, are approximated by the second component of the pair.

The geometric meaning of the finitary data type and consistency is as follows. Each step function that approximates the function value is represented by a finite set of pairs $(a_i, b_i)_{i\in I}$ where $a \subseteq \mathbb{R}^n$ is a rational hyper-rectangle and $b \subseteq \mathbb{R}$ is a compact interval such that b_i and b_j have non-empty intersection whenever this is the case for the interiors of a_i and a_j. Similarly, approximations of the n partial derivatives are given as finite sets of pairs $(a_i, b_i)_{i\in I}$ where the a_i are as above but the b_j are now rational

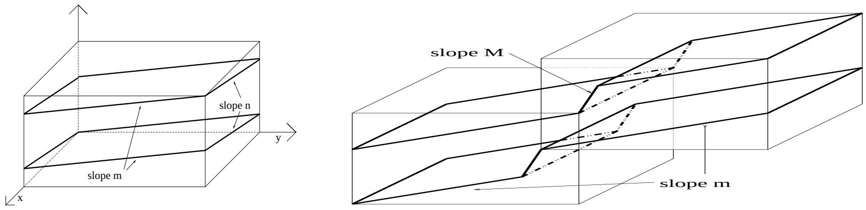

Figure 1: Two examples of consistent function and derivative approximations

polyhedra.

We now call a pair consisting of an approximation to the function value and its n partial derivatives consistent if there exists a piecewise differentiable function which is approximated together with its partial derivatives, where defined, by the pair. For a consistent pair, there are a least and a greatest piecewise differentiable function which satisfy the function and the partial derivative constraints.

Figure 1 shows two examples of consistent tuples for $n = 2$ and in each case the least and greatest functions consistent with the derivative constraints are drawn. In the first case, on the left, there is a single hyper-rectangle for function approximation and the derivative approximations in the x and y directions over the whole domain of the function are given respectively by the constant intervals $[n, N]$ and $[m, M]$ with $n, m > 0$. In the second case, on the right, there are two intersecting hyper-rectangles for the function approximation and the derivative approximations are the constant intervals $[0, 0]$ and $[m, M]$ with $m > 0$.

The main question now is whether consistency of a pair containing function and derivative information is actually decidable. For $n = 1$, it was shown in [11] that consistency is decidable and in [9], a linear algorithm was presented (linear in the number of pairs in the two step functions) which decides the consistency in this case. For finite $n > 1$, the problem is, as we have seen, very simple to state but not easy to solve. The main result of the paper is to show, in two stages, that consistency is decidable on basis elements.

In the first stage we need to develop a decidable condition for the integrability of a compact-convex set-valued function. As in classical differential multi-variable calculus, such a function may fail to be integrable. Borwein et al [4, Theorem 8] have derived a necessary and sufficient condition for the integrability of an upper semi-continuous (equivalently Scott continuous) compact-convex set-valued function on finite dimensional Euclidean spaces in terms of the existence of a measurable selection of the compact-convex set-valued function for which all line integrals over polygonal path vanishes; see Section 4.

In this paper, we provide an equivalent condition for this integrability, which we prove is decidable on basis elements of our domain. With this aim in mind, we introduce a domain-theoretic notion of line integral, which we use to establish the following necessary and sufficient condition for a compact-convex set-valued Scott continuous vector function to be integrable: zero must be contained in the line integral of the interval-valued vector field with respect to any closed path. This gives another extension of the classical Green's Theorem for a vector field to be a gradient, i.e., to be an exact differential, [14, pages 286-291]. We thus obtain a main result of this paper: an algorithm to check integrability for rational step functions.

Finally, we use techniques from the theory of minimal surfaces to construct the least and the greatest piecewise linear functions obtained from a pair of rational step functions, representing function and derivative approximation in which the derivative part is assumed to be integrable. These surfaces are obtained by, respectively, maximalizing and minimalizing the lower and the upper line integrals of the derivative information over piecewise linear paths. The maximalization and minimization are achieved for a piecewise linear path which can all be effectively constructed. The decidability of consistency is then reduced to checking whether the minimal surface is below the maximal surface, a task that can be done in finite time. This leads to an algorithm to check consistency of a pair representing function and derivative approximation and shows that consistency is decidable on the rational basis elements of the domain for locally Lipschitz functions, giving an effective framework for multi-variable differential calculus.

1.1 Related work

The domain for real-valued Lipschitz functions on the one dimensional real line has led to applications in solving initial value problems [9, 12]. The domain for the Lipschitz functions on finite dimensional Euclidean spaces has been used to develop domain-theoretic inverse and implicit function theorems for Lipschitz functions [13].

In computable analysis, Pour-El and Richards [16] relate the computability of a function with the computability of its derivative. The scheme employed by Weihrauch [19] leads to partially defined representations, but there is no general result on decidability. Interval analysis [15] also provides a framework for verified numerical computation. There, differentiation is performed by symbolic techniques [17] in contrast to our sequence of approximations of the functions.

1.2 Notations and terminology

We use the standard notions of domain theory as in [1]. For an open subset $U \subset \mathbb{R}^n$, let C^0U be the set of all continuous functions of type $U \to \mathbb{R}$ be the set of continuous functions and $D^0U = U \to \mathbf{I}\mathbb{R}$ the domain of all Scott continuous functions of type $U \to \mathbf{I}\mathbb{R}$; we often write D^0 for D^0U if no ambiguity can arise. A function $f \in D^0$ is given by a pair of respectively lower and upper semi-continuous functions $f^-, f^+ : U \to \mathbb{R}$ with $f(x) = [f^-(x), f^+(x)]$. We denote the continuous Scott domain of the nonempty compact subsets of $\mathbb{R}^n$ ordered by reverse inclusion by $\mathbf{C}\mathbb{R}^n$. We will use a canonical base of this domain, consisting of rational polyhedra together with the set $\mathbb{R}^n$ as a least element. We will consider the extension $- \cdot - : \mathbf{C}\mathbb{R}^n \times \mathbb{R}^n \to \mathbf{I}\mathbb{R}$ of the scalar product which is defined pointwise $b \cdot x = \{y \cdot x : y \in b\}$.

We use standard operations of interval arithmetic on interval matrices. We write $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ for the standard Euclidean norm of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. The Euclidean norm is extended to $b \in \mathbf{C}\mathbb{R}^n$ by $\|b\| = \max\{\|x\| : x \in b\}$. Recall that the derivative of a map $f : U \to \mathbb{R}$ at $y \in U$, when it exists, is defined as the linear map

$T : \mathbb{R}^n \to \mathbb{R}$ with

$$\lim_{\|x-y\|\to 0} \frac{|f(x) - f(y) - T(x-y)|}{\|x-y\|} = 0.$$

The linear map T is denoted by $f'(y)$. Let ∇f denote the gradient of f, when it exists, i.e.,

$$(\nabla f)_i(x) = \frac{\partial f}{\partial x_i} =$$

$$\lim_{x'_i \to x_i} \frac{f(x_1, \ldots, x_i, \ldots, x_n) - f(x_1, \ldots, x'_i, \ldots, x_n)}{x_i - x'_i},$$

for $1 \leq i \leq n$. Recall that if the derivative exists at a point then the gradient also exists at that point and has the same value. We will reserve the notation $\mathcal{L}d$ exclusively in this paper for the L-derivative which will be introduced later. The interior of a set $A \subset \mathbb{R}^n$ is denoted by A° and its closure by $\text{cl}(A)$.

We next aim to define the generalized (Clarke) gradient of a function [5, Chapter two] and explain its properties. Let $U \subset X$ be an open subset of a Banach space X and let $f : U \to \mathbb{R}$ be Lipschitz near $x \in U$ and $v \in X$. The *generalized directional derivative* of f at x in the direction of v is

$$f^\circ(x; v) = \limsup_{y \to x \ \ t \downarrow 0} \frac{f(y + tv) - f(y)}{t}.$$

Let us denote by X^* the dual of X, i.e. the set of real-valued continuous linear functions on X. We consider X^* with its weak* topology, i.e., the weakest topology on X^* in which for any $x \in X$ the map $f \mapsto f(x) : X^* \to \mathbb{R}$ is continuous.

The *generalized gradient* of f at x, denoted by $\partial f(x)$ is the subset of X^* given by

$$\{A \in X^* : f^\circ(x; v) \geq A(v) \text{ for all } v \in X\}.$$

It is shown in [5, page 27] that

- $\partial f(x)$ is a non-empty, convex, weak* compact subset of X^*.
- For $v \in X$, we have:

$$f^\circ(x; v) = \max\{A(v) : A \in \partial f(x)\}.$$

2 Domain for Lipschitz functions

This section reviews the necessary background on the L-derivative and the domain of Lipschitz maps, specialized to finite dimensions, from [8] to which we also refer for all proofs. The local differential property of a function is formalized in the domain-theoretic framework by the notion of an interval Lipschitz constant. Assume $U \subset \mathbb{R}^n$ is an open subset.

Definition 2.1. The continuous function $f : U \to \mathbb{R}$ has *a non-empty, convex and compact set-valued Lipschitz constant* $b \in \mathbf{C}\mathbb{R}^n$ in an open subset $a \subset U$ if for all $x, y \in a$ we have: $b \cdot (x - y) \sqsubseteq f(x) - f(y)$. The *single-step tie* $\delta(a, b) \subseteq C^0U$ of a with b is the collection of all partial functions f on U with $a \subset \mathrm{dom}(f) \subset U$ in C^0U which have b as a non-empty convex compact set-valued Lipschitz constant in a.

For example, if $n = 2$ and $b = b_1 \times b_2 \subseteq \mathbb{R}^2$, the information relation above reduces to $b_1(x_1 - y_1) + b_2(x_2 - y_2) \sqsubseteq f(x) - f(y)$. For a single-step tie $\delta(a, b)$, one can think of b as the non-empty compact-set Lipschitz constant for the family of functions in $\delta(a, b)$. A classical Lipschitz would require $k = \overline{b_i} = -\underline{b_i} \geq 0$ for all $i = 1 \cdots n$. By generalizing the concept of a Lipschitz constant in this way, one is able to obtain essential information about the differential properties of the function. In particular, if $f \in \delta(a, b)$ for $a \neq \emptyset$ and $b \neq \bot$, then $f(x)$ is maximal for each $x \in a$ and the induced function $f : a \to \mathbb{R}$ is Lipschitz: for all $x, y \in a$ we have $|f(x) - f(y| \leq \|b\||x - y|$.

For $f \in C^1U$, the following three conditions are equivalent: (i) $f \in \delta(a, b)$, (ii) $\forall z \in a.\ f'(z) \in b$ and (iii) $a \searrow b \sqsubseteq f'$.

For the rest of this section, we assume we are in dimension $n \geq 2$.

Definition 2.2. A *step tie* of C^0 is any finite intersection $\bigcap_{i \in I} \delta(a_i, b_i) \subset C^0$, where I is a finite indexing set. A *tie* of C^0 is any intersection $\Delta = \bigcap_{i \in I} \delta(a_i, b_i) \subset C^0$, for an arbitrary indexing set I. The *domain* of a non-empty tie Δ is defined as $dom(\Delta) = \bigcup_{i \in I}\{a_i \,|\, b_i \neq \bot\}$.

A non-empty step tie with rational intervals gives us a family of functions with a *finite* set of consistent differential properties, and a non-empty general tie gives a family of functions with a consistent set of differential properties. Recall that a function $f : U \to \mathbb{R}$ defined on the open set $U \subseteq \mathbb{R}^n$ is *locally Lipschitz* if it is Lipschitz in a neighborhood of any point in U. If $\Delta \subset C^0$ is a tie and $f \in \Delta$, then $f(x)$ is maximal for $x \in dom(\Delta)$ and f is locally Lipschitz on $dom(\Delta)$.

We now collect some simple properties of step ties, which we will use later and refer to [8] for proofs. For any indexing set I, the family of step functions $(a_i \searrow b_i)_{i \in I}$ is consistent if $\bigcap_{i \in I} \delta(a_i, b_i) \neq \emptyset$. One important corollary of this is that consistency of a family of step functions can be determined from the associated ties in a finitary manner: The family $(a_i \searrow b_i)_{i \in I}$ is consistent if for any finite subfamily $J \subseteq I$ we have $\bigcap_{i \in J} \delta(a_i, b_i) \neq \emptyset$.

Let $(T^1[0, 1], \supseteq)$ be the dcpo of ties of C^0 ordered by reverse inclusion. We are finally in a position to define the primitives of a Scott continuous function; in fact now we can do more and define:

Definition 2.3. The *primitive map* $\int : (U \to \mathbf{C}\mathbb{R}^n) \to T^1$ is defined by $\int(g) = \bigcap_{i \in I} \delta(a_i, b_i)$, where $g = \sup_{i \in I} a_i \searrow b_i$. We usually write $\int(f)$ as $\int f$ and call it the *primitives* of f.

The primitive map is well-defined, onto and continuous. For $n \geq 2$, as we are assuming here, the primitive map will have the empty tie in its range, a situation which does not occur for $n = 1$.

Example 2.4. Let $g \in [0, 1]^2 \to (\mathbf{C}\mathbb{R})^2$ be given by $g = (g_1, g_2) = (\lambda x_1.\lambda x_2.1, \lambda x_1.\lambda x_2.x_1)$. Then $\frac{\partial g_1}{\partial x_2} = 0 \neq 1 = \frac{\partial g_2}{\partial x_1}$, and it will follow that $\int g = \emptyset$.

Therefore, we have the following important notion in dimensions $n \geq 2$.

Definition 2.5. A map $g \in U \to \mathbf{C}\mathbb{R}^n$ is said to be *integrable* if $\int g \neq \emptyset$.

Given a Scott continuous function $f : U \to \mathbb{R}$, the relation $f \in \delta(a,b)$ provides, as we have seen, finitary information about the local differential properties of f. By collecting all such local information, we obtain the complete differential properties of f, namely its derivative.

Definition 2.6. The *derivative* of a continuous function $f : U \to \mathbb{R}$ is the map

$$\mathcal{L}f = \bigsqcup_{f \in \delta(a,b)} a \searrow b : U \to \mathbf{C}\mathbb{R}^n.$$

We have the following properties, which are established in [8] for the case of arbitrary (possibly infinite) dimension.

Theorem 2.7. (i) *$\mathcal{L}f$ is well-defined and Scott continuous.*

(ii) *If $f \in C^1U$ then $\mathcal{L}f = f'$.*

(iii) *$f \in \delta(a,b)$ iff $a \searrow b \sqsubseteq \mathcal{L}f$.*

(iv) *If f is differentiable at $x \in U$, then $f'(x) \in \mathcal{L}f$.*

In finite dimensional Euclidean spaces, the L-derivative coincides with the Clarke gradient [7, Corollary 8.2].

We also obtain the generalization of Theorem 2.7(iii) to ties, which provides a duality between the domain-theoretic derivative and integral and can be considered as a variant of the fundamental theorem of calculus:

Corollary 2.8. *$f \in \int g$ iff $g \sqsubseteq \mathcal{L}f$.*

The notion of g-Lipschitz functions due to Borwein et al [4] is defined as follows; we restrict to finite dimensions. Let $g \in (U \to \mathbf{C}\mathbb{R}^n)$ be Scott continuous. Then, the set of *g-Lipschitz maps* is defined as

$$\chi_g = \{f : U \to \mathbb{R} : f \text{ is locally Lipschitz and } \partial f(x) \subset g(x) \text{ for all } x \in U\}.$$

By the equivalence of the Clarke gradient and the L-derivative in finite dimensions, it follows immediately from Corollary 2.8 that $\chi_g = \int g$.

A domain for locally Lipschitz functions and for C^1U is constructed as follows. The idea is to use D^0U to represent the function and $U \to \mathbf{C}\mathbb{R}^n$ to represent the differential properties (partial derivatives) of the function. Consider the *consistency* relation

$$\mathsf{Cons} \subset D^0 \times (U \to \mathbf{C}\mathbb{R}^n),$$

defined by $(f,g) \in \mathsf{Cons}$ if $\uparrow\! f \cap \int g \neq \emptyset$. For a consistent (f,g), we think of f as the *function part* or the *function approximation* and g as the *derivative part* or the *derivative approximation*. We will show that the consistency relation is Scott closed.

Proposition 2.9. *Let $g \in U \to \mathbf{C}\mathbb{R}^n$ and $(f_i)_{i\in I}$ be a non-empty family of functions $f_i : dom(g) \to \mathbb{R}$ with $f_i \in \int g$ for all $i \in I$. If $h_1 = \inf_{i\in I} f_i$ is real-valued then $h_1 \in \int g$. Similarly, if $h_2 = \sup_{i\in I} f_i$ is real-valued, then $h_2 \in \int g$.*

Let $R[0,1]$ be the set of partial maps of $[0,1]$ into the extended real line. Consider the two dcpo's $(R[0,1], \leq)$ and $(R[0,1], \geq)$. Define the maps $s : D^0O \times (U \to \mathbf{C}\mathbb{R}^n) \to (R, \leq)$ and $t : D^0O \times (U \to \mathbf{C}\mathbb{R}^n) \to (R, \geq)$ by

$$s : (f,g) \mapsto \inf\{h : dom(g) \to \mathbb{R} \,|\, h \in \int g \;\&\; h \geq f^-\}$$

$$t : (f,g) \mapsto \sup\{h : dom(g) \to \mathbb{R} \,|\, h \in \int g \;\&\; h \leq f^+\}.$$

We use the convention that the infimum and the supremum of the empty set are ∞ and $-\infty$, respectively. Note that given a connected component A of $dom(g)$ with $A \cap dom(f) = \emptyset$, then $s(f,g)(x) = -\infty$ and $t(s,f)(x) = \infty$ for $x \in A$. In words, $s(f,g)$ is the least primitive map of g that is greater than the lower part of f, whereas $t(f,g)$ is greatest primitive map of g less that the upper part of f.

It then follows that the following three conditions are equivalent: (i) $(f,g) \in \mathsf{Cons}$, (ii) $s(f,g) \leq t(f,g)$ and (iii) There exists a continuous function $h : dom(g) \to \mathbb{R}$ with $g \sqsubseteq \mathcal{L}f$ and $f \sqsubseteq h$ on $dom(g)$.

Moreover, the maps s and t are Scott continuous and the relation Cons is Scott closed. We can sum up the situation for a consistent pair of function and derivative information.

Corollary 2.10. *Let $(f,g) \in \mathsf{Cons}$. Then in each connected component O of the domain of definition of g which intersects the domain of definition of f, there exist two locally Lipschitz functions $s : O \to \mathbb{R}$ and $t : O \to \mathbb{R}$ such that $s,t \in \uparrow f \cap \int g$ and for each $u \in \uparrow f \cap \int g$, we have with $s(x) \leq u(x) \leq t(x)$ for all $x \in O$.*

The central notion of this paper is now presented as follows:

Definition 2.11. Define

$$D^1U = \{(f,g) \in D^0U \times (U \to \mathbf{C}\mathbb{R}^n) : (f,g) \in \mathsf{Cons}\}.$$

The poset D^1U is a continuous Scott domain, i.e. a bounded complete countably based continuous dcpo. Now, let C^0U and C^1U be, respectively, the collection of real-valued C^0 and C^1 functions, and let $\Gamma : C^0U \to D^1U$ be defined by $\Gamma(f) = (f, \mathcal{L}f)$ and let Γ^1 be the restriction of Γ to C^1U. Then, the maps Γ and Γ^1 are respectively continuous injections of C^0U and C^1U into the set of maximal elements of D^1U.

Furthermore, Γ restricts to give a continuous injection for locally Lipschitz functions (where $\mathcal{L}f \neq \bot$ for all x) and it restricts to give a continuous injection for piecewise C^1 functions (where $\mathcal{L}f$ is maximal except for a finite set of points).

3 Interval-valued line integration

In this section, we extend the theory of line integration to interval-valued vector fields. The collection of step functions of the form $(f,g) \in D^0U \times \mathbf{C}\mathbb{R}^n$, where $f \in D^0U$

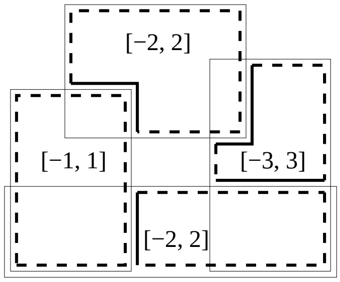

Figure 2: Crescents of a step function (left); the corners and their coaxial points (right)

and $g \in \mathbf{C}\mathbb{R}^n$ are step functions, forms a basis of D^1U. For a countable base, we fix a Cartesian coordinate system and take, as the countable base of topology of U, the hyper-rectangles in U, with faces parallel to this coordinate system, whose vertices have rational coordinates. This then provides us with rational single-step functions of the form $a \searrow r \in D^0U$ where $a \subset U$ is a rational hyper-rectangle with faces parallel to the coordinate system, whose vertices have rational coordinates and $r \in \mathbf{I}\mathbb{R}$ is a rational interval. It also gives us rational single-step functions of the form $a \searrow b$ where $a \subset U$ is as before and $b \in \mathbf{C}\mathbb{R}^n$ is a polyhedron with rational vertices. Note that the faces of b are not necessarily parallel to the coordinate planes. The *rational* basis of D^1U with respect to the given coordinate system is the collection of all step functions (f, g) built from such rational single-step functions.We will show in Section 6 that for rational step functions $f \in D^0U$ and $g \in \mathbf{C}\mathbb{R}^n$, the maps s and t will be piecewise linear, which can be effectively constructed to test the consistency of (f, g).

Let $g = (g_1, \ldots, g_n) \in \mathbf{C}\mathbb{R}^n$ be a rational step function. Recall that a *crescent* is the intersection of an open set and a closed set. The domain $\mathrm{dom}(g)$ of g is partitioned into a finite set of disjoint crescents $\{C_j : j \in I_i\}$, in each of which the value of g_i is constant, where we assume that the indexing sets I_i are pairwise disjoint for $i = 1, \ldots, n$. The collection

$$\{\bigcap_{1 \leq i \leq n} C_{k_i} : k_i \in I_i, 1 \leq i \leq n\}$$

of crescents partition $\mathrm{dom}(g)$ into regions in which the value of g is a constant interval vector; they are called the *associated crescents* of g, which play a main part in deciding integrability as we will see later in this section. Each associated crescent has boundaries parallel to the coordinate planes and these boundaries intersect at points, which are called the *corners* of the crescent.

In Figure 2, an example of a step function g is given with its associated crescents, the interval in each crescent gives the value of g in that crescent. A solid line on the boundary of a crescent indicates that the boundary is in the crescent, whereas a broken line indicates that it is not.

A path in a connected region $R \subset \mathbb{R}^n$ is a continuous map $p : [0, 1] \to R$ with endpoints $p(0)$ and $p(1)$. If p is piecewise C^1, respectively piecewise linear, then the path is called a piecewise C^1, respectively piecewise linear. The space $P(U)$ of piecewise C^1 paths in the region $U \subset \mathbb{R}^n$ is equipped with the C^1 norm:

$$\|p\| = \max\{\max_{r\in[0,1]} \|p(r)\|, \max_{p'(r) \text{ exists}} \|p'(r)\|\}. \tag{1}$$

A path p is *non-self-intersecting* if $p(r) = p(r')$ for $r < r'$ implies $r = 0$ and $r' = 1$. We will be mainly concerned with piecewise linear paths in this paper. For these paths, there exists a strictly increasing sequence of points $(r_i)_{0\leq i\leq k}$ for some $k \in \mathbb{N}$ with $0 = r_0 < r_1 < \cdots r_{k-1} < r_k = 1$ such that p is linear in $[r_i, r_{i+1}]$ for $0 \leq i \leq k-1$. The points $p(r_i)$ for $i = 0, \cdots, k$, are said to be the *nodes* of p; the nodes $p(r_i)$ for $i = 1, \cdots, k-1$ are called the *inner* nodes. The line segment $\{p(r) : r_i \leq r \leq r_{i+1}\}$ is denoted by $p([r_i, r_{i+1}])$. If $p(0) = p(1)$, the path is said to be *closed*.

A *simple* path in a region $R \subset \mathbb{R}^n$ is a non-self-intersecting piecewise C^1 map. We now consider simple paths in the closure cl(O) of a connected component $O \subseteq$ dom(g).

Recall that given a vector field $F : R \to \mathbb{R}^n$ in a region $R \subset R^n$ and a piecewise C^1 path $p : [0,1] \to R$, the line integral of F with respect to p from 0 to $w \in [0,1]$ is defined as $\int_0^1 F(p(r)) \cdot p'(r)\,dr$, when the integral exists. Here, $u \cdot v = \sum_{i=1}^n u_i v_i$ denotes the usual scalar product of two vectors $u, v \in \mathbb{R}^n$. For any $C \in \mathbf{C}\mathbb{R}^n$ and $\delta > 0$, we consider the open set $C_\delta = \{x \in \mathbb{R}^n : d(x, C) < \delta\} \subset \mathbb{R}^n$, where $d(x, C)$ is the minimum distance from the point x to C.

Lemma 3.1. *The map* $- \cdot - : \mathbb{R}^n \times \mathbf{C}\mathbb{R}^n \to \mathbf{I}\mathbb{R}$ *given by* $r \cdot A = \{r \cdot x : x \in A\}$ *is Scott continuous.*

Proof. Assume $r \in \mathbb{R}^n$ and $A \in \mathbf{C}\mathbb{R}^n$. Suppose $\epsilon > 0$ is given and consider the open ball $B(r, \delta)$ centered at r and of radius $\delta = \min(1, \epsilon/2(M+1)$ with $M = \sup\{|r| : r \in A\}$. Let $B \in \mathbf{C}\mathbb{R}^n$ with $B \subset A_{\epsilon/2(|r|+1)}$. For any $x \in B$, take $y \in A$ with $|x - y| \leq \epsilon/2(|r|+1)$. Then, for any $s \in B(r, \delta)$, we have: $s \cdot x = s \cdot x - s \cdot y + s \cdot y - r \cdot y + r \cdot y = s \cdot (x - y) + (s - r) \cdot y + r \cdot y < \epsilon/2 + \epsilon/2 + r \cdot y$ and thus $s \cdot x < r \cdot y + \epsilon \leq \overline{r \cdot A} + \epsilon$ and it follows that $\overline{s \cdot B} < \overline{r \cdot A} + \epsilon$. Similarly, we have: $\underline{s \cdot B} > \underline{r \cdot A} + \epsilon$. □

Corollary 3.2. *For a Scott continuous* $g : U \to \mathbf{C}\mathbb{R}^n$*, where* $U \subset \mathbb{R}^n$ *is an open set, and a piecewise smooth path* $p : [0,1] \to U$*, the map* $t \mapsto g(p(t)) \cdot p'(t) : [0,1] \to \mathbf{I}\mathbb{R}$ *is Scott continuous.*

Corollary 3.3. *For any piecewise linear map* $p \in P(U)$*, the map* $g \mapsto \lambda t.g(p(t)) \cdot p'(t) : (U \to \mathbf{C}(\mathbb{R}^n)) \to D^0$ *is Scott continuous.*

We now define the notion of line integral of the compact-convex polygon valued rational step function $g \in \mathbf{C}\mathbb{R}^n$ with respect to any piecewise C^1 path from y to x in the closure of a connected component of dom(g).

Definition 3.4. Given $g \in \mathbf{C}\mathbb{R}^n$ and a piecewise C^1 path $p : [0,1] \to \mathbb{R}^n$ in the closure of a connected component of the domain of g, the *line integral* of g over $p$is defined as:

$$\int_{p[0,w]} g(r)\,dr = [\mathsf{L}\int_{p[0,1]} g(r)\,dr, \mathsf{U}\int_{p[0,1]} g(r)\,dr]$$

where

$$\mathsf{U}\int_{p[0,1]} g(r)\,dr = \int_0^1 \overline{g(p(r))\cdot p'(r)}\,dr, \quad \mathsf{L}\int_{p[0,1]} g(r)\,dr = \int_0^1 \underline{g(p(r))\cdot p'(r)}\,dr.$$

Note that since $\lambda t.\overline{g(p(t))\cdot p'(t)}$ and $\lambda t.\underline{g(p(t))\cdot p'(t)}$ are, by Corollary 3.2, respectively upper and lower semi-continuous functions, the Lebesgue integrals in the above definition exist.

The interval-valued line integral $\int_{p[0,1]} g(p(t))\cdot p'(t)\,dt$ for a step function in $g \in (U \to \mathbf{C}\mathbb{R}^n)$ and a piecewise linear path p is easy to compute. For each associated crescent T of g with value K and each straight line segment $p : [a,b] \to T$ of p with $p'(t) = c$ for some $c \in \mathbb{R}^n$ for $t \in (a,b) \subset [0,1]$, the extended scalar product $c\cdot K = c\cdot x : x \in K = [m.M]$ where m and M are respectively the least and greatest values of $c\cdot x$ for $x \in K$, which will be the least and greatest values of $c\cdot x$ for x ranging over the vertices of K, since K is a convex polygon. In fact the problem is precisely the (standard) linear optimization problem: minimize and maximize $c\cdot x$ subject to the k conditions $c\cdot d^j \leq 0$ where d^j for $j = 1, ..., k$ are the normals to the k faces of K. The above scheme for computing m and M is precisely the fundamental theorem of linear programming which says that the min and max occur at vertices of the polygon K.

4 A generalization of Green's Theorem

Borwein et al [4] give a necessary and sufficient condition for a Scott continuous function $g : U \to \mathbf{C}\mathbb{R}^n$ where $U \subset \mathbb{R}^n$ is a non-empty open connected subset to be integrable (see also [3, Theorem 8]). We will now explain their condition.

The *line integral* of a measurable map $f : U \to \mathbb{R}^n$ on the line segment $[a,b] \subset U$ is given by the Lebesgue integral:

$$\int\limits_{[a,b]} f(z)\,dz := \int\limits_0^1 f(tb + (1-t)a)(b-a)\,dt.$$

The line integral of f on a piecewise linear path P in U is simply the sum of its line integrals on the line segments of P. For any fixed $\epsilon > 0$, an ordered collection of line segments $P(\epsilon) = \{[ai, bi] : 1 < i < n-1\}$ is an ϵ-path from a to b provided:

$$\|a - a_1\| + \sigma_{i=1}^{n-1}\|a_{i+1} - b_i\| + \|b_n - b\| < \epsilon.$$

Such a path is closed if $a = b$. For a Borel subset $E \subset U$, an ϵ-path P is an E-admissible ϵ-path from a to b if $\lambda(\{t \in [0,1] : tbi + (1-t)a_i \notin E\}) = 0$ for $1 < i < n-1$. Line integrals on an ϵ-path are defined similarly as above.

Theorem 4.1. *[4, Theorem 8] Let U be a non-empty open connected subset of $\mathbb{R}^n$ and let $g : U \to \mathbf{C}(\mathbb{R}^n)$ be a bounded Scott continuous map. Then g is integrable if and only if there exists a Borel set $E \subset U$ with $\lambda(U \setminus E) = 0$ and a measurable selection $f : E \to \mathbb{R}^n$ of g so that $\lim_{\epsilon\to 0^+}\int_{P(\epsilon)} f(z)\,dz = 0$, where $P(\epsilon)$ is any closed E-admissible ϵ-path in U.*

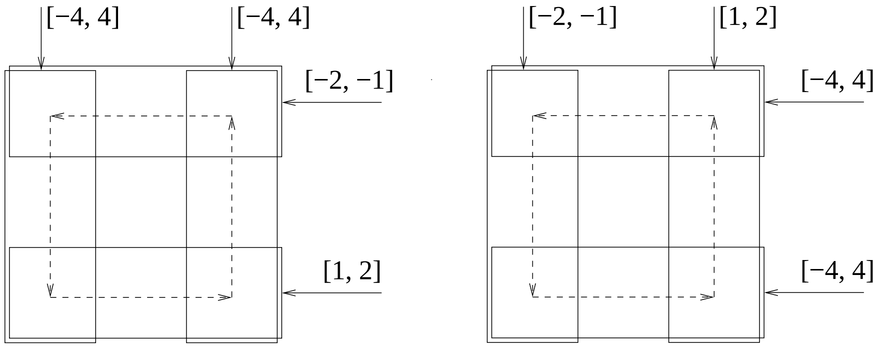

Figure 3: Failure of zero-containment: g_1 (left) and g_2 (right)

The existence of a measurable selection as above is in general non-decidable. In this section, we will derive an alternative necessary and sufficient condition for the integrability of a Scott continuous function $g : U \to \mathbf{C}\mathbb{R}^n$, so that the condition is decidable when g is in fact a rational step function.

We now introduce the domain-theoretic generalization of Green's celebrated condition for the integrability of a vector field.

Definition 4.2. Given $g \in (U \to \mathbf{C}\mathbb{R}^n)$ and a closed simple path p in the closure of a connected component of dom(g), we say that g satisfies the *zero-containment loop condition for* p if

$$0 \in \int_{p[0,1]} g(r)\, dr.$$

We say that $g \in (U \to \mathbf{C}\mathbb{R}^n)$ satisfies the *zero-containment loop condition* if it satisfies the zero-containment loop condition for any closed simple path p in the closure of any connected component of dom(g).

We note that if g is a step function then in the zero-containment condition above, it suffices to consider piecewise linear closed simple paths. If g only takes point (maximal) values, then the zero-containment loop condition is simply the standard condition for g to be a gradient i.e., that the line integral of g vanishes on any closed path. Figure 3 gives an example of a step function $g = (g_1, g_2)$, with dom(g) = $((0,3)\times(0,3))\setminus([1,2]\times[1,2])$ which does not satisfy the zero-containment loop condition. The values of g_1 (left) and g_2 (right) are given for each of the four single-step functions. Denote the dashed path by p; it has nodes at $p(0) = p(1) = (1/2, 1/2)$, $p(1/4) = (5/2, 0)$, $p(1/2) = (5/2, 5/2)$ and $p(3/4) = (1/2, 5/2)$. The lower line integral of g over p gives a strictly positive value:

$$\mathsf{L}\int_p g(r)dr = 8.$$

Recall that $g \in (U \to \mathbf{C}\mathbb{R}^n)$ is called integrable if $\int g \neq \emptyset$. The following is an extension of Green's Theorem also called the Gradient Theorem in classical differential calculus [14].

Theorem 4.3. *Suppose $g \in (U \to \mathbf{C}\mathbb{R}^n)$ is an integrable step function. Then g satisfies the zero-containment loop condition.*

Proof. Assume $h \in \int g$ and thus, by Corollary 2.8,

$$g \sqsubseteq \mathcal{L}h. \tag{2}$$

Take any closed piecewise linear path p_0 in a connected component O of dom(g) with $p_0(0) = p_0(1) = y_0$ and nodes at $0 = r_0 < r_1 < \cdots\cdots r_{k-1} < r_k = 1$ say. Then h is locally Lipschitz in cl(O) and thus by Rademacher's theorem [6, p 148] it is differentiable almost everywhere in O. We take small parallel transversal (Poincaré) $(n-1)$-dimensional sections [18] A_i at $p_0(r_i)$, for $i = 0, \cdots, k-1$, to the path p_0 and consider the family of closed piecewise linear paths p_y with $p_y(0) = p_y(1)$, where $y \in A_0$, such that, for each $y \in A_0$ and each i with $0 \le i \le k-1$, $p_y(r_i) \in A_i$, for $i = 1, \cdots k-1$, and the line segments $p_y[r_i, r_{i+1}]$ are parallel with $p_{y_0}[r_i, r_{i+1}]$. By Fubini's theorem, it follows that for almost all $y \in A_0$, with respect to the $(n-1)$-dimensional Lebesgue measure on A_0, the map h is differentiable on the path p_y almost everywhere with respect to the 1-dimensional Lebesgue measure on p_y. Let $y \in A_0$ be such a point. Thus $h'(x)$ exists almost everywhere for $x \in p_y[0,1]$ with respect to the 1-dimensional Lebesgue measure on $p_y[0,1]$ and, by Theorem 2.7 (iv), at these points we have $h'(x) \in \mathcal{L}h$. This together with Equation 2, implies that

$$h'(x) \in g(x), \tag{3}$$

for almost all $x \in p_y([0,1])$. Now the composition $h \circ p : [0,1] \to \mathbb{R}$ of locally Lipschitz maps is locally Lipschitz and thus equal to the integral of its derivative. Thus, the ordinary path integral of the gradient of h along p_y vanishes:

$$\begin{aligned}\int_0^1 h'(p_y(r)) \cdot p_y'(r)\, dr = \int_0^1 (h \circ p)'(r)\, dr = \\ h(p_y(1)) - h(p_y(0)) = h(y) - h(y) = 0.\end{aligned} \tag{4}$$

On the other hand, from the definitions of the lower and upper path integrals, using Equation 3, we have:

$$\mathsf{L}\int_{p_y} g(r)\, dr \le \int_0^1 h'(p_y(r)) \cdot p_y'(r)\, dr,$$

and

$$\int_0^1 h'(p_y(r)) \cdot p_y'(r)\, dr \le \mathsf{U}\int_{p_y} g(r)\, dr. \tag{5}$$

Thus,

$$\mathsf{L}\int_{p_y} g(r)\, dr \le 0 \le \mathsf{U}\int_{p_y} g(r)\, dr.$$

Since $\mathsf{L}\int_{p_y} g(r)\, dr$ and $\mathsf{U}\int_{p_y} g(r)\, dr$ depend continuously on $y \in A_0$ and since Equations (4) and (5) hold for almost all $y \in A_0$ with respect to the (n-1)-dimensional

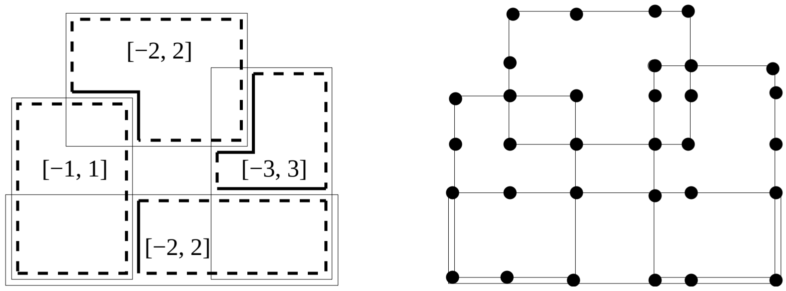

Figure 4: Crescents of a step function (left); the corners and their coaxial points (right)

Lebesgue measure on A_0, it follows that these equations hold actually for all $y \in A_0$ and in particular for $y = y_0$. This establishes that g satisfies the zero-containment loop condition for any closed non-self-intersecting piecewise linear path in O. By continuity, it follows that g also satisfies the zero-containment loop condition for any closed non-self-intersecting piecewise linear path in the closure of O. □

5 Integrability of derivative information

We will now show that if a step function $g \in (U \to \mathbf{C}\mathbb{R}^n)$ satisfies the zero-containment loop condition, then it is integrable. Let O be a connected component of dom(g). Note that any step function g can be extended to the boundary of dom(g) by the lower and upper semi continuity of g^- and g^+ respectively. We adopt the following convention. If two crescents have a common boundary, we consider their common boundary as infinitesimally separated so that they have distinct boundaries. This means that a line segment of a piecewise linear simple path on a common boundary of two different crescents is always regarded as the limit of a sequence of parallel segments contained on one side of this boundary.

A point of the boundary of an associated crescent of a step function is a *coaxial point* of a point in some associated crescent if the two points have precisely $n-1$ coordinates in common. Clearly, each point has a finite number of coaxial points. In Figure 4, the coaxial points of the corners of the crescents of the step function Figure 2, reproduced on the left, are illustrated on the picture on the right.

We are now ready to introduce a key concept of this paper. For $x, y \in$ cl(O)), we put

$$V_g(x,y) = \sup\{\mathsf{L}\int_{p[0,1]} g(r)\,dr : p \text{ a piecewise linear path in cl}(O) \text{ from } y \text{ to } x\},$$

$$W_g(x,y) = \inf\{\mathsf{U}\int_{p[0,1]} g(r)\,dr : p \text{ a piecewise linear path in cl}(O) \text{ from } y \text{ to } x\}.$$

We collect a few technical lemmas before we are able to prove the equivalence of integrability and the zero-loop containment condition. The first lemma shows that the supremum of lower path integrals is always attained for a path whose inner nodes are at the corners of the crescents. In presence of the zero-loop condition, this allows to effectively construct a piecewise linear path that maximizes the lower path integral.

Lemma 5.1. *Suppose $O = \bigcup_{i=0\ldots k} O_i$ is the disjoint union of rectangular crescents O_i and $g : O \to \mathbf{C}\mathbb{R}^n$ is constant on every crescent O_i. If $x, y \in O$, then*

$$\sup\{\mathsf{L}\int_q g(r)dr \mid q \text{ piecewise linear path in } O \text{ from } x \text{ to } y\}$$

is attained for a piecewise linear path with inner nodes at the corners of the crescents.

Proof. Suppose that $c_i \in \mathbf{C}\mathbb{R}^n$ is the (constant) value of g on the crescent O_i for all $i = 0, \ldots, k$. We show that every piecewise linear path p from x to y can be modified to a piecewise linear path q, also from x to y such that

- the inner nodes of q are at the corners of the crescents
- $\mathsf{L}\int_q g(r)dr \geq \mathsf{L}\int_p g(r)dr$.

Suppose p is a piecewise linear path in O with nodes $p_0, \ldots, p_l$. We may assume without loss of generality that each line segment (p_{j-1}, p_j) lies within a single crescent O_{i_j}. We have, for the upper line integral, that

$$\mathsf{U}\int_p g(r)dr = \sum_{j=1}^{l} x_j \cdot (p_j - p_{j-1})$$

for a collection of values $x_j \in c_{i_j}$. Now assume that an inner node $p_j \in O_{j_j}$ does not lie at the corner of any crescent O_{i_j}. As a consequence, both the line segment (p_{j-1}, p_j) and (p_j, p_{j+1}) lie within the same crescent O_{i_j}. Moreover, the function

$$f : \mathrm{cl}(O_{i_j}) \to \mathbb{R}, \quad f(v) = x_j(v - p_{j-1}) + x_{j+1}(p_{j+1} - v)$$

is a linear function of v and therefore attains its maximum at an extremal point, say p'_j, of its domain of definition. Replacing p_j by p'_j in the definition of the path p therefore gives a path p' such that $\mathsf{L}\int_{p'} g(r)dr \geq \mathsf{L}\int_p g(r)dr$ so that the j-th inner node lies at the corner of a crescent O_{i_j}. Repeating this process for all other inner points that do not lie on the corners of the crescents $O_0, \ldots, O_k$ produces the desired path. ☐

Proposition 5.2. *Suppose g satisfies the zero-containment loop condition and $x, y \in$ cl(O), then there are piecewise linear simple paths p and q from y to x such that:*

$$V_g(x, y) = \mathsf{L}\int_{p[0,1]} g(r)\, dr \qquad W_g(x, y) = \mathsf{U}\int_{q[0,1]} g(r)\, dr.$$

Proof. We prove the statements for the lower integral; those for the upper integral are dual to these. Consider $V_g(x_0, y)$ for some $x_0 \in \mathrm{cl}(O)$ and a piecewise linear path p^{x_0} in cl(O) with $p^{x_0}(0) = y$ and $p^{x_0}(1) = x_0$ and nodes $p^{x_0}(r_i)$ for $0 = r_0 < r_1 < \cdots < r_{k-1} < r_k$. By Lemma 5.1 it follows that the extremal values of $\mathsf{L}\int_p g(r)dr$ are attained when the inner nodes of the path are at the corners of the crescents and their coaxial points.

It then follows from the zero-containment loop condition that the maximum value of $\mathsf{L}\int_{p^{x_0}} g(r)\,dr$ is reached for a piecewise linear simple path, with inner nodes at the corners of the crescents and their coaxial points. Since there are a finite number of corners of the crescents of O and their coaxial points, it follows that there are a finite number of piecewise linear simple paths, say $p_1, \cdots, p_N$ for some some $N \geq 1$, from y to x with inner nodes at the corners of the crescents of g in O and their coaxial points, one of which necessarily maximizes $\mathsf{L}\int_{p[0,1]} g(r)dr$. $\square$

Proposition 5.3. *If g satisfies the zero-containment loop condition, then, for all $y \in \mathrm{cl}(O)$, the two maps given by $V_g(\cdot, y), W_g(\cdot, y) : cl(O) \to \mathbb{R}$ are continuous, piecewise linear and satisfy $V_g(y, y) = W_g(y, y) = 0$.*

Proof. For a fixed $y \in \mathrm{cl}(O)$, as x varies locally near x_0, each path p^x_m, for $1 \leq m \leq N$, depends continuously on x, with respect to the norm given in Formula (1), and thus $\mathsf{L}\int_{p^{x_0}_m} g(r)\,dr$ depends piecewise linearly on the coordinates of x, with the linearity coefficients changing only at the corners of the crescents boundary of the associated crescents of g in O and their coaxial points. Thus, for each $m = 1, \cdots, N$, we obtain for x near x_0 a family of N paths p^x_m ($1 \leq m \leq M$) with $p^x_m(0) = y$ and $p^x_m(1) = x$ such that the mapping $\lambda x.\mathsf{L}\int_{p^x_m} g(r)\,dr$ is a continuous piecewise linear surface. Therefore the map $V_g(\cdot, y)$, which is locally the maximum of N continuous piecewise linear local surfaces is itself a continuous piecewise linear surface.

In order to show that $V_g(y, y) = 0$, we note that the trivial constant path p with constant value y is a piecewise linear simple path from y to y with $\mathsf{L}\int_{p[0,1]} g(r)\,dr = 0$. By the zero-containment loop condition, any other closed piecewise linear simple path q from y to y satisfies $\mathsf{L}\int_q g(r)\,dr \leq 0$ and thus $V_g(y, y) = \mathsf{L}\int_{p[0,1]} g(r)\,dr = 0$.

The statement for the upper line integral is entirely dual. $\square$

Intuitively speaking, we have an N valued multi-surface and we can move continuously on this multi-surface from the point $(x, \mathsf{L}\int_{p^x_m} g(r)\,dr)$ on the graph of $\lambda x.\mathsf{L}\int_{p^x_m} g(r)\,dr$ to the point $(x, \mathsf{L}\int_{p^x_\ell} g(r)\,dr)$ on the graph of $\lambda x.\mathsf{L}\int_{p^x_\ell} g(r)\,dr$ (for $m \neq \ell$) by moving x continuously, around the holes of O, along some closed path back to its original position. The importance of the maps V_g and W_g lies in the fact that their derivatives provide a refinement of g.

Proposition 5.4. *If g satisfies the zero-containment loop condition, then the maps $V_g(\cdot, y)$ and $W_g(\cdot, y)$ satisfy*

$$g \sqsubseteq \mathcal{L}V_g(\cdot, y) \qquad \textit{and} \qquad g \sqsubseteq \mathcal{L}W_g(\cdot, y)$$

for all $y \in \mathrm{cl}(O)$.

Proof. We prove that $g \sqsubseteq \mathcal{L}V_g(\cdot, y)$ where the statement concerning the upper line integral is entirely analogous. First we show that there exists small $a > 0$ such that this relation holds for all y and x with in any closed n-dimensional disc of radius a with center in cl(O). For any $w \in \mathrm{cl}(O)$ there exists some $b_w > 0$ such for all y and x in the closed n-dimensional disc $\mathsf{D}_{b_w}(w)$ of radius b_w and center w, the path p from y to x with $V_g(x,y) = \mathsf{L}\int_{p[0,1]} g(r)\,dr$ will either be,

(i) the straight path from y to x, or,

(ii) the piecewise linear simple path from y to x with two line segments and inner node w.

In fact if w is not a corner, we can choose b_w small enough so that $\mathsf{D}_{b_w}(w)$ does not contain any corners and that it satisfies (i) above, whereas if w is a corner we can choose b_w such that (ii) above holds. Now, we consider the covering of the compact set cl(O) by the collection

$$\{\mathsf{D}^{\circ}_{b_w/2}(w) : w \in \text{cl}(O)\}$$

of the open discs $\mathsf{D}^{\circ}_{b_w/2}(w)$. Let

$$\{\mathsf{D}^{\circ}_{b_{w_i/2}}(w_i) : i = 1, \cdots, M\}$$

be a finite subcovering and let

$$a = \min\{b_{w_i}/2 : i = 1, \cdots, M\}.$$

Then, for any $w \in \text{cl}(O)$, we have $w \in \mathsf{D}^{\circ}_{b_{w_i/2}}(w_i)$ for some $i = 1, \cdots, M$. We have $\mathsf{D}_a(w) \subset \mathsf{D}^{\circ}_{b_{w_i}}(w_i)$ and thus for any $x, y \in \mathsf{D}_a(w)$, the piecewise linear simple path p from y to x with $V_g(x,y) = \mathsf{L}\int_{p[0,1]} g(r)\,dr$ will be either of type (i) or (ii) above, and $g \sqsubseteq \mathcal{L}V_g(\cdot, y)$ will hold as it is easily seen from the definition of lower path integral.

Now consider any $x, y \in \text{cl}(O)$ and consider the n-dimensional sphere $\mathsf{S}_a(x)$ and the disc $\mathsf{D}_a(x)$ of radius a with center x. For any $z \in \mathsf{S}_a(x)$ and $u \in \mathsf{D}_a(x)$, we have: $V_g(u,y) \geq V_g(z,y) + V_g(u,z)$ since any pair of piecewise linear paths p_1 (from y to z) and p_2 (from z to u) gives rise, by concatenation, to a piecewise linear path (p_1 followed by p_2) from y to u. Moreover,

$$V_g(u,y) = \sup\{V_g(z,y) + V_g(u,z)) : z \in \mathsf{S}_a(x)\},$$

since $V_g(u,y)$ is the maximum value of the lower path integral over all piecewise linear paths from y to u and any path p_0 as such will intersect $\mathsf{S}_a(x)$ at some point z and thus gives rise to a piecewise linear path p_1 from y to z and a piecewise linear path p_2 from z to u. Now for fixed $y, x \in \text{cl}(O)$ and fixed $z \in \mathsf{S}_a(x)$, the map $\lambda u.V_g(z,y) + V_g(u,z)$ satisfies

$$g \sqsubseteq \mathcal{L}(V_g(z,y) + V_g(\cdot, z))$$

since $V_g(z,y)$ is a constant and $u \in \mathsf{D}_a(x)$. Thus by Proposition 2.9, we have $g \sqsubseteq \mathcal{L}V_g(\cdot, y)$. Since $u \in \mathsf{D}_a(x)$ is an arbitrary point, in particular we have $g \sqsubseteq \mathcal{L}V_g(\cdot, y)$ and the proof is complete. $\square$

Thus, we obtain the following main result:

Theorem 5.5. *A function $g \in (U \to \mathbf{C}\mathbb{R}^n)$ is integrable iff it satisfies the zero-containment loop condition.*

Proof. If g is a step function then the result follows from Theorem 4.3 and Proposition 5.2. To extend the result to all functions we only need to show that the set of integrable functions in $(U \to \mathbf{C}\mathbb{R}^n)$ and the set of functions in $(U \to \mathbf{C}\mathbb{R}^n)$ that satisfy the zero-containment loop condition are both Scott closed. This will imply that they are continuous Scott subdomains of $(U \to \mathbf{C}\mathbb{R}^n)$ with a common basis consisting of step functions and thus they are the same domains. Now if $g \in (U \to \mathbf{C}\mathbb{R}^n)$ is integrable and $h \sqsubseteq g$, then $\emptyset \neq \int g \subseteq \int h$ and thus h is integrable. If $(g_i)_{i\geq 0}$ is an increasing sequence of integrable functions, then by Proposition 5.2 we have for any fixed $y \in \mathrm{cl}(O)$:

$$\cdots, V_{g_i} \leq V_{g_{i+1}} \leq \cdots W_{g_{i+1}} \leq V_{g_i} \leq \cdots,$$

where for all $i \geq 0$,

$$g_i \sqsubseteq \mathcal{L}dV_{g_i}(\cdot, y) \qquad \text{and} \qquad g_i \sqsubseteq \mathcal{L}W_{g_i}(\cdot, y).$$

Let $h_1 = \sup_{i\geq 0} V_{g_i}$. Then $h_1 : \mathrm{cl}(O) \to \mathbb{R}$ is real-valued and thus by Proposition 2.9, we have $g_i \sqsubseteq \mathcal{L}Vh_1$ for each $i \geq 0$. It follows that $g \sqsubseteq \mathcal{L}Vh_1$ and thus g is integrable. This shows that the set of integrable functions in $(U \to \mathbf{C}\mathbb{R}^n)$ is Scott closed.

Next assume that $g \in D^0$ satisfies the zero-containment loop condition and $h \sqsubseteq g$. By monotonicity of the path integral map it follows that h satisfies the zero-containment loop condition as well. If $(g_i)_{i\geq 0}$ is an increasing sequence of functions satisfying the zero-containment loop condition, then $0 \in \int_{p[0,1]} g_i \, dr$ for any closed path p and any $i \geq 0$ it follows that from the Scott continuity of the path integral map that $0 \in \int_{p[0,1]} g \, dr$. ☐

Proposition 5.6. *For a rational step function $g \in (U \to \mathbf{C}\mathbb{R}^n)$ defined over rational numbers, the zero-containment loop condition is decidable.*

Proof. There are a finite number of connected components of $\mathrm{dom}(g)$. In each connected component O of $\mathrm{dom}(g)$, the values of $\mathsf{L}\int_{p[0,1]} g(r)\, dr$ and $\mathsf{U}\int_{p[0,1]} g(r)\, dr$, for a closed piecewise linear simple path in $\mathrm{cl}(O)$ depend piecewise linearly on the coordinates of any given node of the path. It follows that the maximum value of the lower integral and the minimum value of the upper integral are reached for a path p with nodes at the corners of the crescents of O and their coaxial points. Since the number of such closed piecewise linear simple paths is finite and since for each such path $\mathsf{L}\int_{p[0,1]} g(r)\, dr$ is a rational number, we can decide in finite time if the zero-containment loop condition holds for g. ☐

For an associated crescent a of a step function g we write $v(a)$ for the value of g on a, i.e. $v(a) = g(x)$ where $x \in a^o$ is some point in the interior of a. To check whether a rational step function g is integrable, the proof of Proposition 5.6 shows that it suffices to check that g satisfies the zero-containment loop condition on all paths with nodes in the finite set of corners of the associated crescents and their coaxial points.

6 Consistency of function and derivative information

We will now show that for a pair of rational step functions $(f, g) \in D^1$, with g integrable, the consistency relation $(f, g) \in \mathsf{Cons}$ is decidable. For this, we explicitly construct $s(f, g)$ and $t(f, g)$, which will be piecewise linear functions that enable us to decide if $s(f, g) \leq t(f, g)$. Let x and y be in the same connected component O of dom(g) with $O \cap \text{dom}(f) \neq \emptyset$.

Theorem 6.1. *The maps $V_g(\cdot, y), W_g(\cdot, y) : cl(O) \to \mathbb{R}$ are respectively the least and the greatest continuous maps $L, G : O \to \mathbb{R}$ with $L(y) = 0$ and $G(y) = 0$ such that $g \sqsubseteq \mathcal{L}L$ and $g \sqsubseteq \mathcal{L}G$.*

Proof. We prove that $L = V_g(\cdot, y)$ by showing that the set $A = \{x \in \text{cl}(O) : L(x) = V_g(x, y)\}$ is non-empty, open and closed in cl(O), from which the result follows. Clearly $y \in A$ so $A \neq \emptyset$. Also since L and $V_g(\cdot, y)$ are both continuous, A is closed. It remains to show that A is open. Suppose A is not open. Then cl$(O) \setminus A \neq \emptyset$. We can take $u \in \text{cl}(O) \setminus A$ close to the boundary of A and away from the boundary of O such that the following condition holds: the locally minimal path p from y to u with $V_g(u, y) = \mathsf{L} \int_{p[0,1]} g(r)\, dr$ intersects the boundary of A at a point x such that the subpath q of p from y to x gives: $V_g(x, y) = \mathsf{L} \int_q g(r)\, dr$ and the the subpath of p from x to u is a straight line. Thus, $V_g(u, y) = V_g(x, y) + V_g(u, x)$. On the other hand,

$$L(u) \geq L(x) + V_g(u, x) = V(x, y) + V_g(u, x)$$

and it follows that $L(u) \geq V_g(u, y)$, i.e., $L(u) = V_g(u, y)$, which gives a contradiction. This establishes the proof. The case for G is similar. □

Let $S_{(f,g)}(x, y) = V_g(x, y) + \underline{\lim} f^-(y)$.

Corollary 6.2. *Let O be a connected component of dom(g) with non-empty intersection with dom(f). For $x \in O$, we have:*

$$s(f, g)(x) = \sup_{y \in O \cap \textit{dom(f)}} S_{(f,g)}(x, y). \tag{6}$$

Proof. By Proposition 6.1, the map $h_y = \lambda x. S_{(f,g)}(x, y)$ is the least function with $h_y(y) = \underline{\lim} f^-(y)$ such that $g \sqsubseteq \mathcal{L}h_y$. By definition, $s(f, g)$ is precisely the upper envelop of h_y for $y \in O$. □

Proposition 6.3. *There exist a finite number of points $y_0, y_1, \ldots, y_i \in cl(O \cap \textit{dom(f)})$ with*

$$s(f, g)(x) = \max\{S_{(f,g)}(x, y_j) : j = 0, 1, \ldots, i\}$$

for $x \in O$.

Proof. For fixed (f, g) and x, the value of $S_{(f,g)}(x, y)$ depends piecewise linearly on the coordinates of y, and thus its maximum value is reached for a piecewise linear simple path with modes at the corners of the crescents of O and x and their coaxial points. □

Results dual to those above are obtained for $t(f,g)$ as follows. We put $T_{(f,g)}(x,y) = W_g(x,y) + \overline{\lim} f^+(y)$. Then, we have

$$t(f,g)(x) = \inf_{y \in O \cap \text{dom(f)}} T_{(f,g)}(x,y),$$

and there exist $y_0, y_1, \ldots, y_i \in \text{cl}(O \cap \text{dom(f)})$ with

$$t(f,g)(x) = \min\{T_{(f,g)}(x,y_j) : j = 0,1,\ldots,i\},$$

for $x \in O$.

Corollary 6.4. *The predicate Cons is decidable on basis elements (f,g) consisting of rational step functions.*

The algorithm for deciding consistency of a rational step function $f : [0,1]^n \to \mathbf{I}\mathbb{R}$ and a rational step function $g : [0,1]^n \to (\mathbf{I}\mathbb{R})^n_s$ works as follows: Recall that f and g are consistent iff $s(f,g) \leq t(f,g)$. By the proof of Proposition 6.3, both functions can be constructed by evaluating line integrals over piecewise linear simple paths with inner nodes in the set of corners of the crescents of g, the endpoint of the line integrals and the coaxial points of these.

Note that $s(x)$ and $t(x)$ are piecewise linear functions in x with rational coefficients, hence we can decide $s(x) \leq t(x)$ on cl(a) by first computing the rectangles on which both s and t are linear and then checking for $s \leq t$ on the corners of those.

Theorem 6.5. *The domain D^1 can be given an effective structure using a standard enumeration of its rational basis.*

Proof. As consistency is decidable, an effective structure for $D^1(U)$ is obtained from an effective structure for $(U \to \mathbf{I}\mathbb{R}) \times (U \to \mathbf{C}\mathbb{R}^n)$ by removing the non-consistent pairs. □

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