A Language for Differentiable Functions

Pietro Di Gianantonio
Dip. di Matematica e Informatica
Università di Udine
33100 Udine, Italy
Email: pietro.digianatonio@uniud.it

Abbas Edalat
Department of Computing
Imperial College London
London SW7 2RH, UK
Email: ae@ic.ac.uk

***Abstract*—We introduce a typed lambda calculus in which real numbers, real functions, and in particular continuously differentiable and more generally Lipschitz functions can be defined. Given an expression representing a real-valued function of a real variable in this calculus, we are able to evaluate the expression on an argument but also evaluate the generalised derivative, i.e., the L-derivative, equivalently the Clarke gradient, of the expression on an argument. The language is an extension of PCF with a real number data-type, similar to Real PCF and RL, but is equipped with primitives for min and weighted average to capture computable continuously differentiable or Lipschitz functions on real numbers. We present an operational semantics and a denotational semantics based on continuous Scott domains and several logical relations on these domains. We then prove an adequacy result for the two semantics. The denotational semantics is closely linked with Automatic Differentiation also called Algorithmic Differentiation, which has been an active area of research in numerical analysis for decades, and our framework can also be considered as providing denotational semantics for Automatic Differentiation. We derive a definability result showing that for any computable Lipschitz function there is a closed term in the language whose evaluation on any real number coincides with the value of the function and whose derivative expression also evaluates on the argument to the value of the generalised derivative of the function.**

INTRODUCTION

Real-valued locally Lipschitz maps on finite dimensional Euclidean spaces enjoy a number of fundamental properties which make them the appropriate choice of functions in many different areas of applied mathematics and computation. They contain the class of continuously differentiable functions and more generally the class of differentiable functions with locally bounded derivatives. They are closed under composition and the absolute value, min and max operations, and thus contain the important class of piecewise polynomial functions, which are widely used in geometric modelling, approximation and interpolation and are supported in MatLab [4]. Lipschitz maps with uniformly bounded Lipschitz constants are also closed under convergence with respect to the sup norm. In the theory and application of ordinary differential equations, Lipschitz maps represent the most fundamental class of maps in view of their basic and essentially unrivalled property that a Lipschitz vector field in $\mathbb{R}^n$ has a unique solution in the initial value problem [3].

In the past thirty years, motivated by applications in non-smooth analysis, optimisation and control theory, the notion of Clarke gradient has been developed as a convex and compact set-valued generalized derivative for real-valued locally Lipschitz maps [2]. For example, the absolute value function, which is not classically differentiable at zero, is a Lipschitz map which has Clarke gradient $[-1, 1]$ at zero. The Clarke gradient extends the classical (Fréchet) derivative for continuously differentiable functions and is moreover always defined and continuous with respect to what is in fact the Scott topology on a domain.

Independently, a domain-theoretic Lipschitz derivative, later called the L-derivative, was introduced in [7] for interval-valued functions of an interval variable and was used to construct a domain for locally Lipschitz maps; these results were then extended to higher dimensions [8]. The L-derivative was later defined and studied for real-valued functions on Banach spaces and it was shown that on finite dimensional Euclidean spaces the L-derivative actually coincides with the Clarke gradient [5]. In finite dimensions, therefore, the L-derivative provides a simple and finitary representation for the Clarke gradient, which in its original form was defined using an infinitary double limit superior operation.

Since the mid 1990's, a number of typed lambda calculi, namely extensions of PCF with a real number data type, have been constructed, including Real PCF, RL and LPR [10], [11], [17], which are essentially equivalent and in which computable continuous functions can be defined. Moreover, IC-Reals, a variant of LPR with seven digits, has been implemented with reasonable efficiency in C and Haskell [13].

The aim of this work is to take the current extensions of PCF with a real number data type into a new category and define a typed lambda calculus, in which real numbers, real functions and in particular continuously differentiable and Lipschitz functions are definable objects. Given an expression e representing a function from real numbers to real numbers in this language, we would not only be able to evaluate e on an argument, but also to evaluate the derivative of e on a given argument.

To develop such a language, we need to find a suitable replacement for the test for positiveness ($(0<)$), which is used in the current extensions of PCF with real numbers to define functions by cases. In fact, a function defined using the conditional with this constructor will not be differentiable at zero even if the two outputs of the conditional are both differentiable: Suppose we have two real computable functions f and g whose derivatives $D\,f$ and $D\,g$ are also computable,

and consider $l = \lambda x.\ \mathsf{if}\ (\mathsf{0}<)\ x\ \mathsf{then}\ f\,x\ \mathsf{else}\ g\,x$. The function l is computable and there is an effective way to obtain approximations of the value of $l(x)$ including at 0. However, there is no effective way to generate any approximation for the derivative of l, i.e., $D\,l$, at the point 0. In fact, it is correct to generate an approximation of Dl on 0 only if $f(0) = g(0)$, but this equality is undecidable, i.e., it cannot be established by observing the computation of f and g at 0 for any finite time.

In this paper, instead of the test $(\mathsf{0}<)$, we will use the functions minimum, negation and weighted average when defining continuously differentiable or Lipschitz maps. These primitives are of course definable in Real PCF, RL and LPR, but the definitions are based on the test $(\mathsf{0}<)$, which means that the information about the derivative is lost. We show that when the functions minimum, negation and average are introduced as primitives, the language becomes sufficiently rich to define any computable, continuously differentiable or Lipschitz function on reals.

Note that by a simple transfer of the origin and a rescaling of coordinates we can take the interval $[-1, 1]$ as the domain of definition of Lipschitz maps. Furthermore, by a rescaling of the values of Lipschitz maps (i.e., multiplying them with the reciprocal of their Lipschitz constant) we can convert them to non-expansive maps, i.e., we can take their Lipschitz constant to be one. Concretely, we take digits similar to those in Real PCF as constructors and develop an operational semantics and a denotational semantics based on three logical relations, and prove an adequacy result. The denotational semantics for first order types is closely related but different from the domain constructed in [7] in that we capture approximations to the function part and to the derivative part regarded as a sublinear map on the tangent space. Finally, we prove a definability result and show that every computable Lipschitz map is definable in the language.

A. Related work

Given a programme to evaluate the values of a function defined in terms of a number of basic primitives, Automatic Differentiation (also called Algorithmic Differentiation) seeks to use the chain rule to compute the derivative of the function. AD is distinct from symbolic differentiation and from numerical differentiation. Our work can be regarded as providing denotational semantics for *forward* Automatic Differentiation and can be used to extend AD to computation of the generalised derivative of Lipschitz functions.

In [9], the *differential lambda calculus* has been introduced which syntactically models the derivative operation on power series in a typed lambda calculus or a full linear logic. It is however only applicable to analytical maps which have power series expansion and as the authors point out the usual denotational semantics using domain theory is lost. Computable Analysis [18], [19] and Constructive Analysis [1] are not directly concerned with computation of the derivative and both only deal with continuously differentiable functions. In fact, a computable real-valued function with a continuous derivative has a computable derivative if and only if the derivative has a recursive modulus of uniform continuity [14, p. 191], [18, p. 53], which is precisely the definition of a differentiable function in constructive mathematics [1, p. 44].

I. Syntax

We denote the new language with PCDF (Programming language for Computable and Differentiable Functions).

The types of PCDF are the types of PCF together with a new type ι, an expression e of type ι denotes a real number in the interval $[-1, 1]$ or a partial approximation of a real number, represented by a closed intervals contained in $[-1, 1]$. The set T of type expressions is defined by the grammar:

$$\sigma ::= o \mid \nu \mid \iota \mid \sigma \to \sigma$$

where o is the type of booleans and ν is the type of natural numbers.

The expressions of PCDF are the expressions of PCF together with a new set of constants for dealing with real numbers. This set of constants is composed by the following elements:

(i) A set of constructors for real numbers,

$$\{\mathsf{d}_{\mathsf{a,b}} \mid -1 \le b-a \le b+a \le 1 \wedge a \ne 1 \wedge a, b \text{ rational } \}.$$

The constructor $\mathsf{d}_{\mathsf{a,b}} : \iota \to \iota$, represents the affine transformation $\lambda x.ax + b$. The condition

$$-1 \le b - a \le b + a \le 1,$$

implies that the affine transformation $\mathsf{d}_{\mathsf{a,b}}$ ($\lambda x.ax + b$) maps the interval $[-1, 1]$ into itself and has a non-negative slope or derivative $0 \le a \le 1$ while the condition $a \ne 1$ excludes the identity map.
The constructors $\mathsf{d}_{\mathsf{a,b}}$ are also called generalised digits. We associate to any finite sequence of generalised digits $\langle \mathsf{d}_{\mathsf{a_i,b_i}} \rangle_{i<n}$ the rational interval $\mathsf{d}_{\mathsf{a_0,b_0}}\,(\mathsf{d}_{\mathsf{a_0,b_0}}\,(\ldots(\mathsf{d}_{\mathsf{a_{n-1},b_{n-1}}}\,([-1, 1])$. Through a limiting process we associate to a stream of generalised digits a real interval, that can also be also a singleton interval $[r, r]$, in this case the stream of digit represents the real number r.

(ii) The negation function $\mathsf{t}_{-1} : \iota \to \iota$.

(iii) $(+) : \nu \to \iota \to \iota$ representing the function $\lambda n.x.\mathsf{min}\,((x + r(n)), 1))$, where r is an injective enumeration of the rational dyadic numbers in $(0, 1)$, given by the formula $r(n) = \frac{2(n-2^{\lfloor \log_2 n+1 \rfloor})+3}{2^{\lfloor \log_2 n+1 \rfloor+1}}$, the first element in the enumeration are: $1/2, 1/4, 3/4, 1/8, 3/8, 5/8, \ldots$. To improve readability, in the following we sometime substitute a natural number with the corresponding dyadic number, as given by the enumeration r, for example we write $(+)\,\frac{1}{2}\,x$ instead of $(+)\,0\,x$. Moreover we write $(+) - nx$ as an abbreviation for the expression $\mathsf{t}_{-1}\,((+)\,n(\mathsf{t}_{-1}\,x))$ that returns the value $\mathsf{max}\,((x - r(n)), -1))$.

(iv) A set of weighted average functions: for any rational number c in the interval $(0, 1)$, the weighted average

function $\oplus_{\mathsf{c}} : \iota \to \iota \to \iota$ is defined as $\oplus_{\mathsf{c}} = \lambda x.y\ .\ cx + (1-c)y$. The infix notation is used with the constant $\oplus_{\mathsf{c}}$, that is we write $e_1 \oplus_{\mathsf{c}} e_2$ instead of $\oplus_{\mathsf{c}} e_1 e_2$.

(v) The minimum function

$$\mathsf{min} : \iota \to \iota \to \iota$$

with the obvious action on pairs of real numbers. We do not introduce the maximum function since it can be defined by the minimum and the negation functions, $\mathsf{max} = \lambda x.y.\mathsf{t}_{-1}\,(\mathsf{min}\,(\mathsf{t}_{-1}\,x)(\mathsf{t}_{-1}\,y))$.

(vi) A test function $(0<) : \iota \to o$, which checks if the argument is greater than zero. The test function can be used for constructing functions that are not differentiable, an example being the function $\lambda x.\mathsf{if}\ (0<)\ (x)\ \mathsf{then}\ 1\ \mathsf{else}\ 0$; as a consequence we impose some restriction in its use.

(vii) The if-then-else constructor on reals, $\mathsf{if} : o \to \iota \to \iota \to \iota$, and the parallel if-then-else constructor $\mathsf{pif} : o \to \iota \to \iota \to \iota$.

(viii) A new binding operator D. The operator D can bind only variables of type ι and can be applied only to expressions of type ι. In our language, $\mathsf{D}x.e$ represents the derivative of the real function $\lambda x.e$.
The differential operator D can be applied only to expressions that contain neither the constant $(0<)$ nor the differential operator D itself.

Note that, with the exception of the test functions $(0<)$, all the new constants represent functions on reals that are non-expansive; the if-then-else constructors are also non-expansive if the distance between true (tt) and false (ff) is defined to be equal to two, while is the test function $(0<)$ cannot be non-expansive, whatever metric is defined on the Boolean values,. The expressions containing neither the constant $(0<)$ nor the differential operator D are called *non-expansive* since they denote functions on real numbers that are non-expansive. This fact, intuitively true, is formally proved by Proposition 2. The possibility to syntactically characterise a sufficiently rich set of expressions representing non-expansive functions is a key ingredient in our approach that allows us to obtain information about the derivative of a function expression without completely evaluating it. For example, from the fact that $e : \iota$ is a non-expansive expression, one can establish that the derivative of $\lambda x.e$, at any point, is contained in the interval $[-1, 1]$ and that the derivative of $\lambda x.\mathsf{d}_{\mathsf{a,b}}\,e$ is contained in the smaller interval $[-a, a]$.

II. Operational semantics

The operational semantics is given by an *small step reduction relation*, $\to$, which is obtained by adding to the PCF reduction rules the following set of extra rules for the new constants.

The operational semantics of $(+)$ and min operators uses the extra constants:

$$\{\mathsf{t}_{\mathsf{a,b}} \mid a \geq 0 \wedge a, b \text{ rational}\},$$

representing general (including expansive) affine transformations with a non-negative derivative, that is $\mathsf{t}_{\mathsf{a,b}}$ is intended to represent the affine transformation $\lambda x.ax + b$. A property preserved (i.e., an invariant) by the reduction rules is that the constants $\mathsf{t}_{\mathsf{a,b}}$ appear only as the head of one argument of the constants min and $\mathsf{t}_{\mathsf{a,b}}$. It follows that in any expression e' in the reduction chain of a standard expression e (without the extra constants $\mathsf{t}_{\mathsf{a,b}}$), the constants $\mathsf{t}_{\mathsf{a,b}}$ can appear only in the above positions.

The generalised digit $\mathsf{d}_{\mathsf{a,b}}$ is a special case of an affine transformation. Therefore, in applying the reduction rules, we use the convention that any reduction rule containing a general affine transformation $\mathsf{t}_{\mathsf{a,b}}$ can be instantiated to a term where the affine transformation $\mathsf{t}_{\mathsf{a,b}}$ is substituted by a generalised digit $\mathsf{d}_{\mathsf{a,b}}$.

On affine transformations we will use the following notations:

- $\mathsf{t}_{\mathsf{a_1,b_1}} \circ \mathsf{t}_{\mathsf{a_2,b_2}}$ stands for $\mathsf{t}_{\mathsf{a_1a_2,a_1b_2+b_1}}$, that is the composition of affine transformations $\mathsf{t}_{\mathsf{a_1,b_1}}$ and $\mathsf{t}_{\mathsf{a_2,b_2}}$
- $\mathsf{t}_{\mathsf{a,b}}{}^{-1}$ stands for $\mathsf{t}_{\mathsf{a^{-1},-b/a}}$, that is the inverse of the affine transformation $\mathsf{t}_{\mathsf{a,b}}$.
- An affine transformation with non-negative slope is uniquely characterised by the image of the interval $[-1, 1]$ under the map. The affine transformation $\mathsf{t}_{\mathsf{a,b}}$ sends the interval $[-1, 1]$ into the interval $[b - a,\ b + a]$, and the only affine transformation, with non-negative derivative, that maps $[-1, 1]$ into $[a, b]$ is the affine transformation $\mathsf{t}_{\mathsf{(b-a)/2,(b+a)/2}}$. There are cases in which the operational rules can be better expressed if an affine map is represented by the image of the interval $[-1, 1]$ under the map. Therefore we use the symbol $\mathsf{t}_{\mathsf{[a,b]}}$ to represent $\mathsf{t}_{\mathsf{(b-a)/2,(b+a)/2}}$. Given this notation, the expression $\mathsf{t}_{\mathsf{[a_1,b_1]}} \sqcap \mathsf{t}_{\mathsf{[a_2,b_2]}}$ stands for $\mathsf{t}_{\mathsf{[min\{a_1,a_2\},max\{b_1,b_2\}]}}$.

The reduction rules are the PCF reduction rules together the following extra rules:

1) $\mathsf{d}_{\mathsf{a_1,b_1}}\,(\mathsf{d}_{\mathsf{a_2,b_2}}\,e) \to \mathsf{d}_{\mathsf{a_1,b_1}} \circ \mathsf{d}_{\mathsf{a_2,b_2}}\ e$
2) $\mathsf{t}_{-1}\,(\mathsf{d}_{\mathsf{a,b}}\,e) \to \mathsf{d}_{\mathsf{a,-b}}\,(\mathsf{t}_{-1}\,e)$
3) $(+)\,n\,e \to \mathsf{min}\,(\mathsf{t}_{\mathsf{1,r(n)}}\,e)(\mathsf{d}_{\mathsf{0,1}}\,e)$
4) $(\mathsf{d}_{\mathsf{a,b}}\,e_1) \oplus_{\mathsf{c}} e_2 \to \mathsf{d}_{\mathsf{a',b'}}\,(e_1 \oplus_{\mathsf{c'}} e_2)$
where $a' = ac + (1 - c)$, $\quad b' = bc$ and $c' = ca/a'$. It is an easy exercise to check that the left and the right parts of the reduction rules represent the same affine transformation with arguments e_1, e_2.
5) $e_1 \oplus_{\mathsf{c}} (\mathsf{d}_{\mathsf{a,b}}\,e_2) \to \mathsf{d}_{\mathsf{a',b'}}\,(e_1 \oplus_{\mathsf{c'}} e_2)$
where $a' = a(1 - c) + c$, $\quad b' = b(1 - c)$, and $c' = c/a'$.
6) $\mathsf{min}\,(\mathsf{d}_{\mathsf{[a_1,b_1]}}\,e_1)(\mathsf{t}_{\mathsf{[a_2,b_2]}}\,e_2) \to \mathsf{d}_{\mathsf{[a_1,b_1]}}\,e_1 \qquad$ if $b_1 \leq a_2$
7) $\mathsf{min}\,(\mathsf{t}_{\mathsf{[a_1,b_1]}}\,e_1)(\mathsf{d}_{\mathsf{[a_2,b_2]}}\,e_2) \to \mathsf{d}_{\mathsf{[a_2,b_2]}}\,e_1 \qquad$ if $b_2 \leq a_1$
8) $\mathsf{min}\,(\mathsf{d}_{\mathsf{[a,b]}}\,e_1)e_2 \to$
$\mathsf{d}_{\mathsf{[-1,b]}}\,(\mathsf{min}\,(\mathsf{d}_{\mathsf{[-1,b]}}{}^{-1} \circ \mathsf{d}_{\mathsf{[a,b]}}\ e_1)(\mathsf{d}_{\mathsf{[-1,b]}}{}^{-1} e_2))$
9) $\mathsf{min}\,e_1(\mathsf{d}_{\mathsf{[a,b]}}\,e_2) \to$
$\mathsf{d}_{\mathsf{[-1,b]}}\,(\mathsf{min}\,(\mathsf{d}_{\mathsf{[-1,b]}}{}^{-1} e_1)(\mathsf{d}_{\mathsf{[-1,b]}}{}^{-1} \circ \mathsf{d}_{\mathsf{[a,b]}}\ e_2))$
10) $\mathsf{min}\,(\mathsf{t}_{\mathsf{[a_1,b_1]}}\,e_1)(\mathsf{t}_{\mathsf{[a_2,b_2]}}\,e_2) \to$
$\mathsf{d}_{\mathsf{[a,1]}}\,(\mathsf{min}\,(\mathsf{d}_{\mathsf{[a,1]}}{}^{-1} \circ \mathsf{t}_{\mathsf{[a_1,b_1]}}\ e_1)$
$(\mathsf{d}_{\mathsf{[a,1]}}{}^{-1} \circ \mathsf{t}_{\mathsf{[a_2,b_2]}}\ e_2))$
where $a = \mathsf{min}(a_1, a_2)$ satisfies $-1 < a < 1$.
11) $\mathsf{t}_{\mathsf{a_1,b_1}}\,(\mathsf{t}_{\mathsf{a_2,b_2}}\,e) \to (\mathsf{t}_{\mathsf{a_1,b_1}} \circ \mathsf{t}_{\mathsf{a_2,b_2}})\,e$
12) $\mathsf{t}_{\mathsf{[a,b]}}\,e \to \mathsf{d}_{\mathsf{[a,b]}}\,e$

if $[a,b] \subset [-1,1]$ and e is not in the form $\mathsf{t}_{a',b'}\, e$.

13) $(0<)\, \mathsf{d}_{[a,b]}\, e \to \mathsf{tt}$ if $a > 0$
14) $(0<)\, \mathsf{d}_{[a,b]}\, e \to \mathsf{ff}$ if $b < 0$
15) $\mathsf{if}\ \mathsf{tt}\ \mathsf{then}\ e_1\ \mathsf{else}\ e_2 \to e_1$
16) $\mathsf{if}\ \mathsf{ff}\ \mathsf{then}\ e_1\ \mathsf{else}\ e_2 \to e_2$
17) $\mathsf{pif}_\iota\ \mathsf{tt}\ \mathsf{then}\ e_1\ \mathsf{else}\ e_2 \to e_1$
18) $\mathsf{pif}_\iota\ \mathsf{ff}\ \mathsf{then}\ e_1\ \mathsf{else}\ e_2 \to e_2$
19) $\mathsf{pif}_\iota\ e\ \mathsf{then}\ \mathsf{d}_{[a_1,b_1]}\, e_1\ \mathsf{else}\ \mathsf{d}_{[a_2,b_2]}\, e_2 \to$
$\mathsf{d}_{[a,b]}\, (\mathsf{pif}\ e\ \ \mathsf{then}\ (\mathsf{d}_{[a,b]}{}^{-1} \circ \mathsf{d}_{[a_1,b_1]})e_1$
$\mathsf{else}\ (\mathsf{d}_{[a,b]}{}^{-1} \circ \mathsf{d}_{[a_2,b_2]})e_2)$
where $\mathsf{d}_{[a,b]} = \mathsf{d}_{[a_1,b_1]} \sqcap \mathsf{d}_{[a_2,b_2]}$
20) $\frac{N \to N'}{MN \to MN'}$ if M is a constant different from Y or is the expression $\min M'$, $M' \oplus_{\mathsf{c}}$, $\mathsf{pif}_\iota\ M'\ \mathsf{then}$, $\mathsf{pif}_\iota\ M'\ \mathsf{then}\ M''\ \mathsf{else}$

The reduction rules for the derivative operator are:

1) $\mathsf{D}x.x \to \lambda y.\mathsf{d}_{0,1}\, y$
2) $\mathsf{D}x.\mathsf{d}_{\mathsf{a},\mathsf{b}}\, e \to \lambda y.\mathsf{d}_{\mathsf{a},0}\, (\mathsf{D}x.e)y$
3) $\mathsf{D}x.\mathsf{t}_{-1}\, e \to \lambda y.\mathsf{t}_{-1}\, (\mathsf{D}x.e)y$
4) $\mathsf{D}x.(+)\, n\, e \to$
$\lambda y.\mathsf{pif}_\iota\ (0<)\, ((+)\, m(\mathsf{t}_{-1}\, e))\ \mathsf{then}\ (\mathsf{D}x.e)y\ \mathsf{else}\ \mathsf{d}_{0,0}\, y$
where $r(m) = 1 - r(n)$
5) $\mathsf{D}x.e_1 \oplus_{\mathsf{c}} e_2 \to \lambda y.(\mathsf{D}x.e_1)y \oplus_{\mathsf{c}} (\mathsf{D}x.e_2)y$
6) $\mathsf{D}x.\mathsf{min}\ e_1 e_2 \to \lambda y.$
$\mathsf{pif}\ (\lambda x.(0<)\, ((\mathsf{t}_{-1}\, e_1) \oplus_{1/2} e_2))y$
$\mathsf{then}\ (\mathsf{D}x.e_1)y\ \mathsf{else}\ (\mathsf{D}x.e_2)y$
7) $\mathsf{D}x.\mathsf{pif}_\iota\ e_1\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3 \to$
$\lambda y.\mathsf{pif}_\iota\ (\lambda x.e_1)y\ \mathsf{then}\ (\mathsf{D}x.e_1)y\ \mathsf{else}\ (\mathsf{D}x.e_2)y$
8) $\mathsf{D}x.\mathsf{if}\ e_1\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3 \to$
$\lambda y.\mathsf{if}\ (\lambda x.e_1)y\ \mathsf{then}\ (\mathsf{D}x.e_1)y\ \mathsf{else}\ (\mathsf{D}x.e_2)y$
9) $\mathsf{D}x.Ye \to \mathsf{D}x.e(Ye)$
10) $\mathsf{D}x.(\lambda y.e)e_1 \ldots e_n \to \mathsf{D}x.e[e_1/y]e_2 \ldots e_n$

Note that the rules for the derivative operator almost coincide with the usual rules for the symbolic computation of the derivative of a function.

III. Denotational Semantics

The denotational semantics for PCDF is given in the standard way as a family of continuous Scott domains, $UD := \{\mathcal{D}_\sigma \mid \sigma \in T\}$ The basic types are interpreted using the standard flat domains of integers and booleans. The domain associated to real numbers is the product domain $\mathcal{D}_\iota = \mathcal{I} \times \mathcal{I}$, where $\mathcal{I}$ is the continuous Scott domain consisting of the compact subintervals of the interval $I = [-1,1]$ partially ordered with reverse inclusion. Elements of $\mathcal{I}$ can represent either a real number x, the degenerated interval $[x,x]$, or a partial information about a real number x, the intervals $[a,b]$, with $x \in (a,b)$. On the element of $\mathcal{I}$, we consider both the set theoretic operation of intersection ($\cap$), the pointwise extensions of the arithmetic operations, and the lattice operations on the domain information order ($\sqcap, \sqcup$), [10]. Function types have the usual interpretation of call-by-name programming languages: $\mathcal{D}_{\sigma\to\tau} = \mathcal{D}_\sigma \to \mathcal{D}_\tau$.

A hand waiving explanation for the definition of the domain $\mathcal{D}_\iota = \mathcal{I} \times \mathcal{I}$, is that the first component is used to define the value part of the function while the second component is used to define the derivative part. More precisely, a (non-expansive) function f from I to I, is described, in the domain, by the product of two functions $\langle f_1, f_2\rangle : (\mathcal{I}\times\mathcal{I}) \to (\mathcal{I}\times\mathcal{I})$: the function $f_1 : (\mathcal{I}\times\mathcal{I}) \to \mathcal{I}$ represents the value part of f, in particular $f_1(i,j)$ is the image of the interval i under f for all intervals j, i.e., f_1 depends only on the first argument. The second function $f_2 : (\mathcal{I}\times\mathcal{I}) \to \mathcal{I}$ represents the derivative part. If $D\, f$ denotes the derivative of f, then $f_2(i,j)$ is the image of the intervals i and j under the function $\lambda x, y.D\, f\, (x)\cdot y$. Thus, f_2 is linear in its second component and $f_2(\{x\},\{1\})$ is the derivative of f at the point x.

Note that with respect to the above interpretation, composition behaves correctly, that is if the pair $\langle f_1, f_2\rangle : (\mathcal{I}\times\mathcal{I}) \to (\mathcal{I}\times\mathcal{I})$ describes the value part and the derivative part of a function $f : I \to I$ and $\langle g_1, g_2\rangle : (\mathcal{I}\times\mathcal{I}) \to (\mathcal{I}\times\mathcal{I})$ describes a function $g : I \to I$ then $\langle h_1, h_2\rangle$ describes, by the chain rule, the function $f \circ g$ with $h_1(i,j) = f_1(g_1(i,j), g_2(i,j))$ and $h_2(i,j) = f_2(g_1(i,j), g_2(i,j))$.

The L-derivative of the non-expansive map $f : I \to I$ is the Scott continuous function $\mathcal{L}(f) : I \to \mathcal{I}$ defined by [5]:

$$\mathcal{L}(f)(x) = \bigcap\{b \in \mathcal{I} : \exists \text{ open interval } O \subset I \text{ with } \tfrac{f(x)-f(y)}{x-y} \in b \text{ for all } x, y \in O, x \neq y\}.$$

Consider now the case of functions in two arguments. Given a function $g : I \to I \to I$, its domain description will be an element in $(\mathcal{I}\times\mathcal{I}) \to (\mathcal{I}\times\mathcal{I}) \to (\mathcal{I}\to\mathcal{I})$, which is isomorphic to $((\mathcal{I}\times\mathcal{I})\times(\mathcal{I}\times\mathcal{I})) \to (\mathcal{I}\times\mathcal{I})$. Thus again, the domain description of g consists of a pair of functions $\langle g_1, g_2\rangle$, with g_1 describing the value part. If $D\, g\, (x_1, x_2)$ is the linear transformation representing the derivative of g at (x_1, x_2), then the function g_2 is a domain extension of the real function $\lambda x_1, y_1, x_2, y_2.D\, g\, (x_1, x_2)\cdot(y_1, y_2)$.

This approach for describing functions on reals is also used in (forward mode) Automatic Differentiation [12]. While Automatic Differentiation is different from our method in that it does not consider the domain of real numbers and the notion of partial reals, it is similar to our approach in that it uses two real numbers as input and a pair of functions to describe the derivative of functions on reals. The idea of using two separated components to describe the value part and the derivative part in the domain-theoretic setting was introduced in [7], which is implemented in a different way than here.

The semantic interpretation function $\mathcal{E}$ is defined, by structural induction, in the standard way:

$$\begin{array}{lcl} \mathcal{E}[\![c]\!]_\rho & = & \mathcal{B}[\![c]\!] \\ \mathcal{E}[\![x]\!]_\rho & = & \rho(x) \\ \mathcal{E}[\![e_1 e_2]\!]_\rho & = & \mathcal{E}[\![e_1]\!]_\rho(\mathcal{E}[\![e_2]\!]_\rho) \\ \mathcal{E}[\![\lambda x^\sigma.e]\!]_\rho & = & \lambda d \in \mathcal{D}_\sigma.\mathcal{E}[\![e]\!]_{(\rho[d/x])} \end{array}$$

The semantic interpretation of any PCF constant is the usual one, while the semantic interpretation of the new constants on reals is given by:

$$\mathcal{B}[\![\mathsf{d}_{\mathsf{a},\mathsf{b}}]\!](\langle i,j\rangle) = \langle ai + b,\ aj\rangle$$

$$\mathcal{B}[\![\mathsf{t}_{-1}]\!](\langle i,j\rangle) = \langle -i,\ -j\rangle$$

$$\mathcal{B}[\![(+)\,]\!]n(\langle i,j\rangle) = \begin{cases} \bot & \text{if } n = \bot \\ \langle i + r(n), j\rangle \text{ if } i + r(n) < 1 \text{ and } n \neq \bot \\ \langle [1,1],[0,0]\rangle \text{ if } i + r(n) > 1 \text{ and } n \neq \bot \\ \langle i + r(n) \cap [-1,1],\ j \sqcap [0,0]\rangle \text{ otherwise} \end{cases}$$

$$\mathcal{B}[\![\oplus_{\mathsf{c}}]\!](\langle i_1,j_1\rangle,\langle i_2,j_2\rangle) = \langle ci_1 + (1-c)i_2,\ cj_1 + (1-c)j_2\rangle$$

$$\mathcal{B}[\![\mathsf{min}\,]\!](\langle i_1,j_1\rangle,\langle i_2,j_2\rangle) = \begin{cases} \langle i_1,j_1\rangle & \text{if } i_1 < i_2 \\ \langle i_2,j_2\rangle & \text{if } i_1 > i_2 \\ \langle i_1 \mathsf{min}\, i_2,\ j_1 \sqcap j_2\rangle & \text{otherwise} \end{cases}$$

$$\mathcal{B}[\![(0<)\,]\!](\langle i,j\rangle) = \begin{cases} tt & \text{if } i > 0 \\ ff & \text{if } i < 0 \\ \bot & \text{otherwise} \end{cases}$$

The interpretation of the derivative operator is given by:

$$\mathcal{E}[\![\mathsf{D}x.e]\!]_\rho = \lambda d \in \mathcal{I}\times\mathcal{I}\,.\,\langle \pi_2(\mathcal{E}[\![e]\!]_\rho[\langle \pi_1 d, 1\rangle/x]), \bot\rangle$$

Note that the function $\mathcal{B}[\![(0<)\,]\!]$ loses the information given by the derivative part, while the function $\mathcal{E}[\![\mathsf{D}x.e]\!]_\rho$, is a sort of translation of the function $\mathcal{E}[\![\lambda x.e]\!]_\rho$: The value of $\mathcal{E}[\![\mathsf{D}x.e]\!]_\rho$ is obtained from the derivative part of $\mathcal{E}[\![\lambda x.e]\!]_\rho$, while the derivative part of $\mathcal{E}[\![\mathsf{D}x.e]\!]_\rho$ is set to $\bot$.

Consider some examples. The absolute value function can be implemented through the term $\mathrm{Ab} = \lambda x.\mathsf{max}\,(\mathsf{t}_{-1}\,x)x$ with the following semantic interpretation:

$$\mathcal{E}[\![\mathrm{Ab}]\!]_\rho(\langle i,j\rangle) = \begin{cases} \langle i,j\rangle & \text{if } i > 0 \\ \langle -i,-j\rangle & \text{if } i < 0 \\ \langle [k^-,k^+],[-1,1]j\rangle & \text{otherwise,} \end{cases}$$

where $k^- = \max(i^-,-i^+)$, $k^+ = \max(i^+,-i^-)$ with $i = [i^-,i^+]$.

When the absolute value function evaluated at 0, where it is not differentiable, the derivative part of the semantic interpretation returns a partial value: $\pi_2(\mathcal{E}[\![\mathrm{Ab}]\!]_\rho(\{0\},\{1\}) = [-1,1]$. This partial value coincides with the Clarke gradient, equivalently the L-derivative, of the absolute value function.

The function $\frac{|x-y|}{2}$, is represented by the expression

$$\text{Ab-dif} = \lambda x.y.\mathsf{max}\,(x \oplus_{1/2} (\mathsf{t}_{-1}\,y))((\mathsf{t}_{-1}\,x)\oplus_{1/2} y)$$

whose semantics is the function:

$$\mathcal{E}[\![\text{Ab-dif}]\!]_\rho(\langle i_1,j_1\rangle,\langle i_2,j_2\rangle) =$$

$$\begin{cases} \langle \frac{i_1-i_2}{2}, \frac{j_1-j_2}{2}\rangle & \text{if } i_1 > i_2 \\ \langle \frac{i_2-i_1}{2}, \frac{j_2-j_1}{2}\rangle & \text{if } i_1 < j_1 \\ \langle [k^-,k^+],[-1/2,1/2](j_1-j_2)\rangle & \text{otherwise,} \end{cases},$$

where $k^- = \max(i_1^- - i_2^+, i_2^- - i_1^+)$ and $k^+ = \max(i_1^+ - i_2^-, i_2^+ - i_1^-)$.

From $[\![\text{Ab-dif}]\!]$ it is possible to evaluate the partial derivative of the function $\frac{|x-y|}{2}$, not only along the axes x and y, but along any direction. Considering the Euclidean distance, the derivative of the function at $(0,0)$ in the direction of the unit vector $(u/\sqrt{u^2+v^2}, v/\sqrt{u^2+v^2})$ is given by $\mathcal{E}[\![\text{Ab-dif}]\!]_\rho(\langle\{0\},\{u/\sqrt{u^2+v^2}\}\rangle,\langle\{0\},\{v/\sqrt{u^2+v^2}\}\rangle)$, that is the the interval $[-1/2,1/2]\frac{u-v}{\sqrt{u^2+v^2}}$. Again this value coincides with the value of the Clarke gradient of the function $\frac{|x-y|}{2}$ at $(0,0)$ in the direction $(u/\sqrt{u^2+v^2}, v/\sqrt{u^2+v^2})$.

A. Logical relations characterization

In the present approach we choose to define the semantic domains in the simplest possible way. As a consequence, our domains contain also points that are not consistent with the intended meaning, for example, the domain $\mathcal{D}_{\iota\to\iota} = (\mathcal{I}\times\mathcal{I}) \to (\mathcal{I}\times\mathcal{I})$ contains also the product of two functions $\langle f_1, f_2\rangle$ where the derivative part f_2 is not necessarily linear in its second argument and is not necessarily consistent with the value part, i.e., the function f_1; moreover the value part f_1 can be a function depending also on its second argument.

However the semantic interpretation of (non-expansive) PCDF expressions will not have this pathological behaviour. A proof of this fact and a more precise characterisation of the semantic interpretation of expressions can be obtained through the technique of logical relations [16]. In particular we define a set of logical relations on the semantic domains and prove that, for any non-expansive PCDF expression e, the semantic interpretation of e satisfies these relations. Using this method, we can establish a list of properties for the semantic interpretation of PCDF expressions.

Definition 1: The following list of relations are defined on the domain $\mathcal{D}_\iota$.

- **Independence**: A binary relation R^i_ι consisting of the pairs of the form $(\langle i,j_1\rangle,\langle i,j_2\rangle)$. The relation R^i_ι is used to establish that, for a given function, the value part of the result is independent from the derivative part of the argument: $f_1(i,j_1) = f_1(i,j_2)$.
- **Sub-linearity**: A family of relations $R^{l,r}_\iota$ indexed by a rational number $r \in [-1,1]$. The family $R^{l,r}_\iota$ consists of pairs of the form $(\langle i,j_1\rangle,\langle i,j_2\rangle)$ where $j_1 \sqsubseteq r\cdot j_2$. These relations are used to establish the sublinearity of the derivative part: $f_2(i,r\cdot j) \sqsubseteq r\cdot f_2(i,j)$.
- **Consistency**: A family of ternary relation $R^{d,r}_\iota$ indexed by a rational number $r \in (0,2]$, consisting of triples of the form $(\langle i_1,j_1\rangle,\langle i_2,j_2\rangle,\langle i_3,j_3\rangle)$ with $i_3 \sqsubseteq i_1 \sqcap i_2$ and $(r\cdot j_3)$ consistent with $(i_1 - i_2)$, that is the intervals $(r\cdot j_3)$ and $(i_1 - i_2)$ have a non-empty intersection. This relation is used to establish the consistency of the derivative part of a function with respect to the value part.

The above relations are defined on the other ground domains $\mathcal{D}_o$ and $\mathcal{D}_\nu$ as the diagonal relations in two or three arguments, e.g., $R^{d,r}_\nu(l,m,n)$ iff $l = m = n$. The relations are extended inductively to higher order domains by the usual definition on logical relations: $R^i_{\sigma\to\tau}(f,g)$ iff for every $d_1,d_2 \in \mathcal{D}_\sigma$, $R^i_\sigma(d_1,d_2)$ implies $R^i_\tau(f(d_1),g(d_2))$, and similarly for the other relations.

Proposition 1: For any closed expression $e : \sigma$, for any rational number $r \in [-1,1]$, the semantic interpretation $\mathcal{E}[\![e]\!]$ of e, is self-related by R^i_σ, $R^{l,r}_\sigma$, i.e. $R^i_\sigma(\mathcal{E}[\![e]\!]_\rho,\mathcal{E}[\![e]\!]_\rho)$, and similarly for $R^{l,r}_\sigma$. Moreover, if the expression $e : \sigma$ is non-expansive, the semantic interpretation $\mathcal{E}[\![e]\!]$, is self-related by $R^{d,r}_\sigma$,.

Proof: The proof is quite standard, and is based on the fact that the relations R^i_σ, $R^{l,r}_\sigma$, $R^{d,r}_\sigma$ are logical. First one proves that the semantic interpretation of (non-expansive) constants are self-related by $R^{d,r}_\sigma$, R^i_σ, and $R^{l,r}_\sigma$. Then, to show that the fixed-point operator preserves the relations, one shows that the bottom elements are self-related by R^i_σ, $R^{l,r}_\sigma$, and $R^{d,r}_\sigma$, and that the relations are closed under the lub of chains. Finally, by the basic lemma of logical relations, one obtains the result. ∎

We now show how the three relations ensure the three properties of independence, sublinearity and consistency. To any element $f = \langle f_1, f_2\rangle$ in the domain $\mathcal{D}_{\iota\to\iota} = (\mathcal{I}\times\mathcal{I}) \to (\mathcal{I}\times\mathcal{I})$ we associate a partial function $f_v : I \to I$ with

$$f_v(x) = \begin{cases} y & \text{if } f_1(\langle\{x\},\bot\rangle) = \{y\} \\ \text{undefined} & \text{if } f_1(\langle\{x\},\bot\rangle) \text{ is a proper interval} \end{cases}$$

and a total function

$$f_d : I \to \mathcal{I} = \lambda x.f_2(\langle\{x\},\{1\}\rangle))$$

The preservation of the relations R^i_ι, $R^{l,r}_\iota$ has the following straightforward consequences:

Proposition 2: (i) For any function $f = \langle f_1, f_2\rangle$ in $\mathcal{D}_{\iota\to\iota}$ self-related by $R^i_{\iota\to\iota}$, for every i, j_1, j_2, $f_1(\langle i, j_1\rangle) = f_1(\langle i, j_2\rangle)$, the return value part in independent from the derivative argument.

(ii) For any function $f = \langle f_1, f_2\rangle$ in $\mathcal{D}_{\iota\to\iota}$ self-related by $R^{l,r}_{\iota\to\iota}$ for every i, j, and for every rational $r \in [-1, 1]$, $f_2(\langle i, r\cdot j\rangle) \sqsubseteq r \cdot f_2(\langle i, j\rangle)$. Therefore:

- $(f_2(\langle i,\{r\}\rangle))/r \sqsubseteq f_2(\langle i,\{1\}\rangle)$, i.e., the most precise approximation of the L-derivative is obtained by evaluating the function with 1 as its second argument,
- for every i, j, $f_2(\langle i, -j\rangle) = -f_2(\langle i, j\rangle)$, i.e., the derivative part is an odd function.

The preservation of the relation $R^{d,r}_\iota$ induces the following properties (see the Appendix for the proof):

Proposition 3: For any function $f = \langle f_1, f_2\rangle : \mathcal{D}_{\iota\to\iota}$ self-related by $R^{d,r}_{\iota\to\iota}$:

(i) the function f_v is non-expansive;

(ii) on the open sets where the functions f_v is defined, the function f_d is an approximation to the L-derivative of the function f_v;

(iii) if f is a maximal element of $\mathcal{D}_{\iota\to\iota}$ then f_v is a total function and f_d is the associated L-derivative.

B. Subdomains

By definition, the logical relations are closed under directed lubs, and as a consequence also the sets of elements self-related by them are also closed under directed lubs.

For any ground type σ the relations R^i_σ, $R^{l,r}_\sigma$, $R^{d,r}_\sigma$ are closed under arbitrary meets, meaning that if $\forall j \in J\ .\ R^i_\sigma(d_j, e_j)$ then $R^i_\sigma(\bigsqcap_{j\in J} d_j, \bigsqcap_{j\in J} e_j)$ and similarly for the other relations $R^{l,r}_\sigma$, $R^{d,r}_\sigma$. The proof is immediate for $\sigma = o, \nu$, and is a simple check for $\sigma = \iota$. The following result shows that this closure property holds also for $\sigma = \iota \to \iota$.

Proposition 4: The set of elements in $\mathcal{D}_{\iota\to\iota}$ self-related by any of the three relations $R^i_{\iota\to\iota}$, $R^{l,r}_{\iota\to\iota}$, and $R^{d,r}_{\iota\to\iota}$ is closed under arbitrary meets.

Proof: For the independence relation $R^i_{\iota\to\iota}$, the closure property is trivial to check. For the consistency relation $R^{d,r}_{\iota\to\iota}$, the closure under non-empty meets follows immediately from the fact that this relation is downward closed. The closure property for the sublinearity relation $R^{l,r}_{\iota\to\iota}$ is given in the Appendix.

∎

We now employ the following result whose proof can be found in the Appendix.

Proposition 5: In a continuous Scott domain, a non-empty subset closed under lubs of directed subsets and closed under non-empty meets is a continuous Scott subdomain.

Corollary 1: If σ is a ground type or first order type, then the set of elements in $\mathcal{D}_\sigma$ self-related by the three logical relations is a continuous Scott subdomain of $\mathcal{D}_\sigma$.

As we do not deal with second or higher order real types in this extended abstract, we will not discuss the corresponding subdomains here.

C. Adequacy

As usual once an operational and denotational semantics are defined, it is necessary to present an adequacy theorem stating that the two semantics agree.

Let us denote by $\mathsf{d}_{[\mathsf{a},\mathsf{b}]} \ll Eval(e)$ the fact that there exits a rational interval $[a', b']$ such that $e \to^\star \mathsf{d}_{[\mathsf{a}',\mathsf{b}']}\, e'$ and $[a', b'] \subset (a, b)$. The proof of the following theorem is presented in the Appendix.

Theorem 1 (Adequacy): For every closed term e with type ι and environment ρ, we have:

$$\mathsf{d}_{[\mathsf{a},\mathsf{b}]} \ll Eval(e) \text{ iff } [a, b] \ll \pi_1(\mathcal{E}[\![e]\!]_\rho)$$

In the operational semantics that we have proposed, the calculus of the derivative is performed through a sort of symbolic computation: the rewriting rules specified how to evaluate the derivative of the primitive functions and the application of the derivative rules essentially transforms a function expression into the function expression representing the derivative. The denotational semantics provides an alternative approach to the computation of the derivative, which almost exactly coincides with the computation performed by Automatic Differentiation. We can interpret our adequacy result as a proof that symbolic computation of the derivative and the computation of the derivative through Automatic Differentiation coincide. We remark in passing that, inspired by the denotational semantics, it is possible to define an alternative operational semantics that will perform the computation of the derivative in the same way that is performed by Automatic Differentiation.

IV. Function definability

We will show in this section that for any maximal computable function f in $\mathcal{D}_{\iota\to\iota}$ preserving the logical relations,

there exists a closed PCDF expression f with type $\iota \to \iota$ whose semantics, $\mathcal{E}[\![\mathsf{f}]\!]_\rho$, coincides with f on maximal elements, i.e. real numbers. More precisely, we show that for any real number $x \in [-1,1]$, we have: $f_v(x) = (\mathcal{E}[\![\mathsf{f}]\!]_\rho)_v(x)$ and $f_d(x) = (\mathcal{E}[\![\mathsf{f}]\!]_\rho)_d(x)$

We do not consider the problem of defining PCDF expressions whose semantics coincides with f on non-maximal partial elements. PCDF is not rich enough for such a definability result, and the introduction of new constants, simply to allow the definability of partial elements, will make the language less natural.

Here we will not give a detailed proof but present a general construction that can be used to define, inside PCDF, any computable non-expansive function. The presentation is quite lengthy and proceeds incrementally showing, in several steps, how to consider to define larger and larger classes of computable maximal elements in $\mathcal{D}_{\iota\to\iota}$. Each step will introduce a new ingredient in the construction. More precisely, we first present a construction that can deal with any piecewise continuously differentiable function (i.e., a function that is continuously differentiable except for a finite number of points at which the left and right derivatives exist), then we extend it to treat functions that are piecewise continuously differentiable except for a finite number of points (of essential discontinuities of the derivative at which the left and right derivatives do not exist), and finally we give a definability result for general Lipschitz maps.

Notation. Given two real numbers x, r we denote with $x \pm r$ the interval having center in x and diameter $2r$. Given a total function f, we denote by $f \pm r$ the partial function $\lambda x.f(x) \pm r$. With some abuse of notation given a PCDF constant c representing a function on reals, we will use the symbol c to denote the functional obtained by pointwise application of the function c. For example, min denotes the functional $\lambda f.g\,.\,\lambda x\,.\,\mathsf{min}\,(fx)(gx)$.

For start, we present a series of functions, and functionals definable by PCDF expressions.

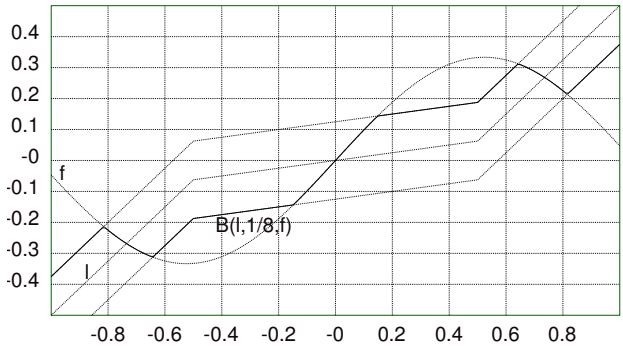
- It is easy to see that any non-expansive piecewise rational linear function l is definable using the functions $\mathsf{d_{a,b}}\,, \mathsf{t_{-1}}\,, (+)\,, \mathsf{min}\,, \mathsf{max}$, in the sense that there exists a PCDF function expression l such that $l = (\mathcal{E}[\![\mathsf{l}]\!]_\rho)_v$ and $\frac{dl}{dx} = (\mathcal{E}[\![\mathsf{l}]\!]_\rho)_d$. For example a piecewise linear interpolation of the function $\lambda x.x^3/2$ coinciding with the function on the points with x equal to $-1, -1/2, 0, 1/2, 1$ can be defined as

$$\lambda x.\mathsf{max}\,(\mathsf{min}\,((+)\,\tfrac{3}{8}(\mathsf{d}_{\frac{7}{8},0}\,x))(\mathsf{d}_{\frac{1}{8},0}\,x))\\ ((+)\,\tfrac{-3}{8}(\mathsf{d}_{\frac{7}{8},0}\,x))$$

  We use $\mathsf{l}_1, \mathsf{l}_2, \ldots$ as metavariables over expressions defining piecewise rational linear functions, with $l_1, l_2, \ldots$ denoting the corresponding functions on reals, i.e., $l_1 = (\mathcal{E}[\![\mathsf{l}_1]\!]_\rho)_v$
- In the following we will use the functional:

$$\mathsf{B} = \lambda l^{\iota\to\iota}.c^{\nu}.f^{\iota\to\iota}\lambda x^{\iota}.\\ \mathsf{min}\,(\mathsf{max}\,((+)\,-c\,(l\,x))(f\,x))((+)\,c\,(l\,x)).$$

  Note that, given an expression l defining a (piecewise linear) function l, $\mathcal{E}[\![\mathsf{B\,l\,n}]\!]_\rho fx$ is the interval $(l(x) \pm r(n)) \cap f(x)$, if the interval $l(x) \pm r(n)$ and $f(x)$ intersect, otherwise $\mathcal{E}[\![\mathsf{B\,l\,n}]\!]_\rho fx$ coincides with one of the two bounds of the interval $l(x) \pm r(n)$. The following diagram illustrates the behaviour of the functional B on maximal points for a function preserving these points.



  On partial elements, $\mathcal{E}[\![\mathsf{B\,l\,n}]\!]_\rho f$ is a sort of *projection* of the function f on the function $\lambda x\,.\,l(x) \pm r(n)$. Given an expression Ω denoting the completely undefined function, the value part of $\mathcal{E}[\![\mathsf{B\,l\,n}\,\Omega]\!]_\rho)$ is the function $\lambda x\,.\,l(x) \pm r(n)$, while the derivative part $(\mathcal{E}[\![\mathsf{B\,l\,n}\Omega]\!]_\rho)_d$ is the completely undefined function.
- A PCDF expression L such that $\mathcal{E}[\![\mathsf{Lf_1nf_2}]\!]_\rho = \mathcal{E}[\![\mathsf{f_1} \oplus_{\mathsf{r(n)}} \mathsf{f_2}]\!]_\rho$ can be defined as

$$\begin{aligned}\mathsf{L} = \;& \mathsf{Y}(\lambda \mathsf{F.f_1.n.f_2.\ if\ (n = 1/2)\ then\ f_1} \oplus_{\frac{1}{2}} \mathsf{f_2}\\ & \mathsf{else\ if\ (n > 1/2)\ then\ f_1} \oplus_{\frac{1}{2}} (\mathsf{Ff_1(2n-1)f_2})\\ & \mathsf{else\ f_2} \oplus_{\frac{1}{2}} (\mathsf{Ff_1(2n)f_2}))\end{aligned}$$

  where with $(n > 1/2)$ we indicate a suitable expression evaluating to tt if $r(n) > 1/2$ and to ff otherwise. Similar considerations hold for the other abbreviations, $(n = 1/2)$, $(2n)$, $(2n-1)$.
  Given an expression l defining a piecewise linear function l, it is readily seen that the value part of $\mathcal{E}[\![\mathsf{Ll n}\Omega_{\iota\to\iota}]\!]_\rho$ is the function $\lambda x\,.\,(1 - r(n)) \cdot l(x) \pm r(n)$, while $(\mathcal{E}[\![\mathsf{Ll n}\Omega_{\iota\to\iota}]\!]_\rho)_d$ is the function $\lambda x\,.\,(1 - r(n)) \cdot \frac{dl}{dx}(x) \pm r(d)$.

It is easy to show that for any non-expansive function $f : I \to I$ there exists a sequence of piecewise linear functions $\langle l_i \rangle_{i\in\mathbb{N}}$ converging fast to f, in the sense that for any i, we have $f \in l_i \pm 2^{-i+1}$.

If the sequence of piecewise linear functions $\langle l_i \rangle_{i\in\mathbb{N}}$ is definable in the sense that there exists a term l such that $\mathsf{l\,n}$ defines the function l_n, denoting by (2^{-n-1}) a suitable term converging, for any instantiation of the variable n, to a value h such that $r(h) = (2^{-n-1})$, the term $\mathsf{f} = (\mathsf{Y}\ \lambda \mathsf{F}.\lambda \mathsf{n}.\mathsf{B(l\,n)}(2^{-\mathsf{n}-1})(\mathsf{F(n+1)}))\mathsf{0}$ is such that:

$$\mathcal{E}[\![\mathsf{f}]\!]_\rho = \bigsqcup_{i\in\mathbb{N}} \mathcal{E}[\![\mathsf{B(l\,0)}\frac{1}{2}(\mathsf{B(l\,1)}\frac{1}{4}(\ldots \mathsf{B(l\,i)}2^{-\mathsf{i}+1}\Omega)\ldots)]\!]_\rho$$

It is then not difficult to see that $f = (\mathcal{E}[\![\mathsf{f}]\!]_\rho)_v$. However, $(\mathcal{E}[\![\mathsf{f}]\!]_\rho)_d$ is the bottom function, i.e., the completely undefined

approximation of the derivative of the function f. We now proceed in three steps of increasing complexity to define various classes of Lipschitz functions in PCDF.

A. Piecewise continuously differentiable

If $f : I \to I$ is piecewise continuously differentiable, then there exists a sequence of piecewise linear functions $\langle l_i \rangle_{i\in\mathbb{N}}$ such that for all i the function $l_1 \oplus_{1/2} (l_2 \oplus_{1/2} (\dots l_i \oplus_{1/2} 0) \dots)$ approximates the function f with precision 2^{-i}, both for the value and for the derivative part. If the sequence of piecewise linear function is definable by a term l then we can construct a term f such that:

$$\mathcal{E}[\![\mathsf{f}]\!]_\rho = \bigsqcup_{i\in\mathbb{N}} \mathcal{E}[\![\mathsf{L}(\mathsf{l}\,0)\frac{1}{2}(\mathsf{L}(\mathsf{l}\,1)\frac{1}{2}(\dots \mathsf{L}(\mathsf{l}\,\mathsf{i})\frac{1}{2}(\Omega)\dots)]\!]_\rho$$

and one can prove that $\mathcal{E}[\![\mathsf{f}]\!]_\rho$ describes both the value part and the derivative part of f.

B. Piecewise continuously differentiable except for isolated points

The above construction can be applied only if the function $f : I \to I$, together with its derivative, is globally approximable by a sequence of piecewise linear functions. In general, if f is not piecewise continuously differentiable, this is not always possible. For example consider $f(x) = x^2 \cdot \sin(1/x)/4$, in $[-1, 1]$. Then f has a Lipschitz constant $3/4$ and is differentiable at every point, but in any neighbourhood of 0 its derivative assumes all the values between $-1/4$, $1/4$, i.e., the left and right derivatives at 0 do not exist. It follows that there is no piecewise linear function, whose derivative part approximates the derivative of part of f with an error smaller that $1/4$. To overcome this, we now present a construction where the problem of defining a function on the whole interval $[-1, 1]$ is reduced to the problem of defining suitable approximations to the function on smaller and smaller intervals.

It works as follows: given a non-expansive function $f : I \to I$, we first obtain a piecewise linear function $l_{0,0}$, and a rational number $c_{0,0} \in [0, 1)$ such that $c_{0,0} \cdot l_{0,0}$ globally approximates the value and derivative part of f with an error $1 - c_{0,0}$. Given an expression $\mathsf{l}_{0,0}$ defining the function $l_{0,0}$, an expression defining f can be written in the form $\mathsf{l}_{0,0} \oplus_{\mathsf{c}} \mathsf{g}_{0,0}$, for a suitable expression $\mathsf{g}_{0,0}$, defining the non-expansive function $g_{0,0} = (f - c_{0,0} \cdot l_{0,0})/(1 - c_{0,0})$. In other words we reduce the problem of defining f to the problem of defining $g_{0,0}$. At this stage we do not look for a global piecewise linear approximation of $g_{0,0}$, but we split the domain of $g_{0,0}$ in two overlapping intervals $J_{1,0}$ and $J_{1,0}$, and consider two functions $f_{1,0}$ and $f_{1,1}$ defined as the least non-expansive functions that coincide with $g_{0,0}$ on the intervals $J_{1,0}$ and $J_{1,0}$ respectively. The function $g_{0,0}$ can be expressed as $\mathsf{max}(f_{1,0}, f_{1,1})$ In this way, the problem of defining $g_{0,0}$, it is split into the problem of defining two functions $f_{1,0}, f_{1,1}$ each of them having a complex behaviour just in one restricted part of the domain and in the remaining part behaving as piecewise linear functions. Corecursively, we apply the apply the procedure consider for the function f to the functions $f_{1,0}$ and $f_{1,1}$, constructing an infinitary tree of linear approximations, each of which considers the behaviour of the function f in smaller and smaller intervals.

A more formal presentation of the construction is the following.

First we define two sequences of coverings, I_i, J_i, with $i > 0$, of the interval I by rational intervals. To any pair i, j of non-negative integers with $2^i > j \geq 0$, we associate the real intervals

$$I_{i,j} = [(j - 2^{i-1})/2^{i-1}, (j + 1 - 2^{i-1})/2^{i-1}],$$

and

$$J_{i,j} = [(2j - 1 - 2^i)/2^i, (2j + 3 - 2^i)/2^i] \cap [-1, 1].$$

As a numerical example, the covering I_2 is formed by the intervals

$$[-1, -1/2], [-1/2, 0], [0, 1/2], [1/2, 1]$$

while the overlapping covering J_2 is formed by the intervals

$$[-1, -1/4], [-3/4, 1/4], [-1/4, 3/4], [1/4, 1].$$

By simultaneous induction on $i \geq 0$ we construct three families of double indexed maps $f_{i,j}$, $l_{i,j}$ and $g_{i,j}$, and a double indexed family of rational $c_{i,j}$ as follows:

- A family of functions $f_{i,j}$ from I to I, with $0 \leq i$ and $0 \leq j < 2^i$, is defined by:
  - $f_{0,0} = f_v$
  - $f_{i+1,j}$ is the smallest (wrt the real line order) non-expansive function coinciding with the $g_{i,\lfloor j/2 \rfloor}$ on the interval $J_{i+1,j}$, formally:
    $f_{i+1,j}(x) = \mathsf{min}\,(g_{i,\lfloor j/2 \rfloor}(x),\, x + a_{i+1,j},\, -x + b_{i+1,j})$.
    where, denoting by $J^-_{i,j}$ and $J^+_{i,j}$, respectively, the left and right bound of the interval $J_{i,j}$, we put $a_{i,j} = g_{i,\lfloor j/2 \rfloor}(J^-_{i,j}) - J^-_{i,j}$ and $b_{i,j} = g_{i,\lfloor j/2 \rfloor}(J^+_{i,j}) + J^+_{i,j}$. These definitions imply that the function $\lambda x.\, x + a_{i,j}$ passes through the point with coordinates $(J^-_{i,j}, g_{i,\lfloor j/2 \rfloor}(J^-_{i,j}))$ while the function $\lambda x.\, -x + b_{i,j}$ passes through the point with coordinates $(J^+_{i,j}, g_{i,\lfloor j/2 \rfloor}(J^+_{i,j}))$.

  Aim of this definition is to reduce the definability of $g_{i,j}$ to the definability of $f_{i+1,2j}$ and $f_{i+1,2j+1}$, each of them consider a different region portion of the function domain of $g_{i,j}$.
- A family of piecewise linear functions $l_{i,j}$ and the dyadic rational numbers $c_{i,j} \in [0, 1)$, with $0 \leq i$ and $0 \leq j \leq 2^i - 1$, such that: $f_{i,j} \in c_{i,j} \cdot l_{i,j} \pm (1 - c_{i,j})$ and $\frac{d\, f_{i,j}}{d\, x} \in c_{i,j} \cdot \frac{d\, l_{i,j}}{d\, x} \pm (1 - c_{i,j})$.

  The functions $l_{i,j}$ and the rationals $c_{i,j}$ are not uniquely defined, the construction just chooses them in such a way that $c_{i,j} \cdot l_{i,j}$ is a piecewise approximation of, value and derivative part of, $f_{i,j}$, with error $(1 - c_{i,j})$.
- The family of functions $g_{i,j}$ from I to I, with $0 \leq i$ and $0 \leq j < 2^i$ are defined such that $f_{i,j} = l_{i,j} \oplus_{c_{i,j}} g_{i,j}$;

the conditions pose on the function $l_{i,j}$ assure that the function $f_{i,j}$ exists and it is non-expansive.

After having generated the approximation $l_{i,j}$ of the functions $f_{i,j}$, one is left with the problem of defining the function $g_{i,j}$.

As an example of the above construction, consider the function $f = x^2/2$. We can choose, in the first step of approximation, the function $l_{0,0}(x) = \mathsf{max}\,(-x, x)$ and the constant $c_{0,0} = 1/2$. This choice induces the functions $g_{0,0}(x) = \mathsf{min}\,((x^2 + x), (x^2 - x))$ and $f_{1,0}(x) = \mathsf{min}\,((x^2 + x), (x^2 - x), (-x + 1/4)$. Proceeding with the construction, using similar choices for the next steps, leads to the function $f_{2,1}(x) = \mathsf{min}\,((2x^2 + 3x + 1), (2x^2 + x), (2x^2 - x), (x + 5/8), (-x + 1/8))$. A piecewise linear approximation of function $f_{2,1}$, with precision $1/2$ is given by the function $l_{2,1}(x) = \mathsf{max}\,(\mathsf{min}\,(x + 1/2, -x - 1/2), \mathsf{min}\,(x, -x))$. The following diagram depicts the functions $f_{2,1}$ and $l_{2,1}/2$.

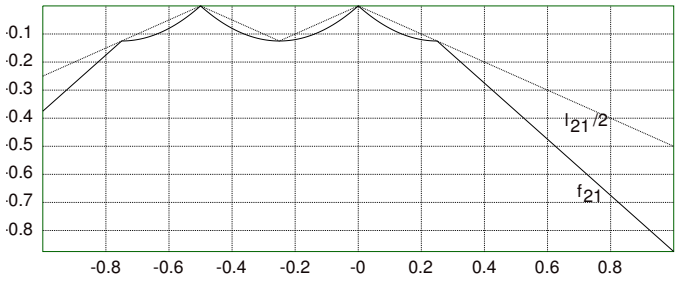


Fig. 1. Functions used in approximating the square function

The function $g_{2,1}$ with $f_{2,1} = l_{2,1}/2 + g_{2,1}/2$ and the function $f_{3,2}(x) = \mathsf{min}\,((4x^2 + 5x + 3/2), (4x^2 + 3x + 1/2), (4x^2 + x), (x + 9/16), (-x - 3/16))$ are illustrated by the following diagram:

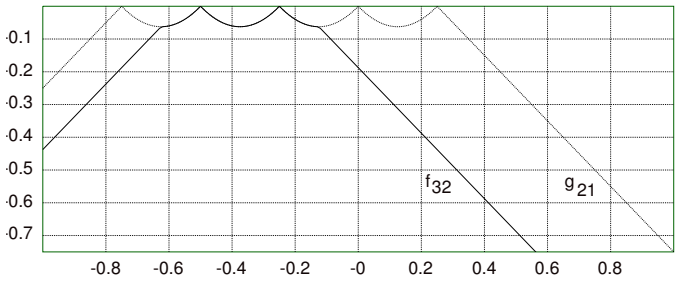


Fig. 2. Approximation of the square function

Coming back to the general construction, at any point on the interval I, we have that $f_{i,j} \geq l_{i,j} \oplus_{c_{i,j}} \mathsf{max}\,(f_{i+1,2j}, f_{i+2,2j})$ while on the interval $J_{i+1,2j} \cap J_{i+1,2j+1}$ equality holds: $f_{i,j} = l_{i,j} \oplus_{c_{i,j}} \mathsf{max}\,(f_{i+1,2j}, f_{i+2,2j})$. Thus, the following infinitary formula gives a correct approximation of the function f:

$$l_{0,0} \oplus_{c_{0,0}} \mathsf{max}\,(\,(l_{1,0} \oplus_{c_{1,0}} \mathsf{max}\,(\,(l_{2,0} \oplus_{c_{2,0}} \mathsf{max}\,\ldots), (l_{2,1} \oplus_{c_{2,1}} \mathsf{max}\,\ldots))), (l_{1,1} \oplus_{c_{1,1}} \mathsf{max}\,(\,(l_{2,2} \oplus_{c_{2,2}} \mathsf{max}\,\ldots), (l_{2,3} \oplus_{c_{2,3}} \mathsf{max}\,\ldots)))).$$

If the families $c_{i,j}$ and $l_{i,j}$ are definable, then it is possible to construct a PCDF expression whose semantics coincides with the formula. Given a real number $x \in I$, denote with $\langle J_{i,h(i)}\rangle_{i\in\mathbb{N}}$ a sequence of J intervals converging to x such that $\forall i.h(i) = \lfloor h(i+1)/2 \rfloor$ (if x is not a dyadic rational this sequence is unique, if x is a dyadic rationals there are two such a sequences). The above formula defines a function converging on x iff $\Pi_{i\in\mathbb{N}}(1 - c_{i,h(i)}) = 0$, for any such a sequence (each level reduces the inaccuracy by a factor $(1 - c_{i,j})$). If there exists an index k such that the function f is continuously differentiable in any interval $J_{k,j}$ containing x, then on these intervals f can be approximated with arbitrary precision by a piecewise linear function and therefore there is a choice for the constants $c_{i,j}$ making the above construction converge on x. But if x is a point of essential discontinuity for the derivative, there is a limit on the level of the precision for any choice for the constants $c_{i,j}$, and we need to consider the next construction to obtain convergence to the value of the function and its derivative at x.

C. General Lipschitz functions

In the previous construction, the finite approximations of the above displayed formula define both the value part and the derivative part of the function with the same level of precision. But there are non-expansive functions whose Clarke gradients (L-derivatives) are partial elements at all points [15], [6]. When applied to this class of functions the above construction can only lead to expressions whose semantics is a partial function also for the value part. To define functions in this class, we have to add an extra ingredient to the construction and to use the "projection" operator B, which increases the information contained in the value part of the partial function without necessarily modifying the information contained in the derivative part. To apply the operator B, it is necessary to build a list of piecewise linear functions $l'_{i,j}$ and dyadic rational numbers $c'_{i,j}$, with $0 \leq j \leq 2^i - 1$ satisfying the following three conditions: $g_{i,j} \in l'_{i,j} \pm c'_{i,j}/4$, $c'_{0,0} \cdot (1 - c_{0,0}) \leq \frac{1}{2}$ and $c'_{i+1,j} \cdot (1 - c_{i+1,j}) \leq \frac{c'_{i,j/2}}{2}$, that is $l'_{i,j}$ is a piecewise linear approximation of the function $g_{i,j}$ such that the value part of $g_{i,j}$ is approximated within an error $c'_{i,j}/4$, while there is no condition on the derivative part of $l'_{i,j}$.

The function f can then be expressed as

$$l_{0,0} \oplus_{c_{0,0}} (B_{c'_{0,0}} l'_{0,0}(\mathsf{max}\ (l_{1,0} \oplus_{c_{1,0}} (B_{c'_{1,0}} l'_{1,0}(\mathsf{max}\ \ldots))), (l_{1,1} \oplus_{c_{1,1}} (B_{c'_{1,1}} l'_{1,1}(\mathsf{max}\ \ldots))))).$$

The conditions on the constants $c'_{i,j}$ are such that the expansion of the above formula until the level i describes the value part of f with precision 2^{-1}. The conditions on $l'_{i,j}$ are such that a further application of the B operator determines the value of the function with an error strictly smaller than the application above it.

Given a maximal computable element f in the function domain $\mathcal{D}_{\iota\to\iota}$, the value part f_v is a total functions. Moreover, by the computability of f, it is possible to effectively generate, with an arbitrary precision, the graphs of the functions f_v and

f_d. Therefore it is possible to effectively generate the families of dyadic rationals $c_{i,j}$, $c'_{i,j}$ and the piecewise linear functions $l_{i,j}$, $l'_{i,j}$ of the construction above. To ensure the convergence of the derivative part, we also require that given a recursive enumeration of the finite elements below f, the rational dyadic number $c_{i,j}$ is chosen as the largest number in the form $\frac{k}{2^i}$ that can be generated after examining the first 2^i elements in the enumeration of f. Since the construction is effective, by Turing completeness of PCF, there exist four PCDF terms $\mathsf{c}, \mathsf{c}', \mathsf{l}, \mathsf{l}$ generating the above families $c_{i,j}$, $c'_{i,j}$, $l_{i,j}$, $l'_{i,j}$.

Let $\mathsf{f} : \nu \to \nu \to \iota \to \iota$ be the expression

$$\mathsf{f} = \mathsf{Y}\ \lambda\mathsf{F}.\lambda\mathsf{i}.\mathsf{jL}(\mathsf{l}\,\mathsf{i}\,\mathsf{j})(\mathsf{c}\,\mathsf{i}\,\mathsf{j})(\mathsf{B}(\mathsf{l}'\,\mathsf{i}\,\mathsf{j})(\mathsf{c}'\,\mathsf{i}\,\mathsf{j})(\mathsf{max}\,(\mathsf{F}(\mathsf{i}+1)(2\mathsf{j}))\\ (\mathsf{F}(\mathsf{i}+1)(2\mathsf{j}+1))))$$

The expression $\mathsf{f}\,0\,0$ defines the function f; in the sense that for any real number $x \in I$, we have $f_v(x) = (\mathcal{E}[\![\mathsf{f}_{0,0}]\!]_\rho)_v(x)$ and $f_d(x) = (\mathcal{E}[\![\mathsf{f}_{0,0}]\!]_\rho)_d(x)$.

Note that above definability result outlines a program expression that computes a function similar to the tradition of numerical analysis: the function f is expressed as the limit of a sequence of piecewise linear functions and the program that computes the value of the function at a given point actually also computes the values of the derivative at that point. Note moreover that the definability constructions do not use the parallel if operator pif. However pif becomes necessary in evaluating the derivative of a generic function since the operational semantics rules reduce the derivative of min to an expression containing pif.

V. Conclusion

In this paper we have presented a language for exact computation in the differentiable calculus. The language is obviously too simple to be practically usable. Even the product operator is not a primitive function in the calculus and needs to be defined.

The aim however has been to show that it is possible to integrate, in a single language, exact lazy computation of real functions with exact lazy computation of their derivatives. Moreover we have selected a small set of primitive functions that are sufficient to define any other function. In real programming languages, practical reasons call for the use of a larger set of primitives.

The present research can be extended in several directions. We outline here some possible future work.

- An obvious problem to consider is whether the definability result presented in the paper can be extended to a larger class of function domains. We claim that the techniques presented here can be easily adapted to functions with several arguments. This is not however the case when considering higher order functions, whose definability is an open problem.
- Another open problem is whether the set of elements in a generic domain $\mathcal{D}_\sigma$ that are self-related by the three logical relations R^i_σ, $R^{l,r}_\sigma$, and $R^{d,r}_\sigma$ forms a continuous Scott subdomain, and if so to find a direct characterisation of these subdomains.
- Another direction for possible further research is the treatment of C^2 or C^∞ functions. Interestingly it is possible to extend the domains of denotational semantics in such a way as to describe not only the derivatives but also the second derivatives of functions. For this, it is sufficient to use as basic domain for real numbers the domain $\mathcal{I} \times \mathcal{I} \times \mathcal{I}$. Moreover using the infinite product of $\mathcal{I}$ as domain for reals, one can deal with C^∞ functions, and allow an arbitrary depth application of the derivative operator. From the language point of view however, it is an open problem to find a set of primitive functions to generate twice differentiable or infinitely differentiable functions.

References

[1] E. Bishop and D. Bridges. *Constructive Analysis*. Springer-Verlag, 1985.
[2] F. H. Clarke. *Optimization and Nonsmooth Analysis*. Wiley, 1983.
[3] E. A. Coddington and N. Levinson. *Theory of Ordinary Differential Equations*. McGraw-Hill, 1955.
[4] T. A. Davis and K. Sigmon. *MATLAB Primer*. CRC Press, seventh edition, 2005.
[5] A. Edalat. A continuous derivative for real-valued functions. In S. B. Cooper, B. Löwe, and A. Sorbi, editors, *New Computational Paradigms, Changing Conceptions of What is Computable*, pages 493–519. Springer, 2008.
[6] A. Edalat. A differential operator and weak topology for Lipschitz maps. *Topology and its Applications*, 157,(9):1629–1650, June 2010.
[7] A. Edalat and A. Lieutier. Domain theory and differential calculus (Functions of one variable). *Mathematical Structures in Computer Science*, 14(6):771–802, December 2004.
[8] A. Edalat, A. Lieutier, and D. Pattinson. A computational model for multi-variable differential calculus. In V. Sassone, editor, *Proc. FoSSaCS 2005*, volume 3441 of *Lecture Notes in Computer Science*, pages 505–519, 2005. Available in `doc.ic.ac.uk/~ae/papers/multi.pdf`.
[9] T. Ehrhard and L. Regnier. The differential lambda-calculus. *Theoretical Computer Science*, 309(1-3), 2003.
[10] M. H. Escardó. PCF extended with real numbers. *Theoretical Computer Science*, 162(1):79–115, August 1996.
[11] P. Di Gianantonio. An abstract data type for real numbers. *Theoretical Computer Science*, 221:295–326, 1999.
[12] A. Griewank and A. Walther. *Evaluating Derivatives*. Siam, second edition, 2008.
[13] IC-Reals. `www.doc.ic.ac.uk/exact-computation/`.
[14] K. Ko. *Complexity Theory of Real Numbers*. Birkhäuser, 1991.
[15] G. Lebourg. Generic differentiability of lipschitzian functions. *Transaction of AMS*, 256:125–144, 1979.
[16] J. C. Mitchell. *Foundations of Programming Languages*. MIT Press, 1996.
[17] P. J. Potts, A. Edalat, and M. Escardó. Semantics of exact real arithmetic. In *Twelfth Annual IEEE Symposium on Logic in Computer Science*. IEEE, 1997.
[18] M. B. Pour-El and J. I. Richards. *Computability in Analysis and Physics*. Springer-Verlag, 1988.
[19] K. Weihrauch. *Computable Analysis (An Introduction)*. Springer, 2000.

APPENDIX

We give the details of several proofs and the alternative logical relations here.

A. Proof of Proposition 3

Proof: (i) Let x and y be two real numbers for which the function f_v is defined. For any rational $r \geq |x-y|$ we have that $R^{d,r}_\iota(\langle\{x\},[-1,1]\rangle, \langle\{y\},[-1,1]\rangle, \langle[x,y],[-1,1]\rangle)$. Therefore $R^{d,r}_\iota(f(\langle\{x\},[-1,1]\rangle), f(\langle\{y\},[-1,1]\rangle), f(\langle[x,y],[-1,1]\rangle))$, which implies that $f_v(x) - f_v(y) \in r \cdot f_2(\langle[x,y],[-1,1]\rangle)$, and thus $-1 \leq \frac{f_v(x)-f_v(y)}{x-y} \leq 1$.
(ii) Given any $x \in I$ and any open interval O containing $f_d(x) = f_2(\langle\{x,\},\{1\}\rangle)$, let $[a,b] \ll \{x\}$ be a rational interval such that f_v is define on $[a,b]$ and $f_2(\langle[a,b],\{1\}\rangle) \subseteq O$, and let $r = b - a$, we have $R^{d,r}_\iota(\langle\{b\},[-1,1]\rangle, \langle\{a\},[-1,1]\rangle, \langle[a,b],\{1\}\rangle)$, by repeating the arguments of the previous point, it follows $\frac{f_v(b)-f_v(a)}{b-a} \in f_2(\langle[a,b],\{1\}\rangle)$. By monotonicity of f it follows that for any pair of rationals $a', b' \in (a,b)$, we have: $\frac{f_v(b')-f_v(a')}{b'-a'} \in f_2(\langle[a,b],\{1\}\rangle)$, and by continuity of f_v for any pair of real numbers $x, y \in (a,b)$ $\frac{f_v(x)-f_v(y)}{x-y} \in O$.
(iii) If f is a maximal element $\mathcal{D}_{\iota\to\iota}$, by point (i) the function f_v is non-expansive on the points where it is defined. It follows that if the function f_v is not defined at a given point x, it is always possible to construct a function f° such that $f \sqsubseteq f^\circ$ and f°_v defined on x, leading to a contradiction. Similar arguments can be used to prove that f_d is the L-derivative of f_v and not only an approximation of the L-derivative. ∎

B. Proof of Proposition 4

The sublinearity relation $R^{l,r}_{\iota\to\iota}$ is closed under non-empty meets.

Proof: To show that sublinearity is closed under meets, assume that $f_k : \mathcal{I} \to \mathcal{I}$ with $k \in K$ is a family of Scott continuous functions satisfying the sublinearity $f_k(r[x,y]) \geq r f_k([x,y])$ for all $[x,y] \in \mathcal{I}$ and some (rational) $r \in [-1,1]$. We show that the meet $\bigsqcap_k f_k$ will also be sublinear.

We use the lower and upper parts of any $f : \mathcal{I} \to \mathcal{I}$ as $f^-, f^+ : T \to [-1,1]$ where $T = \{(x,y) \in [-1,1] \times [-1,1] : x \leq y\}$. Note that f^- and f^+ are lower and upper semi-continuous respectively. Sublinearity of f is equivalent to $f^+(r(x,y)) \geq r f^+((x,y))$ and $f^-(r(x,y)) \leq r f^-((x,y))$ for $r \in [-1,1]$ and $(x,y) \in T$.

We have: $(\bigsqcap_k f_i)^+ = g$ with $g = \limsup g_0$ where $g_0 = \sup_{k\in K} f_k^+$ and similarly $(\bigsqcap_k f_i)^- = h$ with $h = \liminf h_0$ where $h_0 = \inf_{k\in K} f_k^-$.

The sublinearity condition for f_k is equivalent to $f_k^+(r(x,y)) \geq r f_k^+((x,y))$ and $f_k^-(r(x,y)) \leq r f_k^-((x,y))$ for $r \in [-1,1]$ and $(x,y) \in T$.

By taking pointwise sup and inf respectively we get: $g_0(r(x,y)) \geq r g_0((x,y))$ and $h_0(r(x,y)) \leq r h_0((x,y))$. By taking limsup and liminf respectively we obtain: $g(r(x,y)) \geq r g((x,y))$ and $h(r(x,y)) \leq r h((x,y))$ as required. ∎

C. Proof of Proposition 5

In a continuous Scott domain, a non-empty subset closed under lubs of direct subsets and closed under non-empty meets is a continuous Scott subdomain.

Proof: Let D be a continuous Scott domain and $C \subset D$ a non-empty subset with the above closure properties. Given an element of $d \in D$, denote by $i(d)$ the greatest lower bound (meet) in C of the set $\{c \mid d \sqsubseteq c, c \in C\}$, if this set is not empty, otherwise let $i(d)$ be undefined.

Then i, regarded as a partial function from D to C, preserves the well below relation $\ll$. In fact given two elements $x, y \in D$ with $y \ll_D x$ and $i(x)$ defined, we check that $i(y) \ll_C i(x)$. Let A be a directed subset of elements in C, with $i(x) \sqsubseteq \bigsqcup_C A$. Since C is closed under lub of directed sets, $\bigsqcup_C A = \bigsqcup_D A$, and by construction of i, we have $x \sqsubseteq i(x)$. Thus, $x \sqsubseteq \bigsqcup_D C$, and, by hypothesis, there exists $a \in A$ such that $y \sqsubseteq a$. Since i is monotone and coincides with the identity on the elements of C, we have $i(y) \sqsubseteq i(a) = a$ and therefore $i(y) \ll_C i(x)$. It follows that if a set B is a basis of D then $i(B)$ forms is a basis for C. In fact given an element $x \in C$, the set $A = \{a \in B \mid a \ll x\}$, is a directed set with lub x, then $i(A)$ is a directed set of elements well below $i(x) = x$, having x as lub. Therefore C is a continuous dcpo, and since it has non-empty meets, it is also consistently complete. ∎

D. Proof of Adequacy Theorem 1

For every closed term e with type ι and environment ρ, we have:

$$\mathsf{d}_{[\mathsf{a},\mathsf{b}]} \ll Eval(e) \text{ iff } [a,b] \ll \pi_1(\mathcal{E}[\![e]\!]_\rho)$$

Proof: We use the technique of computability predicates to prove both the soundness and the completeness of the operational semantics. Note that the soundness of the operational semantics cannot be proved by simply showing that the reduction rules preserve the denotational semantics, since this is simply not true. A simple example being the expression $(\mathsf{D}x.\mathsf{d}_{1/2,0}\, e_1)e_2$ that reduces to $\mathsf{d}_{1/2,0}\,((\mathsf{D}x.e_1)e_2)$. The elements $\mathcal{E}[\![(\mathsf{D}x.\mathsf{d}_{1/2,0}\, e_1)e_2]\!]_\rho$ and $\mathcal{E}[\![\mathsf{d}_{1/2,0}\,((\mathsf{D}x.e_1)e_2)]\!]_\rho$ do not coincide on their second component (the first is $\perp$, the other is above $[-1/2, 1/2]$. More generally, all the reduction rules for the derivative operator do not preserve the semantics on the second element.

We define a computability predicate Comp for closed terms of type o, ν by requiring that the denotational and operational semantics coincide, in the usual way. A closed term e having type ι satisfies the predicate Comp_ι if for every closed rational interval $[a,b]$ and environment ρ we have: $\mathsf{d}_{[\mathsf{a},\mathsf{b}]} \ll Eval(e)$ iff $[a,b] \ll \pi_1(\mathcal{E}[\![e]\!]_\rho)$

The computability predicate is then extended, by induction on types, to closed elements of any type, and, by closure, to arbitrary elements.

Using the standard techniques for computability predicate, it is possible to prove that all constants are computable, and that λ-abstraction preserves the computability of the expressions. Therefore all expressions not containing the derivative operator are computable.

To prove that the computability predicate is satisfied by expressions containing the derivative operator, we show, by structural induction on the non-expansive expression e, that the expression $\mathsf{D}x.e$ is also computable.

The proof considers many cases. As an example, we take the case where $e = \mathsf{min}\, e_1 e_2$. By the induction hypothesis we can assume the computability of $\mathsf{D}x.e_1$ and $\mathsf{D}x.e_2$. Since the expressions e_1, e_2, pif and $\oplus_{1/2}$ do not contain the derivative operator, we can assume that they are computable.

We need to prove, for any computable expression e', that $\mathsf{d}_{[\mathsf{a},\mathsf{b}]} \ll Eval((\mathsf{D}x.\mathsf{min}\, e_1 e_2)e')$ iff $[a,b] \ll \pi_1(\mathcal{E}[\![(\mathsf{D}x.\mathsf{min}\, e_1 e_2)e']\!]_\rho)$.

On the one hand, we have the following chain of implications:

$\mathsf{d}_{[\mathsf{a},\mathsf{b}]} \ll Eval((\mathsf{D}x.\mathsf{min}\, e_1 e_2)e')$ iff, by the denotational semantics rules,

$\mathsf{d}_{[\mathsf{a},\mathsf{b}]} \ll Eval(\mathsf{pif}\ (\mathsf{0}<)\,(\mathsf{t}_{-1}\, e_1[e'/x] \oplus_{1/2} (e_2[e'/x]))$
$\quad \mathsf{then}\ (\mathsf{D}x.e_1)e'\ \mathsf{else}\ (\mathsf{D}x.e_2)e')$

iff, by computability of the expression right hand side

$[a,b] \ll \pi_1(\mathcal{E}[\![\mathsf{pif}\ (\mathsf{0}<)\,(e_1[e'/x] \oplus_{1/2} (\mathsf{t}_{-1}\, e_2[e'/x]))$
$\quad \mathsf{then}\ (\mathsf{D}x.e_1)e'\ \mathsf{else}\ (\mathsf{D}x.e_2)e']\!]_\rho)$

If we pose: $\mathcal{E}[\![e']\!]_\rho = \langle i, j\rangle$, $\mathcal{E}[\![e_1]\!]_{\rho[\langle i,1\rangle/x]} = \langle i_1, j_1\rangle$, $\mathcal{E}[\![e_1]\!]_{\rho[\langle i,j\rangle/x]} = \langle i'_1, j'_1\rangle$, $\mathcal{E}[\![e_2]\!]_{\rho[\langle i,1\rangle/x]} = \langle i_2, j_2\rangle$, $\mathcal{E}[\![e_2]\!]_{\rho[\langle i,j\rangle/x]} = \langle i'_2, j'_2\rangle$.

by applying the denotational semantics rule we can derive that right hand side in the last relation is equal to j_1 if $i'_1 < i'_2$, to j_2 if $i'_2 < i'_1$, and to $j_1 \sqcup j_2$ otherwise.

On the other hand, by the rules of denotational semantics: $\pi_1(\mathcal{E}[\![(\mathsf{D}x.\mathsf{min}\, e_1 e_2)e]\!]_\rho) = \pi_2(\mathcal{E}[\![\mathsf{min}\, e_1 e_2]\!]_{\rho[i,1\rangle/x]})$ which is equal to j_1 if $i_1 < i_2$, to j_2 if $i_2 < i_1$, and to $j_1 \sqcup j_2$ otherwise.

Since $\mathcal{E}[\![e_1]\!]$ and $\mathcal{E}[\![e_2]\!]$ are self-related by R^i_σ, their value parts are independent from the derivative part so $i_1 = i'_1$ and $i_2 = i'_2$, from which the result follows.

The other cases can be proved in a similar way.

■

E. Alternative Logical Relations

It is possible to give a more precise characterization of definable elements by presenting an alternative set of logical relations. In the new set, the relation for sublinearity $R^{l,r}_\sigma$ is replaced by a relation $S^{l,r}_\sigma$ implying linearity, rather than the sublinearity, of the derivative part. Given a rational number $r \in [-1,1]$, the family $S^{l,r}_\iota$ consists of pairs of the form $(\langle i, j_1\rangle, \langle i, j_2\rangle)$ where $j_1 = r \cdot j_2$. It is now easy to prove that the semantic interpretation of all constants are self-related by $S^{l,r}_\sigma$. The only problem is that the bottom element $\bot_\sigma$ is not self-related by $S^{l,r}_\sigma$, and as a consequence expressions containing the recursive operator Y_σ have semantic interpretations that are not self-related by $S^{l,r}_\sigma$. This problem can be resolved by giving an alternative semantics interpretation for Y_σ: in particular the fixed-point $\mathsf{Y}_\sigma e$ of an expression e is now defined as the least upper bound of a chain whose first element is the least element $\bot^l_\sigma$, called the least linear element, self-related by $S^{l,r}_\sigma$, rather than $\bot_\sigma$, as usual. Therefore, before defining the interpretation of the fixed-point operator Y_σ, one needs to introduce the logical relations, $S^{l,r}_\sigma$, for linearity on an arbitrary type σ and characterise the least linear element in each semantic domain $\mathcal{D}_s\sigma$, and finally prove that the semantic interpretation of expressions are self-related by $S^{l,r}_\sigma$.

Note that the elements of $\mathcal{D}_\iota$ that are self-related by $S^{l,r}_\iota$ are the element in the form $\langle i, \{0\}\rangle$, which form a subdomain isomorphic to $\mathcal{I}$, the standard domain for real numbers.

For a given type σ, the least linear element $\bot^l_\sigma$ is defined using a set function $ext_\sigma : \mathcal{D}_\sigma \to \mathcal{I}$ by mutual induction on the structure of σ as follows:

- $\bot^l_\iota = \langle [-1,1], [0,0]\rangle$,
- if σ has the form $\sigma = \sigma_1 \to \ldots \sigma_n \to \iota$ we define
$$\bot^l_\sigma d_1 \ldots d_n = \langle [-1,1], \bigsqcup_{i\in I} \{ext_{\sigma_i}(d_i)\}$$
- if σ is not in the above form $\bot^l_\sigma = \bot_\sigma$.
- The function ext_σ is defined by: $ext_\iota(\langle i, j\rangle) = j \sqcup -j$, $ext_o(d) = [0,0]$, $ext_\nu(d) = [0,0]$, $ext_{\sigma_1\to\sigma_2}(d) = ext_{\sigma_2}(d(\bot^l_{\sigma_1}))$.

Clearly this solution gives rise to a more complex definition of the semantics. Therefore we preferred to present the simpler semantics in this paper, and just sketched here the possibility of obtaining a more refined and precise semantics.