INFORMATION AND COMPUTATION **124**, 182–197 (1996)
ARTICLE NO. 0014

Power Domains and Iterated Function Systems

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We introduce the notion of weakly hyperbolic iterated function system (IFS) on a compact metric space, which generalises that of hyperbolic IFS. Based on a domain-theoretic model, which uses the Plotkin power domain and the probabilistic power domain respectively, we prove the existence and uniqueness of the attractor of a weakly hyperbolic IFS and the invariant measure of a weakly hyperbolic IFS with probabilities, extending the classic results of Hutchinson for hyperbolic IFSs in this more general setting. We also present finite algorithms to obtain discrete and digitised approximations to the attractor and the invariant measure, extending the corresponding algorithms for hyperbolic IFSs. We then prove the existence and uniqueness of the invariant distribution of a weakly hyperbolic recurrent IFS and obtain an algorithm to generate the invariant distribution on the digitised screen. The generalised Riemann integral is used to provide a formula for the expected value of almost everywhere continuous functions with respect to this distribution. For hyperbolic recurrent IFSs and Lipschitz maps, one can estimate the integral up to any threshold of accuracy.

1. INTRODUCTION

The theory of iterated function system has bee an active area of research since the seminal work of Mandelbrot [32] on fractals and self-similarity in nature in late seventies and early eighties [25, 2, 13, 29, 30, 18, 5]. The theory has found applications in diverse areas such as computer graphics, image compression, learning automata, neural nets, and statistical physics [7, 8, 1, 6, 11, 29, 30, 10, 9].

In this paper, we will be mainly concerned with the basic theoretical work of Hutchinson [25] and a number of algorithms based on this work. We start by briefly reviewing the classical work. See [19] for a comprehensive introduction to iterated function systems and fractals.

1.1. *Iterated Function Systems*

An *iterated function system* (*IFS*) $\{X; f_1, f_2, ..., f_N\}$ on a topological space X is given by a finite set of continuous maps $f_i\colon X \to X$ ($i = 1; ..., N$). If X is a complete metric space and the maps f_i are all contracting, then the IFS is said to be *hyperbolic*. For a complete metric space X, let $\mathbf{H}X$ be the complete metric space of all non-empty compact subsets of X with the *Hausdorff* metric d_{H} defined by

$$d_{\mathrm{H}}(A, B) = \inf\{\delta \mid B \subseteq A_\delta \text{ and } A \subseteq B_\delta\},$$

where, for a non-empty compact subset $C \subseteq X$ and $\delta \geqslant 0$, the set

$$C_\delta = \{x \in X \mid \exists y \in C \cdot d(x, y) \leqslant \delta\}$$

is the *δ-parallel body* of C.

A hyperbolic IFS induces a map

$$F\colon \mathbf{H}X \to \mathbf{H}X, \tag{1}$$

defined by $F(A) = f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A)$. In fact, F is also contracting with contractivity factor $s = \max_i s_i$; where s_i is the contractivity factor of f_i ($1 \leqslant i \leqslant N$). The number s is called the *contractivity* of the IFS. By the contracting mapping theorem, F has a unique fixed point A^* in $\mathbf{H}X$, which is called the *attractor* of the IFS, and we have

$$A^* = \lim_{n \to \infty} F^n(A) \tag{2}$$

for any non-empty compact subset $A \subseteq X$ [25]. The attractor is also called a *self-similar set*.

For applications in graphics and image compression [1, 6, 21], it is assumed that X is the plane $\mathbb{R}^2$ and that the maps are contracting *affine* transformations. Then, the attractor is usually a *fractal*; i.e., it has fine, complicated and non-smooth local structure, some form of self-similarity, and, usually, a non-integral *Hausdorff dimension*. A finite algorithm to generate a discrete approximation to the attractor was first obtained by Hepting *et al.* [24]. (See also [14, 33].) It is described in Section 2.3.

1.2. *IFS with Probabilities*

There is also a probabilistic version of the theory that produces invariant probability distributions and, as a result, coloured images in computer graphics. A hyperbolic IFS with probabilities $\{X; f_1, ..., f_N; p_1, ..., p_N\}$ is a hyperbolic IFS $\{X; f_1, f_2, ..., f_N\}$, with X a compact metric space, such that each f_i ($1 \leqslant i \leqslant N$) is assigned a probability p_i with

$$0 < p_i < 1 \qquad \text{and} \qquad \sum_{i=1}^{N} p_i = 1.$$

Then, the *Markov operator* is defined by

$$T\colon \mathbf{M}^1 X \to \mathbf{M}^1 X \tag{3}$$

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on the set $\mathbf{M}^1X$ of normalised Borel measures on X. It takes a Borel measure $\mu \in \mathbf{M}^1X$ to a Borel measure $T(\mu) \in \mathbf{M}^1X$ given by

$$T(\mu)(B) = \sum_{i=1}^{N} p_i \mu(f_i^{-1}(B))$$

for any Borel subset $B \subseteq X$. When X is compact, the *Hutchinson metric* r_H can be defined on $\mathbf{M}^1X$ as follows [2]:

$$r_H(\mu, \nu) = \sup \left\{ \int_X f\, d\mu - \int_X f\, d\nu \mid f: X \to \mathbb{R}, \right.$$
$$\left. |f(x) - f(y)| \leqslant d(x, y), \forall x, y \in X \right\}.$$

Then, using some Banach space theory, including Alaoglu's theorem, it is shown that the weak* topology and the Hutchinson metric topology on $\mathbf{M}^1X$ coincide, thereby making $(\mathbf{M}^1X, r_H)$ a compact metric space. If the IFS is hyperbolic, T will be a contracting map. The unique fixed point μ^* of T then defines a probability distribution on X whose support is the attractor of $\{X; f_1, ..., f_N\}$ [25]. The measure μ^* is also called a *self similar measure* or a *multifractal.* When $X \subseteq \mathbb{R}^n$, this invariant distribution gives different point densities in different regions of the attractor, and using a colouring scheme, one can colour the attractor accordingly. A finite algorithm to generate a discrete approximation to this invariant measure and a formula for the value of the integral of a continuous function with respect to this measure were also obtained in [24]; they are described in Sections 3.3 and 5, respectively.

The *random iteration algorithm* for an IFS with probabilities [13, 1] is based on the following ergodic theorem of Elton [18]. Let $\{X; f_1, ..., f_N; p_1, ..., p_N\}$ be an IFS with probabilities on the compact metric space X and let $x_0 \in X$ be any initial point. Put $\Sigma_N = \{1, ..., N\}$ with the discrete topology. Choose $i_1 \in \Sigma_N$ at random such that i is chosen with probability p_i. Let $x_1 = f_{i_1}(x_0)$. Repeat to obtain i_2 and $x_2 = f_{i_2}(x_1) = f_{i_2}(f_{i_1}(x_0))$. In this way, construct the sequence $\langle x_n \rangle_{n \geqslant 0}$. Suppose B is a Borel subset of X such that $\mu^*(\gamma(B)) = 0$, where μ^* is the invariant measure of the IFS and $\gamma(B)$ is the boundary of B. Let $L(n, B)$ be the number of points in the set $\{x_0, x_1, ..., x_n\} \cap B$. Then, Elton's Theorem says that, with probability one (i.e., for almost all sequences $\langle x_n \rangle_{n \geqslant 0} \in \Sigma_N^\omega$), we have

$$\mu^*(B) = \lim_{n \to \infty} \frac{L(n, B)}{n+1}.$$

Moreover, for all continuous functions $g: X \to \mathbb{R}$, we have the following convergence with probability one,

$$\int g\, d\mu^* = \lim_{n \to \infty} \frac{\sum_{i=0}^{n} g(x_i)}{n+1}, \tag{4}$$

which gives the expected value of g.

1.3. *Recurrent IFS*

Recurrent iterated function systems generalise IFSs with probabilities as follows [4]. Let X be a compact metric space and $\{X; f_1, f_2, ..., f_N\}$ a (hyperbolic) IFS. Let (p_{ij}) be an indecomposable $N \times N$ row-stochastic matrix, i.e.,

- $\sum_{j=1}^{N} p_{ij} = 1$ for all i,
- $p_{ij} \geqslant 0$ for all i, j, and
- for all i, j there exist $i_1, i_2, ..., i_n$ with $i_1 = i$ and $i_n = j$ such that $p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{n-1} i_n} > 0$.

Then $\{X; f_j; p_{ij}; i, j = 1, 2, ..., N\}$ is called a (hyperbolic) recurrent IFS. For a hyperbolic recurrent IFS, consider a random walk on X as follows. Specify a starting point $x_0 \in X$ and a starting code $i_0 \in \Sigma_N$. Pick a number $i_1 \in \Sigma_N$ such that $p_{i_0 j}$ is the conditional probability that j is chosen, and define $x_1 = f_{i_1}(x_0)$. Then pick $i_2 \in \Sigma_N$ such that $p_{i_1 j}$ is the conditional probability that j is chosen, and put $x_2 = f_{i_2}(f_{i_1}(x_0))$. Continue to obtain the sequence $\langle x_n \rangle_{n \geqslant 0}$. The distribution of this sequence converges with probability one to a measure on X called the *stationary distribution* of the hyperbolic recurrent IFS. This generalises the theory of hyperbolic IFSs with probabilities. In fact, if $p_{ij} = p_j$ is independent of i then we obtain a hyperbolic IFS with probabilities; the stationary distribution is then just the invariant measure and the random walk above reduces to the random iteration algorithm. The first practical software system for fractal image compression, Barnsley's VRIFS (Vector Recurrent Iterated Function System), which is an interactive image modelling system, is based on hyperbolic recurrent IFSs [6].

1.4. *Weakly Hyperbolic IFS*

In [15], power domains were used to construct domain-theoretic models for IFSs and IFSs with probabilities. It was shown that the attractor of a hyperbolic IFS on a compact metric space is obtained as the unique fixed point of a continuous function on the Plotkin power domain of the upper space. Similarly, the invariant measure of a hyperbolic IFS with probabilities on a compact metric space is the fixed point of a continuous function on the probabilistic power domain of the upper space.

We will here introduce the notion of a weakly hyperbolic IFS. Our definition is motivated by a number of applications, for example in neural nets [23, 9, 17], where one encounters IFSs which are not hyperbolic. This situation can arise for example in a compact interval $X \subseteq \mathbb{R}$ if the IFS contains a smooth map $f: X \to X$ satisfying $|f'(x)| \leqslant 1$ but not $|f'(x)| < 1$.

Let (X, d) be a compact metric space; we denote the diameter of any set $a \subseteq X$ by $|a| = \sup\{d(x, y) \mid x, y \in a\}$. As before, let $\Sigma_N = \{1, 2, ..., N\}$ with the discrete topology and let Σ_N^ω be the set of all infinite sequences $i_1 i_2 i_3 \ldots (i_n \in \Sigma_N$ for $n \geqslant 1)$ with the product topology.

DEFINITION 1.1. An IFS $\{X; f_1, f_2, ..., f_N\}$ is *weakly hyperbolic* if for all infinite sequences $i_1 i_2 ... \in \Sigma_N^\omega$ we have $\lim_{n\to\infty} |f_{i_1} f_{i_2} \cdots f_{i_n} X| = 0$.

Weakly hyperbolic IFSs generalise hyperbolic IFSs since clearly a hyperbolic IFS is weakly hyperbolic. One similarly defines a *weakly hyperbolic IFS with probabilities* and a *weakly hyperbolic recurrent IFS*.

There are two other notions of IFSs with non-contracting maps in the literature. We compare these with the notion of weakly hyperbolic IFS in the case of a compact metric space X. An IFS $\{X; f_1, f_2, ..., f_N\}$ is said to be *eventually contracting* [21] if there is some $k \geqslant 1$ such that the N^k maps $g_{i_1 i_2 \cdots i_k} = f_{i_1} f_{i_2} \cdots f_{i_k}$ are contracting for all finite sequences $i_1, i_2, ..., i_k \in \Sigma_N^k$ of length k. It is easy to see that an eventually contracting IFS is weakly hyperbolic as follows. We can write any $n \geqslant 1$ as $n = pk + q$, where p and q are non-negative integers with $0 \leqslant q \leqslant k-1$. Since $\langle f_{i_1} f_{i_2} \cdots f_{i_n} X \rangle_{n \geqslant 0}$ is a decreasing sequence of subsets of X, it follows that $|f_{i_1} f_{i_2} \cdots f_{i_n} X| \leqslant |g_{j_1} g_{j_2} \cdots g_{j_p} X|$ where $j_m = i_{(m-1)k+1} i_{(m-1)k+2} \cdots i_{mk}$ for $1 \leqslant m \leqslant p$. As g_{j_p} is contracting for any $j_p \in \Sigma_N^k$, we conclude that $\lim_{n\to\infty} |f_{i_1} f_{i_2} \cdots f_{i_n} X| = 0$ and that an eventually contracting IFS is weakly hyperbolic. However, a weakly hyperbolic IFS need not be eventually contracting. This can be seen even for the case of a single map ($N = 1$) on $X = [0, 1]$. Let $f_0 \colon [0, 1] \to [0, 1]$ be, say, a twice differentiable map with $f_0(0) = 0$, $f_0'(0) = 1$ and $f_0''(x) < 0$ for all $x \in [0, 1]$ (e.g., $f_0(x) = x(1-x)$ or $f_0(x) = \tanh(x)$). Then, f_0 has a unique weakly attracting fixed point at $x = 0$ and $\lim_{n\to\infty} f^n(x) = 0$ for all $x \in [0, 1]$. It follows that $\lim_{n\to\infty} |f_0^n([0, 1])| = 0$ and, hence, the IFS $\{[0, 1], f_0\}$ is weakly hyperbolic. Since $(f_0^n)'(0) = 1$ for all $n \geqslant 1$, it follows by the mean value theorem that $f_0^n \colon [0, 1] \to [0, 1]$ is not contracting for any $n \geqslant 1$. Therefore, $\{[0, 1], f_0\}$ is not eventually contracting. In fact, for any hyperbolic IFS $\{[0, 1]; f_1, ..., f_N\}$, it can be shown that the extended IFS $\{[0, 1]; f_0, f_1, ..., f_N\}$, where f_0 is as above, is weakly hyperbolic (see Proposition 2.6) but is not eventually contracting. Therefore, weakly hyperbolic IFSs generalise eventually contracting IFSs.

An IFS with probabilities $\{X; f_1, ..., f_N; p_1, ..., p_N\}$ is said to be *contracting on average* [3] if there is $s < 1$ such that

$$\prod_{i=1}^{N} d(f_i(x), f_i(y))^{p_i} \leqslant s d(x, h)$$

for all $x, y \in X$. An eventually contracting IFS (and, hence, a weakly hyperbolic IFS with probabilities) need not be contracting on average. This can be seen even in the trivial IFS $\{[0, 1]; f_1\}$ with $f_1(x) = \max(0, (5x/2) - 2)$. This IFS is clearly not contracting on average, but it is eventually contracting as $f_1^2(x) = 0$ for all $x \in [0, 1]$. We can also add the map $f_2 \colon [0, 1] \to [0, 1]$ with $f_2(x) = (1-x)/2$ to obtain the IFS with probabilities $\{[0, 1]; f_1, f_2; 1/2, 1/2\}$ which is easily shown to be eventually contracting (any composition $f_{i_1} f_{i_2} f_{i_3}$ has contractivity 5/8) but is again not contracting on average. On the other hand, an IFS which is contracting on average need not be weakly hyperbolic (and, hence, need not be eventually contracting). This can be seen by the IFS with probabilities $\{[0, 1]; f_1, f_2; 1/2, 1/2\}$ where $f_1(x) = x/3$ and $f_2(x) = \min(2x, 1)$. It is easily seen that for all $x, y \in [0, 1]$ we have

$$|f_1(x) - f_1(y)|\ |f_2(x) - f_2(y)| \leqslant \tfrac{2}{3} |x - y|^2$$

and, hence, the IFS is contracting on average. However, for all $n \geqslant 1$ we have $f_2^n([0, 1]) = [0, 1]$ and, therefore, the IFS is not weakly hyperbolic. Note that this IFS does *not* have a unique attractor. In fact, the compact subsets $[0, 1]$ and $\{0\}$ are both fixed points of $F \colon \mathbf{H}[0, 1] \to \mathbf{H}[0, 1]$ with $F(A) = f_1(A) \cup f_2(A)$. We therefore conclude that IFSs which are contracting on average represent a totally different class as compared with hyperbolic IFSs.

Since for a weakly hyperbolic IFS, the map $F \colon \mathbf{H}X \to \mathbf{H}X$ is not necessarily contracting, one needs a different approach to prove the existence and uniqueness of the attractor in this more general setting. In this paper, we will use the domain-theoretic model to extend the results of Hutchinson, those of Hepting *et al.* and those in [15] mentioned above to weakly hyperbolic IFSs and weakly hyperbolic IFSs with probabilities. We will then prove the existence and uniqueness of the invariant distribution of a weakly hyperbolic recurrent IFS and obtain a finite algorithm to generate this invariant distribution on a digitised screen. We also deduce a formula for the expected value of an almost continuous function with respect to this distribution and also a simple expression for the expected value of any Lipschitz map, up to any given threshold of accuracy, with respect to the invariant distribution of a hyperbolic recurrent IFS.

The domain-theoretic framework of IFS, we will show, has the unifying feature that several aspects of the theory of IFS, namely (a) the proof of existence and uniqueness of the attractor of a weakly hyperbolic IFS and that of the invariant measure of a weakly hyperbolic IFS with probabilities or recurrent IFS, (b) the finite algorithms to approximate the attractor and the invariant measures, (c) the complexity analyses of these algorithms, and (d) the computation of the expected value of almost everywhere continuous functions (or Lipschitz functions) with respect to these invariant measures, are all integrated uniformly within the domain-theoretic model.

1.5. *Notation and Terminology*

We recall the basic definitions in the theory of continuous posets (poset = partially ordered set).

A non-empty subset $A \subseteq P$ of a poset $(P, \sqsubseteq)$ is *directed* if for any pair of elements $x, y \in A$ there is $z \in A$ with $x, y \sqsubseteq z$. A *directed complete partial order* (*dcpo*) is a partial order in which every directed subset A has a least upper bound (lub), denoted by $\bigsqcup A$. A poset is *bounded complete* if every bounded subset has a lub.

An open set $O \subseteq D$ of the *Scott* topology of a dcpo is a set which is upward closed (i.e., $x \in O$ & $x \sqsubseteq y \Rightarrow y \in O$) and is inaccessible by lubs of directed sets (i.e., $\bigsqcup A \in O \Rightarrow \exists x \in A \cdot x \in O$). It can be shown that a function $f: D \to E$ from a dcpo D to another one E is continuous with respect to the Scott topology iff it is *monotone*, i.e., $x \sqsubseteq y(x) \sqsubseteq f(y)$ and preserves lubs of directed sets, i.e., $\bigsqcup_{i \in I} f(x_i) = f(\bigsqcup_{i \in I} x_i)$, where $\{x_i \mid i \in I\}$ is any directed subset of D. From this it follows that a continuous function $f: D \to D$ on a dcpo D with least element (or bottom) $\perp$ has a *least fixed point* given by $\bigsqcup_{n \geqslant 0} f^n(\perp)$.

Given two elements x, y in a dcpo D, we say x is *way-below* y, denoted by $x \ll y$, if whenever $y \sqsubseteq \bigsqcup A$ for a directed set A, then there is $a \in A$ with $x \sqsubseteq a$. We say that a subset $B \subseteq D$ is a *basis* for D if for each $d \in D$ the set A of elements of B way-below d is directed and $d = \bigsqcup A$. We say D is *continuous* if it has a basis; it is *ω-continuous* if it has a countable basis. The product of (ω-)continuous dcpo's is an (ω-)continuous dcpo in which the Scott topology and the product topology coincide. An (ω-)*algebraic* dcpo is an (ω-)continuous dcpo with a (countable) basis B satisfying $b \ll b$ for all $b \in B$.

For any map $f: D \to E$, any point $x \in D$, any subset $A \subseteq D$ and any subset $B \subseteq E$, we denote, whenever more convenient, the image of x by fx instead of $f(x)$, the forward image of A by fA instead of $f(A)$ and the pre-image of B by $f^{-1}B$ instead of $f^{-1}(B)$. The lattice of open sets of a topological space X is denoted by $\Omega(X)$. For a compact metric space X, we denote by $\mathbf{M}^c X$, $0 \leqslant c \leqslant 1$, the set of all Borel measures μ on X with $\mu(X) = c$.

2. A DOMAIN-THEORETIC MODEL

We start by presenting the domain-theoretic framework for studying IFSs.

2.1. *The Upper Space*

Let X be a compact Hausdorff space. The *upper space* $(\mathbf{U}X, \supseteq)$ of X consists of all non-empty compact subsets of X ordered by reverse inclusion. We recall the following properties of the upper space, for example, from [15]. The partial order $(\mathbf{U}X, \supseteq)$ is a bounded complete continuous dcpo with a bottom element, namely X, in which the least upper bound (lub) of a directed set of compact subsets is their intersection. The way-below relation $B \ll C$ holds if and only if B contains a neighbourhood of C. The Scott topology on $\mathbf{U}X$ has a basis given by the collections $\square a = \{C \in \mathbf{U}X \mid C \subseteq a\}$ $(a \in \Omega(X))$. The singleton map

$$s: X \to \mathbf{U}X$$
$$x \mapsto \{x\}$$

embeds X onto the set $s(X)$ of maximal elements of $\mathbf{U}X$. Any continuous map $f: X \to Y$ of compact Hausdorff spaces induces a Scott-continuous map $\mathbf{U}f: \mathbf{U}X \to \mathbf{U}Y$ defined by $\mathbf{U}f(C) = f(C)$; to keep the notations simple we will write $\mathbf{U}f$ simply as f. If X is in fact a compact metric space, then $(\mathbf{U}X, \supseteq)$ is an ω-continuous dcpo and has a countable basis consisting of finite unions of closures of relatively compact open sets of X. Note that the two topological spaces $(\mathbf{U}X, \supseteq)$ and $(\mathbf{H}X, d_{\mathrm{H}})$ have the same elements (the non-empty compact subsets) but different topologies.

Hayashi used the upper space to note the following result.

Proposition 2.1. [22] *If $\{X; f_1, f_2, ..., f_N\}$ is an IFS on a compact Hausdorff space X, then the map*

$$F: \mathbf{U}X \to \mathbf{U}X$$
$$A \mapsto f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A)$$

is Scott-continuous and has therefore a least fixed point, namely,

$$A^* = \bigsqcup_n F^n(X) = \bigcap_n F^n(X).$$

For convenience, we use the same notation for the map $F: \mathbf{H}X \to \mathbf{H}X$ as in Eq. (1) and the map $F: \mathbf{U}X \to \mathbf{U}X$ above, as they are defined in exactly the same way. Since the ordering in $\mathbf{U}X$ is reverse inclusion, A^* is the largest compact subset of X with $F(A^*) = A^*$. However, in order to obtain a satisfactory result on the uniqueness of this fixed point and in order to formulate a suitable theory of IFS with probabilities, we need to assume that X is a metric space.

On the other hand if X is a locally compact, complete metric space and $\{X; f_1, ..., f_N\}$ a hyperbolic IFS, then there exists a non-empty regular compact set[1] A such that $F(A) = f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A) \subseteq A^\circ$, where A° is the interior of A (see [15, Lemma 3.10]). The unique attractor of the IFS will then lie in A and, therefore, we can simply work with the IFS $\{A; f_1, ..., f_N\}$. In particular, if X is $\mathbb{R}^n$ with the Euclidean metric and s_i, $0 \leqslant s_i < 1$, is the contractivity factor of f_i $(1 \leqslant i \leqslant N)$, then it is easy to check that we have $F(A) \subseteq A$, where A is any closed ball of radius R centred at the origin O with

$$R \geqslant \max_i \frac{d(O, f_i(O))}{1 - s_i},$$

[1] A regular closed set is one which is equal to the closure of its interior.

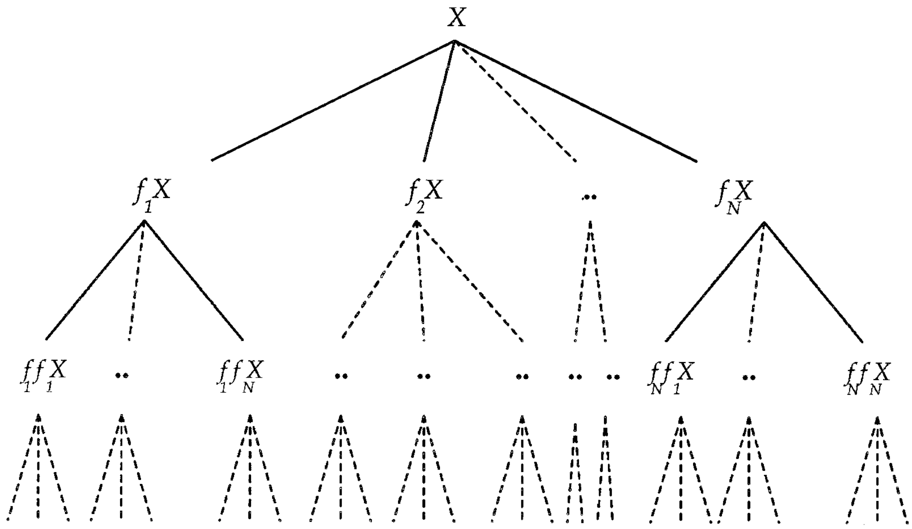

FIG. 1. The IFS tree.

where d is the Euclidean metric. Therefore, as far as a hyperbolic IFS on a locally compact, complete metric space is concerned, there is no loss of generality if we assume that the underlying space X is a compact metric space. We will make this assumption from now on.

Let X be a compact metric space and let $\{X; f_1, ..., f_N\}$ be an IFS. The IFS generates a finitely branching tree as in Fig. 1, which we call the *IFS tree*. Note that each node is a subset of its parent node and therefore the diameters of the nodes decrease along each infinite branch of the tree. The IFS tree plays a fundamental role in the domain-theoretic framework for IFSs: As we will see, all the results in this paper are based on various properties of this tree; these include the existence and uniqueness of the attractor of a weakly hyperbolic IFS, the algorithm to obtain a discrete approximation to the attractor, the existence and uniqueness of the invariant measure of a weakly hyperbolic IFS: with probabilities, the algorithm to generate this measure on a digitised screen, the corresponding results for the recurrent IFSs, and the formula for the expected value of an almost everywhere continuous function with respect to the invariant distribution of a weakly hyperbolic recurrent IFS.

We will now use this tree to obtain some equivalent characterizations of a weakly hyperbolic (IFS) as defined in Definition 1.1.

PROPOSITION 2.2. *For an IFS $\{X; f_1, ..., f_N\}$ on a compact metric space X, the following are equivalent*:

(i) *The IFS is weakly hyperbolic.*

(ii) *For each infinite sequence $i_1 i_2 \cdots \in \Sigma_N^\omega$, the intersection $\bigcap_{n \geqslant 1} f_{i_1} f_{i_2} \cdots f_{i_n} X$ is a singleton set.*

(iii) *For all $\varepsilon > 0$, there exists $n \geqslant 0$ such that $|f_{i_1} f_{i_2} \cdots f_{i_n} X| < \varepsilon$ for all finite sequences $i_1 i_2 \cdots i_n \in \Sigma_N^n$ of length n.*

Proof. The implications (i) $\Leftrightarrow$ (ii) and also (iii) $\Rightarrow$ (i) are all straightforward. It remains to show (i) $\Rightarrow$ (iii). Assume that the IFS does not satisfy (iii). Then there exists $\varepsilon > 0$ such that for all $n \geqslant 0$ there is a node on level n of the IFS tree with diameter at least ε. Since the parent of any such node will also have diameter at least ε, we obtain a finitely branching infinite subtree all whose nodes have diameter at least ε. By König's lemma this subtree will have an infinite branch $\langle f_{i_1} f_{i_2} \cdots f_{i_n} X \rangle_{n \geqslant 0}$. Therefore, the sequence $\langle |f_{i_1} f_{i_2} \cdots f_{i_n} X| \rangle_{n \geqslant 0}$ does not converge to zero as $n \to \infty$ and the IFS is not weakly hyperbolic. ∎

COROLLARY 2.3. *If the IFS is weakly hyperbolic, then for any sequence $i_1 i_2 \cdots \in \Sigma_N^\omega$, the sequence $\langle f_{i_1} f_{i_2} \cdots f_{i_n} x \rangle_{n \geqslant 0}$ converges for any $x \in X$ and the limit is independent of x. Moreover, the mapping*

$$\pi : \Sigma_N^\omega \to X$$

$$i_1 i_2 \cdots \mapsto \lim_{n \to \infty} f_{i_1} f_{i_2} \cdots f_{i_n} x$$

is continuous and its image is $A^* = \bigcap_{n \geqslant 0} F^n X$.

An IFS $\{X; f_1, ..., f_N\}$ also generates another finitely branching tree as in Fig. 2, which we call the *action tree*. Here, a child of a node is the image of the node under the action of some f_i.

Note that the IFS tree and the action tree have the same set of nodes on any level $n \geqslant 0$.

COROLLARY 2.4. *If the IFS is weakly hyperbolic, $\lim_{n \to \infty} |f_{i_n} f_{i_{n-1}} \cdots f_{i_1} X| = 0$ for all infinite sequences $i_1 i_2 \cdots \in \Sigma_N^\omega$.*

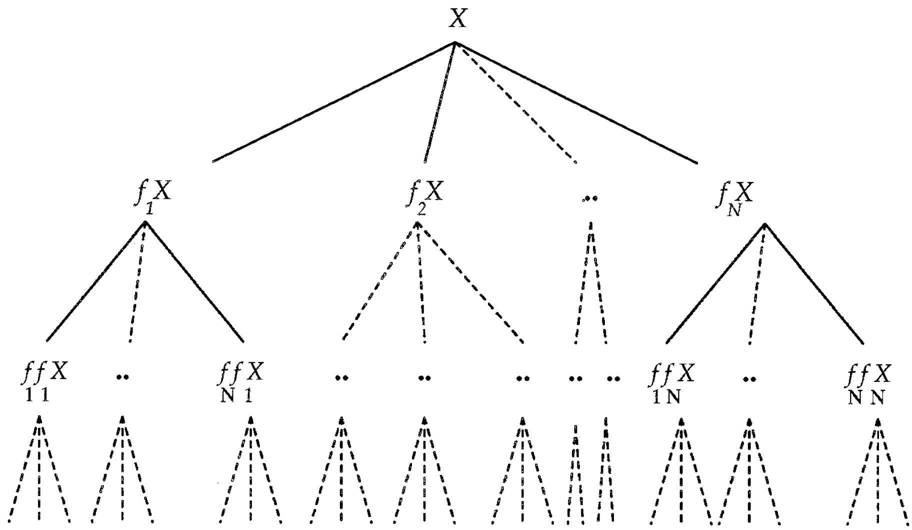

FIG. 2. The action tree.

Conversely, we have the following. Recall that a map $f: X \to X$ is *non-expansive* if $d(f(x), f(y)) \leqslant d(x, y)$ for all $x, y \in X$.

PROPOSITION 2.5. *If each mapping f_i in an IFS is non-expansive and $\lim_{n\to\infty} |f_{i_n} f_{i_{n-1}} \cdots f_{i_1} X| = 0$ for all infinite sequences $i_1 i_2 \ldots \in \Sigma_N^\omega$, then the IFS is weakly hyperbolic.*

Proof. Assume that the IFS is not weakly hyperbolic. Then, by condition (iii) of Proposition 2.2, there exists $\varepsilon > 0$ such that for each $n \geqslant 0$ there is a node $f_{i_n} f_{i_{n-1}} \cdots f_{i_1} X$ on level n of the action tree with diameter at least ε. Since, by assumption, f_{i_n} is non-expansive, it follows that the parent node $f_{i_{n-1}} \cdots f_{i_1} X$ has diameter at least ε. We then have a finitely branching infinite subtree with nodes of diameter at least ε. Therefore, by König's lemma, the action tree has an infinite branch with nodes of diameter at least ε, which gives a contradiction. ∎

PROPOSITION 2.6. *If $\{X; f_1, ..., f_N\}$ is a weakly hyperbolic IFS with non-expansive maps $f_i: X \to X$ $(1 \leqslant i \leqslant N)$ and if $\{X; f_{N+1}, ..., f_M\}$ is a hyperbolic IFS, then $\{X; f_1, ..., f_N, f_{N+1}, ..., f_M\}$ is a weakly hyperbolic IFS.*

Proof. Let $i_1 i_2 \cdots \in \Sigma_M^\omega$. If the set $\{n \geqslant 1 \mid N+1 \leqslant i_n \leqslant M\}$ is infinite, then clearly $\lim_{n\to\infty} |f_{i_1} f_{i_2} \cdots f_{i_n} X| = 0$. If, on the other hand, the above set is finite, then it has a maximum element $m \geqslant 1$ say. Hence for all $n > m$ we have $N+1 \leqslant i_n \leqslant M$ and, therefore, $|f_{i_1} f_{i_2} \cdots f_{i_m} \cdots f_{i_n} X| \leqslant |f_{i_{m+1}} \cdots f_{i_n} X|$ which tends to zero as $n \to \infty$. ∎

By Proposition 2.1, we already know that a weakly hyperbolic IFS has a fixed point given by $A^* = \bigsqcup_{n \geqslant 0} F^n X = \bigcap_{n \geqslant 0} F^n X$. Note that $F^n X$ is the union of the nodes of the IFS tree on level n, and that A^* is the set of lubs of all infinite branches of this tree. Such a set is an example of a *finitely generable subset* of the ω-continuous dcpo $\mathbf{U}X$ as it is obtained from a finitely branching tree of elements of $\mathbf{U}X$. This gives us the motivation to study the *Plotkin power domain* of $\mathbf{U}X$ which can be presented precisely by the set of finitely generable subsets of $\mathbf{U}X$. We will then use the Plotkin power domain to prove the uniqueness of the fixed point of a weakly hyperbolic IFS and deduce its other properties.

2.2. *Finitely Generable Sets*

The following construction of the Plotkin power domain of an ω-continuous dcpo and the subsequent properties are a straightforward generalization of those for an ω-algebraic cpo presented in [35, 36]. Suppose $(D, \sqsubseteq)$ is any ω-continuous dcpo with bottom and $B \subseteq D$ a countable basis for it. Consider any finitely branching tree, whose branches are all infinite and whose nodes are elements of D and each child y of any parent node x satisfies $x \sqsubseteq y$. The set of lubs of all branches of the tree is called a *finitely generable subset* of D. It can be shown that any finitely generable subset of D can also be generated in the above way by a finitely branching tree of elements of the basis B, such that each node is way-below its parents. We denote the set of finitely generable subsets of D by $\mathscr{F}(D)$. It is easily seen that $\mathscr{P}_{\mathrm{f}}(B) \subseteq \mathscr{P}_{\mathrm{f}}(D) \subseteq \mathscr{F}(D)$, where $\mathscr{P}_{\mathrm{f}}(S)$ denotes the set of all finite non-empty subsets of the set S. For $A \in \mathscr{P}_{\mathrm{f}}(B)$ and $C \in \mathscr{F}(D)$, the order $\ll_{\mathrm{EM}}$ is defined by $A \ll_{\mathrm{EM}} C$ iff

$$\forall a \in A \exists c \in C \cdot a \ll c \qquad \text{and} \qquad \forall c \in C \exists a \in A \cdot a \ll c.$$

This induces a pre-order on $\mathscr{F}(D)$ by defining $C_1 \sqsubseteq_{\mathrm{EM}} C_2$ iff for all $A \in \mathscr{P}_{\mathrm{f}}(B)$ whenever $A \ll_{\mathrm{EM}} C_1$ holds we have

$A \ll_{EM} C_2$. Then $(\mathscr{F}(D), \sqsubseteq_{EM})$ becomes an ω-continuous dcpo except that $\sqsubseteq_{EM}$ is a pre-order rather than a partial order. A basis is given by $(\mathscr{P}_f(D), \sqsubseteq_{EM})$ and a countable basis by $(\mathscr{P}_f(B), \sqsubseteq_{EM})$. The *Plotkin power domain* or the *convex power domain* $\mathbf{C}D$ of D is then defined to be the quotient $(\mathscr{F}(D)_{/\cong}, \sqsubseteq_{EM/\cong})$, where the equivalence relation $\cong$ on $\mathscr{F}(D)$ is given by $C_1 \cong C_2$ iff $C_1 \sqsubseteq_{EM} C_2$ and $C_2 \sqsubseteq_{EM} C_1$. If $A \in \mathscr{F}(D)$ and A consists of maximal elements of a bounded complete D, then A will be a maximal element of $(\mathscr{F}(D), \sqsubseteq_{EM})$ and its equivalence class will consist of A only. If D has a bottom element $\bot$, then $(\mathscr{F}(D), \sqsubseteq_{EM})$ has a bottom element, namely $\{\bot\}$, and its equivalence class consists of itself only. Finally, we note that, for any dcpo E, any monotone map $g: \mathscr{P}_f(D) \to E$ has a unique extension to a Scott-continuous map $g: \mathbf{C}D \to E$ which, for convenience, we denote by g.

Now let D be $\mathbf{U}X$ where X is, as before, a compact metric space and $\{X; f_1, ..., f_N\}$ an IFS. Let $F: \mathbf{U}X \to \mathbf{U}X$ be as before and consider the Scott-continuous map $f: \mathbf{CU}X \to \mathbf{CU}X$ which is defined on the basis $\mathscr{P}_f(\mathbf{U}X)$ by the monotone map

$$f: \mathscr{P}_f(\mathbf{U}X) \to \mathbf{CU}X$$
$$\{A_j \mid 1 \leqslant j \leqslant M\} \mapsto \{f_i(A_j) \mid 1 \leqslant j \leqslant M, 1 \leqslant i \leqslant N\}.$$

The set of nodes at level n of the IFS tree is then represented by $f^n\{X\}$. We also consider the Scott-continuous map $\mathsf{U}: \mathbf{CU}X \to \mathbf{U}X$, defined on the above basis by the monotone map

$$\mathsf{U}: \mathscr{P}_f(\mathbf{U}X) \to \mathbf{CU}X$$
$$\{A_j \mid 1 \leqslant j \leqslant M\} \mapsto \bigcup_{1 \leqslant j \leqslant N} A_j.$$

The following properties were shown in [15]; for the sake of completeness, we reiterate them here in the context of our presentation of the Plotkin power domain in terms of finitely generable subsets. The diagram

$$\begin{array}{ccc} \mathbf{U}X & \xleftarrow{\;\mathsf{U}\;} & \mathbf{CU}X \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{U}X & \xleftarrow{\;\mathsf{U}\;} & \mathbf{CU}X \end{array}$$

commutes, which can be easily seen by considering the restriction to the basis $\mathscr{P}_f(\mathbf{U}X)$. It follows that U maps any fixed point of f to a fixed point of F. Moreover, it maps the least fixed point of f to the least fixed point of F, since for each $n \geqslant 0$, $\mathsf{U}f^n\{X\} = F^n\mathsf{U}\{X\} = F^nX$, and, therefore,

$$\mathsf{U} \bigsqcup_{n \geqslant 0} f^n\{X\} = \bigsqcup_{n \geqslant 0} \mathsf{U}f^n\{X\} = \bigsqcup_{n \geqslant 0} F^nX.$$

On the other hand, for $A \in \mathbf{U}X$, let

$$\mathsf{S}(A) = \{s(x) \mid x \in A\} = \{\{x\} \mid x \in A\} \subseteq \mathbf{U}X.$$

It is easy to see that $\mathsf{S}(A)$ is a finitely generable subset of $\mathbf{U}X$. This can be shown for example by constructing a finitely branching tree such that the set of nodes at level $n \geqslant 0$ consists of the closure of open subsets with diameters less than $1/2^n$. It follows that $\mathsf{S}(A)$ is an element of $\mathbf{CU}X$ and, by the above remark, it is a maximal element. Furthermore, the Scott-continuity of f implies that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{U}X & \xrightarrow{\;\mathsf{S}\;} & \mathbf{CU}X \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{U}X & \xrightarrow{\;\mathsf{S}\;} & \mathbf{CU}X \end{array}$$

Therefore, S maps any fixed point of F to a fixed point of f. Note also that S is one-to-one.

PROPOSITION 2.7. *If the IFS $\{X; f_1, ..., f_N\}$ is weakly hyperbolic, then the two maps $F: \mathbf{U}X \to \mathbf{U}X$ and $f: \mathbf{CU}X \to \mathbf{CU}X$ have unique fixed points $A^* = \bigcap_{n \geqslant 0} F^nX$ and $\mathsf{S}A^*$ respectively.*

Proof. For each $n \geqslant 0$, we have

$$f^n\{X\} = \{f_{i_1} f_{i_2} \cdots f_{i_n} X \mid i_1 i_2 \cdots i_n \in \Sigma_N^n\} = \mathsf{S}F^nX.$$

It follows that the least fixed point of f is given by $\bigsqcup_{n \geqslant 0} f^n\{X\} = \{\lim_{n \to \infty} f_{i_1} f_{i_2} \ldots f_{i_n} X \mid i_1 i_2 \cdots \in \Sigma_N^\infty\}$. Since the IFS is weakly hyperbolic, this set consists of singleton sets; in fact we have $\bigsqcup_{n \geqslant 0} f^n\{X\} = \mathsf{S} \bigcap_{n \geqslant 0} F^nX = \mathsf{S}A^*$. However, $\mathsf{S}A^*$ is maximal in $\mathbf{CU}X$, so this least fixed point is indeed the unique fixed point of f. On the other hand, since S is one-to-one and takes any fixed point of F to a fixed point of f, it follows that A^* is the unique fixed point of F. ∎

In order to get the generalization of Eq. (2), we need the following lemma whose straightforward proof is omitted.

LEMMA 2.8. *Let $\{B_i \mid 1 \leqslant i \leqslant M\}$, $\{C_i \mid 1 \leqslant i \leqslant M\}$, $\{D_i \mid 1 \leqslant i \leqslant M\}$ be three finite collections of non-empty compact subset of the metric space X. If $C_i, D_i \subseteq B_i$ and $|B_i| < \varepsilon$ for $1 \leqslant i \leqslant M$, then $d_H(\bigcup_i C_i, \bigcup_i D_i) < \varepsilon$.*

THEOREM 2.9. *If the IFS $\{X; f_1, ..., f_N\}$ is weakly hyperbolic, then the map $F: \mathbf{H}X \to \mathbf{H}X$ has a unique fixed point A^*, the attractor of the IFS. Moreover, for any $A \in \mathbf{H}X$, we have $F^nA \to A^*$ in the Hausdorff metric as $n \to \infty$.*

Proof. Since the set of fixed points of $F: \mathbf{H}X \to \mathbf{H}X$ is precisely the set of fixed points of $F: \mathbf{U}X \to \mathbf{U}X$, the first part follows immediately from Proposition 2.7 and

$A^* = \bigcap_{n \geqslant 0} F^n X$ is indeed the unique fixed point of $F: \mathbf{H}X \to \mathbf{H}X$. Let $A \subseteq X$ be any non-empty compact subset, and let $\varepsilon > 0$ be given. By Proposition 2.2.(iii), there exists $m \geqslant 0$ such that for all $n \geqslant m$ the diameters of all the subsets in the collection $f^n\{X\} = \{f_{i_1} f_{i_2} \cdots f_{i_n} X \mid i_1 i_2 \cdots i_n \in \Sigma_N^n\}$ are less that ε. Clearly, $f_{i_1} f_{i_2} \cdots f_{i_n} A \subseteq f_{i_1} f_{i_2} \cdots f_{i_n} X$ and $A^* \cap f_{i_1} f_{i_2} \cdots f_{i_n} X \subseteq f_{i_1} f_{i_2} \cdots f_{i_n} X$ for all $i_1 i_2 \cdots i_n \in \Sigma_N^n$. Therefore, by the lemma, $d_{\mathrm{H}}(F^n A, A^*) < \varepsilon$. ∎

2.3. *Plotkin Power Domain Algorithm*

Given a weakly hyperbolic IFS $\{X; f_1, ..., f_N\}$, we want to formulate an algorithm to obtain a finite subset A_ε of X which approximates the attractor A^* of the IFS up to a given threshold $\varepsilon > 0$ with respect to the Hausdorff metric.

We will make the assumption that for each node of the IFS tree it is decidable whether or not the diameter of the node is less than ε. For a hyperbolic IFS, we have

$$|f_{i_1} f_{i_2} \cdots f_{i_n} X| \leqslant s_{i_1} s_{i_2} \cdots s_{i_n} |X|,$$

where s_i is the contractivity factor of f_i, and, therefore, the above relation is clearly decidable. However, there are other interesting cases in applications where this relation is also decidable. For example, if $X = [0, 1]^n \subseteq \mathbb{R}^n$ and if, for every $i \in \Sigma_N$, each of the coordinates of the map $f_i: [0, 1]^n \to [0, 1]^n$ is, say, monotonically increasing in each of its arguments, then the diameter of any node is easily computed as

$$|f_{i_1} \cdots f_{i_n} [0, 1]^n| = d(f_{i_1} \cdots f_{i_n}(0, ..., 0), f_{i_1} \cdots f_{i_n}(1, ..., 1)),$$

where d is the Eucleadian distance. It is then clear that the above relation is decidable in this case.

Let $\varepsilon > 0$ be given and fix $x_0 \in X$. We construct a finite subtree of the IFS tree as follows. For any infinite sequence $i_1 i_2 \cdots \in \Sigma_N^\omega$, the sequence $\langle |f_{i_1} f_{i_2} \cdots f_{i_n} X| \rangle_{n \geqslant 0}$ is decreasing and tends to zero, and, therefore, there is a least integer $m \geqslant 0$ such that $|f_{i_1} f_{i_2} \cdots f_{i_m} X| \leqslant \varepsilon$. We truncate the infinite

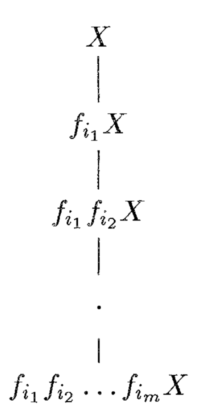

FIG. 3. A branch of the truncated IFS tree.

branch $\langle f_{i_1} f_{i_2} \cdots f_{i_n} X \rangle_{n \geqslant 0}$ of the IFS tree at the node $f_{i_1} f_{i_2} \cdots f_{i_m} X$ which is then a *leaf* of the truncated tree as depicted in Fig. 3, and which contains the distinguished point $f_{i_1} f_{i_2} \cdots f_{i_m} x_0 \in f_{i_1} f_{i_2} \cdots f_{i_m} X$.

By Proposition 2.2, the truncated tree will have finite depth. Let L_ε denote the set of all leaves of this finite tree and let $A_\varepsilon \subseteq X$ be the set of all distinguished points of the leaves. For each leaf $l \in L_\varepsilon$, the attractor satisfies $l \supseteq l \cap A^* \neq \emptyset$ and $A^* = \bigcup_{l \in L_\varepsilon} l \cap A^*$. On the other hand, for each leaf $l \in L_\varepsilon$, we have $l \cap A_\varepsilon \neq \emptyset$ and $A_\varepsilon = \bigcup_{l \in L_\varepsilon} l \cap A_\varepsilon$. It follows, by Lemma 2.8, that $d_{\mathrm{H}}(A_\varepsilon, A^*) \leqslant \varepsilon$. The algorithm therefore traverses the IFS tree in some specific order to obtain the set of leaves L_ε and hence the finite set A_ε which is the required discrete approximation.

For a hyperbolic IFS and for $X = A^*$, this algorithm reduces to that of Hepting *et al.* [24]. We will here obtain an upper bound for the complexity of the algorithm when the maps f_i are contracting affine transformations as this is always the case in image compression. First, we note that there is a simple formula for the contractivity of an affine map. In fact, suppose the map $f: \mathbb{R}^2 \to \mathbb{R}^2$ is given at the point $z \in \mathbb{R}^2$ in matrix notation by $z \mapsto Wz + t$, where the 2×2 matrix W is the linear part and $t \in \mathbb{R}^2$ is the translation part of f. Then, the infimum of numbers c with

$$|f(z) - f(z')| \leqslant c\, |z - z'|$$

is the greatest eigenvalue (in absolute value) of the matrix $W^{\mathrm{t}} W$, where W^{t} is the transpose of W [12]. This greatest eigenvalue is easily calculated for the matrix

$$W = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

to be given by

$$\sqrt{\alpha + \beta + \sqrt{(\alpha - \beta)^2 + \gamma^2}},$$

where $\alpha = (a^2 + c^2)/2$, $\beta = (b^2 + d^2)/2$, and $\gamma = ab + cd$. If f is contracting then this number is strictly less than one and is the contractivity of f. While traversing the tree, the algorithm recursively computes $f_{i_1} f_{i_2} \cdots f_{i_n} x_0$ and $s_{i_1} s_{i_2} \cdots s_{i_n} |X|$, and if $s_{i_1} s_{i_2} \cdots s_{i_n} |X| \leqslant \varepsilon$, then the point $f_{i_1} f_{i_2} \cdots f_{i_n} x_0$ is taken to belong to A_ε. An upper bound for the height of the truncated tree is obtained as follows. We have $s_{i_1} s_{i_2} \cdots s_{i_n} \leqslant s^n$, where $s = \max_{1 \leqslant i \leqslant N} s_i < 1$ is the contractivity of the IFS. Therefore the least integer h with $s^h |X| \leqslant \varepsilon$ is an upper bound, i.e., $h = \lceil \log(\varepsilon / |X|) / \log s \rceil$, where $\lceil a \rceil$ is the least non-negative integer greater than or equal to a. A simple counting shows that there are at most nine arithmetic computations at each node. Therefore, the total number of computations is at most $9(N + N^2 + N^3 + \cdots + N^h) = 9(N^{h+1} - 1)/(N - 1)$, which is $O(N^h)$.

In order to have a similar complexity analysis for the generation of the attractor of a weakly hyperbolic IFS, one needs information on the rate of convergence of $\lim_{n\to\infty} |f_{i_1} f_{i_2} \cdots f_{i_n} X| = 0$ for all sequences $i_1 i_2 \cdots \in \Sigma_N^\omega$. In fact, if there is a uniform constructive rate of convergence, i.e., if for all positive integers m, there is an integer $n = n(m)$, explicitly given in terms of m, such that $|f_{i_1} f_{i_2} \cdots f_{i_n} X| \leqslant 1/m$ for all sequences $i_1 i_2 \cdots \in \Sigma_N^\omega$, then one can easily obtain a complexity result similar to the case of hyperbolic IFSs.

Next we consider the problem of plotting, on the computer screen, the discrete approximation to the attractor of a weakly hyperbolic IFS in $\mathbb{R}^2$. The digitization of the discrete approximation A_ε inevitably produces a further error in approximating the attractor A^*. We will obtain a bound for this error. Suppose we have a weakly hyperbolic IFS $\{X; f_1, ..., f_N\}$ with $X \subseteq \mathbb{R}^2$. By a translation of origin and rescaling if necessary, we can assume that $X \subseteq [0, 1] \times [0, 1]$. Suppose, furthermore, that the computer screen, with resolution $r \times r$, is represented by the unit square $[0, 1] \times [0, 1]$ digitised into a two-dimensional array of $r \times r$ pixels. We regard each pixel as a point so that the distance between nearest pixels is given by $\delta = 1/(r-1)$. We assume that for each point in A_ε the nearest pixel is plotted on the screen. Let A_ε^δ be the set of pixels plotted. Since any point in $[0, 1] \times [0, 1]$ is at most $\sqrt{2}\,\delta/2$ from its nearest pixel, it is easy to see that $d_{\mathrm{H}}(A_\varepsilon^\delta, A_\varepsilon) \leqslant \sqrt{2}\,\delta/2$. It follows that

$$d_{\mathrm{H}}(A_\varepsilon^\delta, A^*) \leqslant d_{\mathrm{H}}(A_\varepsilon^\delta, A_\varepsilon) + d_{\mathrm{H}}(A_\varepsilon, A^*) \leqslant \frac{\sqrt{2}}{2}\,\delta + \varepsilon.$$

In the worst case, the error in the digitization process, for a given resolution $r \times r$ of the screen, is at least $\sqrt{2}\,\delta/2 = \delta/\sqrt{2}$ (i.e., δ divided by the diameter of the screen $[0, 1] \times [0, 1]$) whatever the value of the discrete threshold $\varepsilon > 0$. On the other hand, even in the case of a hyperbolic IFS, the complexity of the algorithm grows as $N^{-\log \varepsilon}$ as $\varepsilon \to 0$. In practice, the optimal balance between accuracy and complexity is reached by taking ε to be of the order of $\delta = 1/(r-1)$.

3. INVARIANT MEASURE OF AN IFS WITH PROBABILITIES

We prove the existence and uniqueness of the invariant measure of a weakly hyperbolic IFS with probabilities by generalizing the corresponding result for an hyperbolic IFS in [15] which is based on the normalised probabilistic power domain. We first recall the basic definitions.

3.1. *Probabilistic Power Domain*

A *valuation* on topological space Y is map $\nu: \Omega(Y) \to [0, \infty)$ which satisfies:

(i) $\nu(a) + \nu(b) = \nu(a \cup b) + \nu(a \cap b)$,
(ii) $\nu(\varnothing) = 0$, and
(iii) $a \subseteq b \Rightarrow \nu(a) \leqslant \nu(b)$.

A *continuous* valuation [31, 27, 26] is a valuation such that whenever $A \subseteq \Omega(Y)$ is a directed set (wrt $\subseteq$) of open sets of Y, then

$$\nu\Big(\bigcup_{O \in A} O\Big) = \sup_{O \in A} \nu(O).$$

For any $b \in Y$, the *point valuation* based at b is the valuation $\delta_b: \Omega(Y) \to [0, \infty)$ defined by

$$\delta_b(O) = \begin{cases} 1, & \text{if} \quad b \in O, \\ 0, & \text{otherwise.} \end{cases}$$

Any finite linear combination

$$\sum_{i=1}^{n} r_i\, \delta_{b_i}$$

of point valuations δ_{b_i} with constant coefficients $r_i \in [0, \infty)$, $(1 \leqslant i \leqslant n)$, is a continuous valuation on Y; we call it a *simple valuation.*

The *normalised probabilistic power domain*, $\mathbf{P}^1 Y$, of a topological space Y consists of the set of continuous valuations ν on Y with $\nu(Y) = 1$ and is ordered as follows:

$$\mu \sqsubseteq \nu \text{ iff for all open sets } O \text{ of } Y, \mu(O) \leqslant \nu(O).$$

The partial order $(\mathbf{P}^1 Y, \sqsubseteq)$ is a dcpo with bottom in which the lub of a directed set $\langle \mu_i \rangle_{i \in I}$ is given by $\bigsqcup_i \mu_i = \nu$, where for $O \in \Omega(Y)$ we have

$$\nu(O) = \sup_{i \in I} \mu_i(O).$$

Moreover, if Y is an ω-continuous dcpo with a bottom element $\bot$, then $\mathbf{P}^1 Y$ is also an ω-continuous dcpo with a bottom element $\delta_\bot$ and has a basis consisting of simple valuations [27, 26, 16]. Therefore, any $\mu \in \mathbf{P}^1 Y$ is the lub of an ω-chain of normalised simple valuations and, hence by a lemma of Saheb-Djahromi [34] can be uniquely extended to a Borel measure on Y which we denote for convenience by μ as well [34, p. 24]. For $0 \leqslant c \leqslant 1$, let $\mathbf{P}^c Y$ denote the dcpo of valuations with total mass c, i.e., $\mathbf{P}^c Y = \{\mu \in \mathbf{P} Y \mid \mu(Y) = c\}$. Since $\mathbf{P}^c Y$ is obtained from $\mathbf{P}^1 Y$ by a simple rescaling, it shares the above properties of $\mathbf{P}^1 Y$; the case $c = 0$ is, of course, trivial.

For two simple valuations

$$\mu_1 = \sum_{b \in B} r_b\, \delta_b \qquad \mu_2 = \sum_{c \in C} s_c\, \delta_c$$

in $\mathbf{P}^1 Y$, where $B, C \in \mathscr{P}_f(Y)$, we have by the *splitting lemma* [27, 16]: $\mu_1 \sqsubseteq \mu_2$ iff, for all $b \in B$ and all $c \in C$, there exists a non-negative number $t_{b,c}$ such that

$$\sum_{c \in C} t_{b,c} = r_b \qquad \sum_{b \in B} t_{b,c} = s_c \tag{5}$$

and $t_{b,c} \neq 0$ implies $b \sqsubseteq c$. We can consider any $b \in B$ as a source with mass r_b, any $c \in C$ as a sink with mass s_c, and the number t_{bc} as the flow of mass from b to c. Then, the above property can be regarded as conservation of total mass.

3.2. *Model for IFS with Probabilities*

Now let X be a compact metric space so that $(\mathbf{U}X, \supseteq)$ is an ω-continuous dcpo with bottom X. Therefore, $\mathbf{P}^1\mathbf{U}X$ is an ω-continuous dcpo with bottom δ_X. Recall that the singleton map $s\colon X \to \mathbf{U}X$ with $s(x) = \{x\}$ embeds X onto the set $s(X)$ of maximal elements of $\mathbf{U}X$. For any open subset $a \subseteq X$, the set $s(a) = \{\{x\} \mid x \in a\} \subseteq \mathbf{U}X$ is a G_δ subset and, hence, a Borel subset [15, Corollary 5.9]. A valuation $\mu \in \mathbf{P}^1\mathbf{U}X$ is said to be *supported* in $s(X)$ if $\mu(\mathbf{U}X \backslash s(X)) = 0$. If μ is supported in $s(X)$, then the *support* of μ is the set of points $y \in s(X)$ such that $\mu(O) > 0$ for any Scott-neighbourhood $O \subseteq \mathbf{U}X$ of y. Any element of $\mathbf{P}^1\mathbf{U}X$ which is supported in $s(X)$ is a maximal element of $\mathbf{P}^1\mathbf{U}X$ [15, Proposition 5.18]; we denote the set of all valuations which are supported in $s(X)$ by $\mathbf{S}^1X$. We can identify $\mathbf{S}^1X$ with the set $\mathbf{M}^1X$ of normalised Borel measures on X as follows. Let

$$e\colon \mathbf{M}^1X \to \mathbf{S}^1X$$
$$\mu \mapsto \mu \circ s^{-1}$$

and

$$j\colon \mathbf{S}^1X \to \mathbf{M}^1X$$
$$\nu \mapsto \nu \circ s.$$

THEOREM 3.1. [15, Theorem 5.21]. *The maps e and j are well-defined and induce an isomorphism between* $\mathbf{S}^1X$ *and* $\mathbf{M}^1X$.

For $\mu \in \mathbf{M}^1X$ and an open subset $a \subseteq X$,

$$\mu(a) = e(\mu)(s(a)) = e(\mu)(\Box a). \tag{6}$$

Let $\{X; f_1, ..., f_N; p_1, ..., p_N\}$ be an IFS with probabilities on the compact metric space X. Define

$$H\colon \mathbf{P}^1\mathbf{U}X \to \mathbf{P}^1\mathbf{U}X$$
$$\mu \mapsto H(\mu)$$

by $H(\mu)(O) = \sum_{i=1}^N p_i \mu(f_i^{-1}(O))$. Note that H is defined in the same way as the Markov operator T in Eq. (3). Then, H is Scott-continuous and has, therefore, a least fixed point given by $\nu^* = \bigsqcup_m H^m \delta_X$, where

$$H^m \delta_X = \sum_{i_1, i_2, ..., i_m = 1}^{N} p_{i_1} p_{i_2} \cdots p_{i_m} \delta_{f_{i_1} f_{i_2} \cdots f_{i_m} X}. \tag{7}$$

Furthermore, we have:

THEOREM 3.2. *For a weakly hyperbolic IFS, the least fixed point* ν^* *of H is in* $\mathbf{S}^1X$. *Hence, it is a maximal element of* $\mathbf{P}^1\mathbf{U}X$ *and therefore the unique fixed point of H. The support of* ν^* *is given by* $\mathsf{S}A^* = \{\{x\} \mid x \in A^*\}$ *where* $A^* \subseteq X$ *is the attractor of the IFS.*

Proof. To show that $\nu^* \in \mathbf{S}^1X$, it is sufficient to show that $\nu^*(s(X)) = 1$. For each integer $k \geqslant 1$, let $\langle b_i \rangle_{i \in I_k}$ be the collection of all open balls $b_i \subseteq X$ of radius less than $1/k$. Let $O_k = \bigcup_{i \in I_k} \Box b_i$. Then $\langle O_k \rangle_{k \geqslant 1}$ is a decreasing sequence of open subsets of $\mathbf{U}X$ and $s(X) = \bigcap_{k \geqslant 1} O_k$. Therefore, $\nu^*(s(X)) = \inf_{k \geqslant 1} \nu^*(O_k)$. By Proposition 2.2(iii), for each $k \geqslant 1$ there exists some integer $n \geqslant 0$ such that all the nodes of the IFS tree on level n have diameter strictly less than $1/k$. Hence, for all finite sequences $i_1 \cdots i_m \in \Sigma_N^m$ with $m \geqslant n$, we have $f_{i_1} f_{i_2} \cdots f_{i_m} X \in O_k$. Therefore, for all $m \geqslant n$,

$$(H^m \delta_X)(O_k) = \sum_{f_{i_1} f_{i_2} \cdots f_{i_m} X \in O_k} p_{i_1} p_{i_2} \cdots p_{i_m} = 1.$$

It follows that $\nu^*(O_k) = \sup_{m \geqslant 0} (H^m \delta_X)(O_k) = 1$, and, therefore, $\nu^*(s(X)) = \inf_{k \geqslant 1} \nu^*(O_k) = 1$, as required. To show that $\mathsf{S}A^*$ is the support of ν^*, let $x \in A^*$ and, for any integer $k \geqslant 1$, let $B_k(x) \subseteq X$ be the open ball of radius $1/k$ centred at x. Then $\{x\}$ is the lub of some infinite branch of the IFS tree: $\{x\} = \bigcap_{n \geqslant 0} f_{i_1} \cdots f_{i_n} X$ for some $i_1 i_2 \cdots \in \Sigma_N^\omega$. As in the above, let $n \geqslant 0$ be such that the diameters of all nodes of the IFS tree on level n are strictly less than $1/k$. Then,

$$\begin{aligned} \nu^*(\Box B_k(x)) &= \sup_{m \geqslant 0} (H^m \delta_X)(\Box B_k(x)) \\ &\geqslant (H^n \delta_X)(\Box B_k(x)) \geqslant p_{i_1} \cdots p_{i_n} > 0. \end{aligned}$$

Since $\langle \Box B_k(x) \rangle_{k \geqslant 1}$ is a neighbourhood basis of $\{x\}$ in $\mathbf{U}X$, it follows that $\{x\}$ is in the support of ν^*. On the other hand, if $x \notin A^*$, there is an open ball $B_\delta(x) \subseteq X$ which does not intersect A^*. Let $n \geqslant 0$ be such that the nodes on level n of the IFS tree have diameters strictly less than δ. Then, for all $m \geqslant n$, we have $(H^m \delta_X)(B_\delta(x)) = 0$ and it follows that

$$\nu^*(B_\delta(x)) = \sup_{m \geqslant 0} (B_\delta(x)) = 0,$$

and $\{x\}$ is not in the support of ν^*. ∎

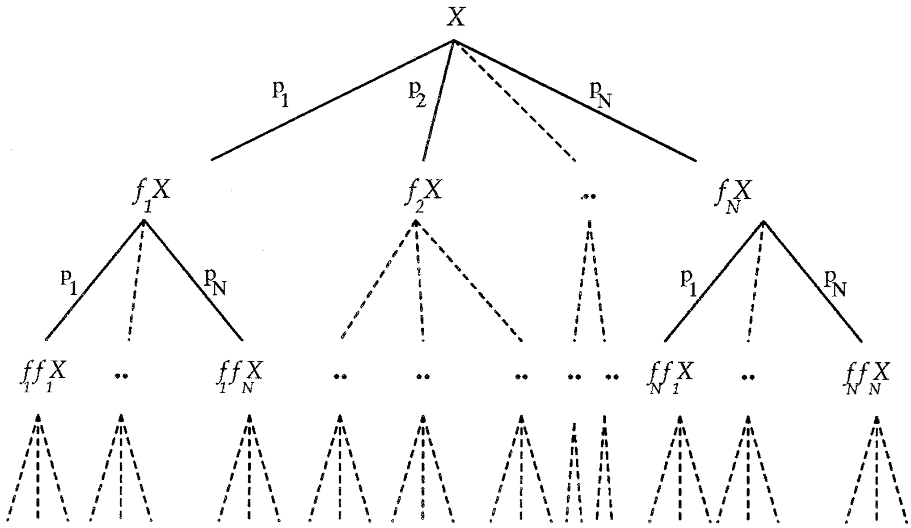

FIG. 4. The IFS tree with transitional probabilities.

COROLLARY 3.3. *For a weakly hyperbolic IFS, the normalised measure* $\mu^* = j(\nu^*) \in \mathbf{M}^1 X$ *is the unique fixed point of the Markov operator* $T: \mathbf{M}^1 X \to \mathbf{M}^1 X$. *Its support is the unique attractor* A^* *of the IFS.*

3.3. *Probabilistic Power Domain Algorithm*

Since the Plotkin power domain algorithm in Section 2.3 provides a digitised discrete approximation A_ε^δ to the attractor A^*, the question is how to render the pixels in A_ε^δ to obtain an approximation to the invariant measure μ^*. We now describe an algorithm to do this, which extends that of Hepting *et al.* for a hyperbolic IFS with probabilities [24]. Assume again that the unit square represents the digitised screen with $r \times r$ pixels. Suppose $\{X; f_1, ..., f_N; p_1, ..., p_N\}$ is a weakly hyperbolic IFS with $X \subseteq [0, 1] \times [0, 1]$ and $\varepsilon > 0$ is the discrete threshold. Fix $x_0 \in X$. The simple valuation $H^m \delta_X$ of Eq. (7) can be depicted by the mth level of the IFS tree labelled with *transitional probabilities* as in Fig. 4.

The root X of the tree has mass one and represents δ_X. Any edge going from a node $t(X)$, where $t = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$ is a finite composition of the maps f_i, to its child $t(f_i(X))$ is labelled with transitional probability p_i for $i = 1, ..., N$. The transitional probability label on each edge gives the flow of mass from the parent node (source) to the child node (sink) in the sense of Eq. (5) in the splitting lemma. The total mass of the node $f_{i_1} f_{i_2} \cdots f_{i_m} X$ on level m is, therefore, the product $p_{i_1} p_{i_2} \cdots p_{i_m}$ of the labels of all the edges leading from the root to the node, in agreement with the expansion of $H^m \delta_X$ in Eq. (7). We again make the assumption that it is decidable that the diameter of any node is less than ε or not. The algorithm then proceeds, as in the deterministic case, to find all the leaves of the IFS tree and, this time, computes the mass of each leaf. The set of all weighted leaves of the truncated IFS tree represents a simple valuation which is a discrete approximation to the invariant measure μ^*. Then the total mass given to each pixel in A_ε^δ is the sum of the masses of all leaves corresponding to that pixel.

In the hyperbolic case, the probabilistic algorithm traverses the finite tree and recursively computes $f_{i_1} f_{i_2} \cdots f_{i_n} x_0$, $p_{i_1} p_{i_2} \cdots p_{i_n}$ and $s_{i_1} s_{i_2} \cdots s_{i_n} |X|$, and if $s_{i_1} s_{i_2} \cdots s_{i_n} |X| \leqslant \varepsilon$, then the weight of the pixel for $f_{i_1} f_{i_2} \cdots f_{i_n} x_0$ is incremented by $p_{i_1} p_{i_2} \cdots p_{i_n}$. A simple counting shows that this takes at most 10 arithmetic computations at each node. Therefore, the total number of computations is at most $10(N + N^2 + N^3 + \cdots + N^h)$, which is $O(N^h)$ as before.

4. A MODEL FOR RECURRENT IFS

In this section, we will construct a domain-theoretic model for weakly hyperbolic recurrent IFSs. Assume that $\{X; f_1, ..., f_N\}$ is an IFS and (p_{ij}) $(1 \leqslant i, j \leqslant N)$ is an indecomposable row-stochastic matrix. Then $\{X; f_j; p_{ij}; i, j = 1, 2, ..., N\}$ is a recurrent IFS. We will see below that this gives rise to a Markov chain on the coproduct of N copies of X. (See [20] for an introduction to Markov chains.)

For a topological space Y, we let $\bar{Y} = \sum_{j=1}^N Y \times \{j\}$ denote the coproduct (disjoint sum) of N copies of Y [37], i.e.,

$$\bar{Y} = \sum_{j=1}^{N} Y \times \{j\} = \{(y, j) \mid y \in Y, 1 \leqslant j \leqslant N\},$$

with its frame of open sets given by $\Omega(\bar{Y}) = (\Omega(Y))^N$, and its Borel subsets by $\mathscr{B}(\bar{Y}) = (\mathscr{B}(Y))^N$, where $\mathscr{B}(Y)$ is the set of Borel subsets of Y.

Any normalised Borel measure $\bar{\mu} \in \mathbf{M}^1 \bar{Y}$ is a mapping

$$\bar{\mu}: \mathscr{B}(Y)^N \to [0, 1]$$

which can be written as $\bar{\mu} = (\mu_j)_j = (\mu_1, \mu_2, ..., \mu_N)$ with $\mu_j \in \mathbf{M}^{c_j} Y$ for some c_j, $(0 \leqslant c_j \leqslant 1$ and $\sum_{j=1}^N c_j = 1)$, such that for $\bar{B} = (B_j)_j = (B_1, B_2, ..., B_N) \in (\mathscr{B}(Y))^N$ we have $\bar{\mu}(\bar{B}) = \sum_{j=1}^N \mu_j(B_j)$.

4.1. *The Generalised Markov Operator*

A recurrent IFS induces a Markov process on $\bar{X} = \sum_{j=1}^N X \times \{j\}$ as follows [4]. Let $i_0, i_1, i_2, ...$ be a Markov chain on $\{1, 2, ..., N\}$ with transition probability matrix (p_{ij}). Let $x_0 \in X$ and consider the process

$$Z_0 = x_0 \qquad Z_n = f_{i_n} Z_{n-1}$$

which gives us the random walk described in Subsection 1.3. Then (Z_n, i_n) is a Markov process on $\bar{X} = \sum_{j=1}^N X \times \{j\}$ with the Markov transition probability function

$$K((x, i), \bar{B}) = \sum_{j=1}^N p_{ij} \chi_{\bar{B}}(f_j x, j)$$

which is the probability of transition from (x, i) into the Borel set $\bar{B} \subseteq \bar{X}$ (here $\chi_{\bar{B}}$ is the characteristic function of the set $\bar{B}$). This transitional probability induces the generalised Markov operator defined by

$$\bar{T}: \mathbf{M}^1 \bar{X} \to \mathbf{M}^1 \bar{X}$$

$$\bar{\mu} \mapsto \bar{T}(\bar{\mu})$$

with

$$\begin{aligned}\bar{T}(\bar{\mu})(\bar{B}) &= \int_{\bar{X}} K((x, i), \bar{B})\, d\bar{\mu} = \int_{\bar{X}} \sum_{j=1}^N p_{ij} \chi_{\bar{B}}(f_j x, j)\, d\bar{\mu} \\ &= \sum_{i=1}^N \int_X \sum_{j=1}^N p_{ij} \chi_{B_j}(f_j x)\, d\mu_i \\ &= \sum_{i=1}^N \sum_{j=1}^N p_{ij} \int_X \chi_{B_j}(f_j x)\, d\mu_i \\ &= \sum_{j=1}^N \sum_{i=1}^N p_{ij} \mu_i(f_j^{-1} B_j).\end{aligned}$$

In other words,

$$(\bar{T}(\bar{\mu}))_j = \sum_{i=1}^N p_{ij} \mu_i \circ f_j^{-1}. \tag{8}$$

Note that $\bar{T}$ is well-defined since

$$\begin{aligned}\bar{T}(\bar{\mu})\, \bar{X} &= \sum_{j=1}^N \sum_{i=1}^N p_{ij} \mu_i(f_j^{-1} X) = \sum_{i=1}^N \sum_{j=1}^N p_{ij} \mu_i(X) \\ &= \sum_{i=1}^N \mu_i(X) = 1\end{aligned}$$

as $\sum_{j=1}^N p_{ij} = 1$ for $1 \leqslant i \leqslant N$. For a hyperbolic recurrent IFS, Barnsley defines the generalised Hutchinson metric $\bar{r}_{\mathrm{H}}$ on $\mathbf{M}^1 \bar{X}$ by

$$\bar{r}_{\mathrm{H}}(\bar{\mu}, \bar{\nu}) = \sup \left\{ \sum_{i=1}^N \left(\int_X f_i\, d\mu_i - \int_X f_i\, d\nu_i \right) \middle| f_i: X \to \mathbb{R}, \right.$$
$$\left. |f_i(x) - f_i(y)| \leqslant d(x, y),\ 1 \leqslant i \leqslant N \right\}$$

and then states in [1, p. 406] that one expects the generalised Markov operator to be a contracting map and therefore to have a unique fixed point. However, he notes that the contractivity factor will depend not only on the contractivity of the IFS but also on the matrix (p_{ij}). Nevertheless, no proof is given that $\bar{T}$ is indeed contracting for a given (p_{ij}); subsequently, the existence and uniqueness of a fixed point is not verified. On the other hand, it is shown in [4, Theorem 2.1] by proving the convergence in distribution of the expected value of real-valued functions on $\bar{X}$ that a hyperbolic recurrent IFS does have a unique stationary distribution. We will show here more generally that for a weakly hyperbolic recurrent IFS the generalised Markov operator has indeed a unique fixed point.

4.2. *The Unique Fixed Point of the Markov Operator*

We will achieve the above task, without any need for a metric, by extending the generalised Markov operator to $\mathbf{P}\overline{\mathbf{U}X}$ where $\overline{\mathbf{U}X} = \sum_{j=1}^N (\mathbf{U}X) \times \{j\}$ is the coproduct of N copies of $\mathbf{U}X$.

If Y is a topological space, a valuation $\mu \in \mathbf{P}^1 \bar{Y}$ is a mapping

$$\bar{\nu}: \Omega(\bar{Y}) \to [0, 1]$$

which can be written as $\bar{\nu} = (\nu_j)_j = (\nu_1, ..., \nu_N)$ with $\nu_j \in \mathbf{P}^{c_j} Y$ for some c_j $(0 \leqslant c \leqslant 1$ and $\sum_{j=1}^N c_j = 1)$, such that for $\bar{O} = (O_j)_j = (O_1, O_2, ..., O_N) \in (\Omega(Y))^N$ we have $\bar{\nu}(\bar{O}) = \sum_{j=1}^N \nu_j(O_j)$ [26, p. 90]. We will work in a subdcpo of $\mathbf{P}^1\overline{\mathbf{U}X}$ which is defined below.

Note that our assumptions imply that (p_{ij}) is the transitional matrix of an ergodic finite Markov chain, and therefore, we have:

PROPOSITION 4.1 [28, p. 100]. *There exists a unique probability vector* (m_j) *with* $m_j > 0$ $(1 \leqslant j \leqslant N)$ *and* $\sum_{j=1}^N m_j = 1$ *which satisfies* $m_j = \sum_{i=1}^N m_i p_{ij}$.

Let $\overline{\nu_0} \in \mathbf{P}^1\overline{\mathbf{U}X}$ be given by $\overline{\nu_0} = (m_1\,\delta_X, m_2\,\delta_X, ..., m_N\,\delta_X)$ where m_j $(1 \leqslant j \leqslant N)$ is the unique probability vector in Proposition 4.1. Put

$$\mathbf{P}_0^1\overline{\mathbf{U}X} = \{\bar{\nu} \in \mathbf{P}^1\overline{\mathbf{U}X} \mid \overline{\nu_0} \sqsubseteq \bar{\nu}\}.$$

Note that for $\bar{\nu} = (\nu_1, \nu_2, ..., \nu_N) \in \mathbf{P}\overline{\mathbf{U}X}$ we have $\bar{\nu} \in \mathbf{P}_0^1\overline{\mathbf{U}X}$ iff $\nu_j(\mathbf{U}X) = m_j$ for $1 \leqslant j \leqslant N$, since $\overline{\nu_0} \sqsubseteq \bar{\nu}$ iff $m_j \leqslant \nu_j(\mathbf{U}X)$ and we have

$$1 = \sum_{j=1}^{N} m_j \leqslant \sum_{j=1}^{N} \nu_j(\mathbf{U}X) = \bar{\nu}(\overline{\mathbf{U}X}) \leqslant 1,$$

which implies $\nu_j(\mathbf{U}X) = m_j$. It also follows that $\mathbf{P}_0^1\overline{\mathbf{U}X} = \prod_{j=1}^{N} \mathbf{P}^{m_j}(\mathbf{U}X)$. Therefore, $\mathbf{P}_0^1\overline{\mathbf{U}X}$ is an ω-continuous dcpo with bottom $\overline{\nu_0}$, and any $(\nu_j)_j \in \mathbf{P}_0^1\overline{\mathbf{U}X}$ extends uniquely to a Borel measure on $\mathbf{P}_0^1\overline{\mathbf{U}X}$ as each ν_j extends uniquely to a Borel measure on $\mathbf{P}_0^1\mathbf{U}X$.

Let

$$\bar{s}: \bar{X} \to \overline{\mathbf{U}X}$$
$$(x, j) \mapsto (\{x\}, j)$$

be the embedding of $\bar{X}$ onto the set of maximal elements of $\overline{\mathbf{U}X}$. Any Borel subset $\bar{B} = (B_j)_j$ of $\bar{X}$ induces a Borel subset $\bar{s}(\bar{B}) = (s(B_j))_j$ of $\overline{\mathbf{U}X}$ since each $s(B_j)$ is a Borel subset of $\mathbf{U}X$. Let $\mathbf{M}_0^1\bar{X} = \{(\nu_j)_j \in \mathbf{M}^1\bar{X} \mid \nu_j(X) = m_j,\ 1 \leqslant j \leqslant N\}$, and let

$$\mathbf{S}_0^1\overline{\mathbf{U}X} = \{\bar{\nu} \in \mathbf{P}_0^1\overline{\mathbf{U}X} \mid \bar{\nu}(\bar{s}\bar{X}) = 1\}$$

and define the two maps

$$\bar{e}: \mathbf{M}_0^1\bar{X} \to \mathbf{S}_0^1\overline{\mathbf{U}X} \quad \text{and} \quad \bar{j}: \mathbf{S}_0^1\overline{\mathbf{U}X} \to \mathbf{M}_0^1\bar{X}$$
$$\bar{\mu} \mapsto \bar{\mu} \circ \bar{s}^{-1} \qquad\qquad \bar{\nu} \mapsto \bar{\nu} \circ \bar{s}.$$

We then have the following generalisation of Theorem 3.1.

THEOREM 4.2. *The two maps $\bar{e}$ and $\bar{j}$ are well-defined and give an isomorphism between $\mathbf{M}_0^1\bar{X}$ and $\mathbf{S}_0^1\overline{\mathbf{U}X}$.*

Given a recurrent IFS $\{X; f_i; p_{ij}; i, j = 1, ..., N\}$ we extend the generalised Markov operator on $\mathbf{P}^1\overline{\mathbf{U}X}$ by

$$\bar{H}: \mathbf{P}^1\overline{\mathbf{U}X} \to \mathbf{P}^1\overline{\mathbf{U}X}$$
$$\bar{\nu} \mapsto \bar{H}(\bar{\nu}),$$

where $\bar{H}(\bar{\nu})(\bar{O}) = \sum_{j=1}^{N}\sum_{i=1}^{N} p_{ij}\nu_i(f_j^{-1}O_j)$; in other words, we have $(\bar{H}(\bar{\nu}))_j = \sum_{i=1}^{N} p_{ij}\nu_i \circ f_j^{-1}$ as in the definition of $\bar{T}$ in Eq. (8). If $\bar{\nu} \in \mathbf{P}_0^1\overline{\mathbf{U}X}$, then $\bar{H}(\bar{\nu}) \in \mathbf{P}_0^1\overline{\mathbf{U}X}$, since

$$(\bar{H}(\bar{\nu}))_j(\mathbf{U}X) = \sum_{i=1}^{N} p_{ij}\nu_i \circ f_j^{-1}(\mathbf{U}X) = \sum_{i=1}^{N} p_{ij}\nu_i(\mathbf{U}X) = \sum_{i=1}^{N} p_{ij}m_i = m_j.$$

PROPOSITION 4.3. *Any fixed point of $\bar{H}$ (respectively $\bar{T}$) is in $\mathbf{P}_0^1\overline{\mathbf{U}X}$ (respectively $\mathbf{M}_0^1\bar{X}$).*

Proof. Let $\bar{\nu} = (\nu_1, \nu_2, ..., \nu_N) \in \mathbf{P}^1\overline{\mathbf{U}X}$ be a fixed point of $\bar{H}$. Then, for each $j \in \{1, 2, ..., N\}$ we have

$$\nu_j(\mathbf{U}X) = (\bar{H}\bar{\nu})_j(\mathbf{U}X) = \sum_{i=1}^{N} p_{ij}\nu_i(f_j^{-1}(\mathbf{U}X)) = \sum_{i=1}^{N} p_{ij}\nu_i(\mathbf{U}X).$$

By Proposition 4.1, we have $\nu_j(\mathbf{U}X) = m_j$, as required. The proof for $\bar{T}$ is similar. ∎

The following lemma shows that, for any recurrent IFS, the generalised Markov operator has a least fixed point.

LEMMA 4.4. *The mapping $\bar{H}: \mathbf{P}_0^1\overline{\mathbf{U}X} \to \mathbf{P}_0^1\overline{\mathbf{U}X}$ is Scott-continuous.*

Proof. It is immediately seen from the definition that $\bar{H}$ is monotone. Let $\langle \bar{\nu}^k \rangle_{k \geqslant 0}$ be an increasing chain in $\mathbf{P}_0^1\overline{\mathbf{U}X}$. Then, for any $\bar{O} = (O_j)_j \in \Omega\overline{\mathbf{U}X}$, we have

$$\begin{aligned}\left(\bar{H} \bigsqcup_k \bar{\nu}^k\right)(\bar{O}) &= \sum_{j=1}^{N}\sum_{i=1}^{N} p_{ij}\left(\bigsqcup_k \nu_i^k\right)(f_j^{-1}O_j)\\ &= \sum_{j=1}^{N}\sum_{i=1}^{N} p_{ij}\sup_k \nu_i^k(f_j^{-1}O_j)\\ &= \sup_k \sum_{j=1}^{N}\sum_{i=1}^{N} p_{ij}\nu_i^k(f_j^{-1}O_j)\\ &= \sup_k (\bar{H}\bar{\nu}^k)(\bar{O})\\ &= \bigsqcup_k (\bar{H}\bar{\nu}^k)(\bar{O}).\end{aligned}$$

The Scott-continuity of $\bar{H}$ follows. ∎

Let us find an explicit formula for the least fixed point $\bar{\nu}^* = \bigsqcup_n \bar{H}^n(\overline{\nu_0})$ of $\bar{H}$. It is convenient to use the inverse transitional probability matrix [20, p. 414] (q_{ij}) which is defined as follows:

$$q_{ij} = \frac{m_j}{m_i} p_{ji}. \tag{9}$$

Note that by Proposition 4.1, $m_j > 0$ for $1 \leqslant j \leqslant N$ and therefore (q_{ij}) is well-defined; it is again row-stochastic, irreducible, and satisfies $\sum_{i=1}^{N} m_i q_{ij} = m_j$ for $j = 1, ..., N$. We can now show by induction that

$$(\bar{H}^n\overline{\nu_0})_j = \sum_{i_1, i_2, ..., i_{n-1}=1}^{N} m_j q_{ji_1} q_{i_1 i_2} \cdots q_{i_{n-2}i_{n-1}} \times \delta_{f_j f_{i_1} f_{i_2} \cdots f_{i_{n-1}} X}. \tag{10}$$

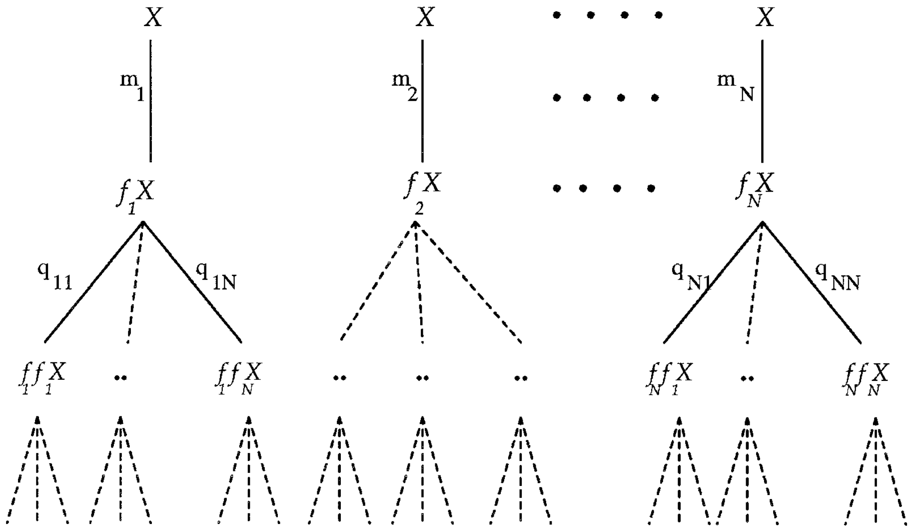

FIG. 5. The recurrent IFS tree with tansitional probabilities.

In fact,

$$\begin{aligned}(\bar{H}\overline{v_0})_j &= \sum_{i=1}^{N} p_{ij} m_i \, \delta_X \circ f_j^{-1} \\ &= \sum_{i=1}^{N} p_{ij} m_i \, \delta_{f_j X} \\ &= m_j \, \delta_{f_j X}.\end{aligned}$$

Assuming the result holds for *n*, we have

$$\begin{aligned}(\bar{H}^{n+1}\overline{v_0})_j &= (\bar{H}(\bar{H}^n \overline{v_0}))_j \\ &= \sum_{i=1}^{N} p_{ij} \sum_{i_1, \ldots, i_{n-1}=1}^{N} m_i q_{ii_1} \cdots q_{i_{n-2} i_{n-1}} \\ &\quad \times (\delta_{f_i f_{i_1} \cdots f_{i_{n-1}} X}) \circ f_j^{-1} \\ &= \sum_{i, i_1, \ldots, i_{n-1}}^{N} m_i p_{ij} q_{ii_1} \cdots q_{i_{n-2} i_{n-1}} \, \delta_{f_j f_i f_{i_1} \cdots f_{i_{n-1}} X} \\ &= \sum_{i, i_1, \ldots, i_{n-1}}^{N} m_j q_{ji} q_{ii_1} \cdots q_{i_{n-2} i_{n-1}} \, \delta_{f_j f_i f_{i_1} \cdots f_{i_{n-1}} X},\end{aligned}$$

as required.

THEOREM 4.5. *For a weakly hyperbolic recurrent IFS, the extended generalized Markov operator* $\bar{H}\colon \mathbf{P}^1\overline{\mathbf{U}X} \to \mathbf{P}^1\overline{\mathbf{U}X}$ *has a unique fixed point* $\bar{v}^* \in \mathbf{S}_0^1\overline{\mathbf{U}X}$ *with support* $(\mathsf{S}(A^* \cap f_j X))_j$ *where* A^* *is the unique attractor of the IFS.*

Proof. We know, by Proposition 4.3 that any fixed point of $\bar{H}$ is in $\mathbf{P}_0^1\overline{\mathbf{U}X}$. Therefore, it is sufficient to show that the least fixed point $\bar{v}^*$ of

$$\bar{H}\colon \mathbf{P}_0^1\overline{\mathbf{U}X} \to \mathbf{P}_0^1\overline{\mathbf{U}X}$$

is unique. Using the explicit form of v^* in Eq. (10), we can show as in the corresponding proof for a weakly hyperbolic IFS with probabilities (Theorem 3.2) that $\bar{v}^* \in \mathbf{S}_0^1\overline{\mathbf{U}X}$. It then follows that $\bar{v}^*$ is maximal in $\mathbf{P}_0^1\overline{\mathbf{U}X}$, and hence is the unique fixed point. By Eq. (10), the support of $\bar{v}^*$ is indeed $(\mathsf{S}(A^* \cap f_j X))_j$. ▮

It then follows, similar to the case of an IFS with probabilities, that $\bar{j}(\bar{v}^*)$ is the unique stationary distribution $\bar{v}^*$ of the generalised Markov operator $\bar{T}\colon \mathbf{M}^1\bar{X} \to \mathbf{M}^1\bar{X}$ of Subsection 4.1, and that the support of $\bar{\mu}^*$ is $(A^* \cap f_j X))_j$.

4.3. *The Recurrent Probabilistic Power Domain Algorithm*

Theorem 4.5 provides us with the *recurrent algorithm* to generate the stationary distribution of a recurrent IFS on the digitised screen. Given the recurrent IFS $\{X; f_j; p_{ij}; i, j = 1, 2, \ldots, N\}$, where X is contained in the unit square, consider the recurrent IFS tree with transitional probabilities in Fig. 5. Let $\varepsilon > 0$ be the discrete threshold.

Initially, the set $X \times \{j\}$ is given mass m_j, which is then distributed amongst the nodes of the tree according to the inverse transitional probability matrix (q_{ij}). The algorithm first computes the unique stationary initial distribution (m_i), by solving the equations $m_j = \sum_{i=1}^{N} m_i p_{ij}$ for m_j $(1 \leqslant j \leqslant N)$ with the Gaussian elimination method, and determines the inverse transition probability matrix (q_{ij}) given by Eq. (9). The number of arithmetic computations for this is $O(N^3)$. Then the algorithm proceeds, exactly as the probabilistic algorithm, to compute, for each pixel, the sum of the weights $m_{i_1} q_{i_1 i_2} \cdots q_{i_{n-1} i_n}$ of the leaves $f_{i_1} f_{i_2} \cdots f_{i_n} X$ of the IFS tree which occupy that pixel. The number of computations for the latter is $O(N^h)$ as before, where $h = \lceil \log(\varepsilon/|X|)/\log s \rceil$. Therefore, the complexity of the algorithm is $O(N^{h'})$ where $h' = \max(h, 3)$.

5. THE EXPECTED VALUE OF CONTINUOUS FUNCTIONS

In this section, we will use the theory of generalised Riemann integration, developed in [16], to obtain the expected value of a continuous real-valued function with respect to the stationary distribution of a recurrent IFS. We first recall the basic notions from the above work.

Let X be a compact metric space and $g: X \to \mathbb{R}$ be a bounded real-valued function which is continuous almost everywhere with respect to a given normalised Borel measure μ on X. By [16, Theorem 6.5], g will be R-integrable, and by [16, Theorem 7.2], its R-integral $\mathbf{R}\int g\, d\mu$ coincides with its Lebesgue integral $\mathbf{L}\int g\, d\mu$. The R-integral can be computed as follows. We know that μ corresponds to a unique valuation $e(\mu) = \mu \circ s^{-1} \in \mathbf{S}^1 X \subseteq \mathbf{P}^1 \mathbf{U} X$, which is supported in $s(X)$. For any simple valuation $v = \sum_{b \in B} r_b\, \delta_b \in \mathbf{P}^1 \mathbf{U} X$, the lower sum of g with respect to v is

$$S^l(g, v) = \sum_{b \in B} r_b \inf g(b).$$

Similarly, the upper sum of g with respect to v is

$$S^u(g, v) = \sum_{b \in B} r_b \sup g(b).$$

Let $\langle v_n \rangle_{n \geqslant 0}$ be an ω-chain of simple valuations in $\mathbf{P}^1 \mathbf{U} X$ with $e(\mu) = \bigsqcup_n v_n$. Then it follows from [16, Corollary 4.9] that $S^l_X(g, v_n)$ is an increasing sequence of $n \geqslant 0$ with limit $\mathbf{R}\int g\, dv$ and $S^l_X(g, v_n)$ is a decreasing sequence with limit $\mathbf{R}\int g\, dv$. We can also compute the R-integral by generalised Riemann sums as follows. Let

$$v_n = \sum_{b \in B_n} r_{n,b}\, \delta_b$$

and $\xi_{n,b} \in b$ for $b \in B_n$ and $n \geqslant 0$. Put

$$S(g, v_n) = \sum_{b \in B_n} r_{n,b}\, g(\xi_{n,b}).$$

Then, we have $S^l(g, v_n) \leqslant S(g, v_n) \leqslant S^u(g, v_n)$ and therefore

$$\lim_{n \to \infty} S(g, v_n) = \mathbf{R}\int g\, dv. \tag{11}$$

Now consider a weakly hyperbolic recurrent IFS $\{X; f_j; p_{ij}; i, j = 1, 2, ..., N\}$. We would like to compute

$$\int g\, d\bar{v}^* = \sum_{j=1}^{N} g\, d\mu_j^*, \tag{12}$$

where $\bar{\mu}^* = (\mu_j^*)_j$ is the unique stationary distribution and $g: X \to \mathbb{R}$ is a bounded real-valued function which is continuous almost everywhere with respect to each component μ_j^*. We know that $\bar{e}(\bar{\mu}^*) = \bar{v}^* = \bigsqcup_n \overline{v_n}$, where $\overline{v_n} = \bar{H}^n(\overline{v_0})$ and each component $(\bar{H}^n \overline{v_0})_j$ is given by Eq. (10). Fix an arbitrary point $x_0 \in X$ and for each component $j = 1, ..., N$, select $f_j f_{i_1} \cdots f_{i_{n-1}} x_0 \in f_j f_{i_1} \cdots f_{i_{n-1}} X$ and define the Riemann sum for the j component by

$$S_j(g, \overline{v_n}) = \sum_{i_1, i_2, ..., i_{n-1}=1}^{N} m_j\, q_{j i_1} q_{i_1 i_2} \cdots q_{i_{n-2} i_{n-1}} \times g(f_j f_{i_1} \cdots f_{i_{n-1}} x_0).$$

Put $S(g, \overline{v_n}) = \sum_{j=1}^{N} S_j(g, \overline{v_n})$. Then by Eqs. (11) and (12), we have

$$\mathbf{R}\int g\, d\bar{\mu}^* = \lim_{n \to \infty} S(g, \overline{v_n}).$$

For an IFS with probabilities, we have $p_{ij} = p_j$ for $1 \leqslant i, j \leqslant N$, which implies $m_j = p_j$ and $q_{ij} = p_j$ for all i, j; the invariant measure μ^* of the IFS with probabilities can be expressed in terms of the stationary distribution $\bar{\mu}^*$ of the recurrent IFS by $\mu^* = \sum_{j=1}^{N} \mu_j^*$ and we obtain

$$\mathbf{R}\int g\, d\mu^* = \lim_{n \to \infty} \sum_{i_1, i_2, ..., i_n = 1}^{N} p_{i_1} \cdots p_{i_n} g(f_{i_1} \cdots f_{i_n} x_0).$$

Compare this formula with Elton's ergodic formula in Eq. (4), which converges with probability one. For a hyperbolic IFS and a continuous function g, the above formula reduces to that of Hepting *et al.* in [24].

For a hyperbolic recurrent IFS and a Lipschitz map g, we can do better; we can obtain a polynomial algorithm to calculate the integral to any given accuracy. Suppose there exist $k > 0$ and $c > 0$ such that g satisfies

$$|g(x) - g(y)| \leqslant c(d(x, y))^k$$

for all $x, y \in X$. Let $\varepsilon > 0$ be given. Then we have $|g(x) - g(y)| \leqslant \varepsilon$ if $d(x, y) \leqslant (\varepsilon/c)^{1/k}$. Put $n = \lceil \log((\varepsilon/c)^{1/k}/|X|)/\log s \rceil$, where s is the contractivity of the IFS. Then, the diameter of the subset $f_{i_1} \cdots f_{i_n} X$ is at most $s^n |X| \leqslant (\varepsilon/c)^{1/k}$ for all sequences $i_1 i_2 \cdots i_n \in \Sigma_N^n$, and hence the variation of g on this subset is at most ε. This implies that

$$\begin{aligned} S^u(g, \overline{v_n}) - S^l(g, \overline{v_n}) &= \sum_{i_1, ..., i_n = 1}^{N} m_{i_1} q_{i_1 i_2} \cdots q_{i_{n-1} i_n} (\sup g(f_{i_1} \cdots f_{i_n} X) \\ &\quad - \inf g(f_{i_1} \cdots f_{i_n} X)) \\ &\leqslant \varepsilon \sum_{i_1, ..., i_n = 1}^{N} m_{i_1} q_{i_1 i_2} \cdots q_{i_{n-1} i_n} = \varepsilon. \end{aligned}$$

Since $S(g, \overline{\nu_n})$ and $\int g\, d\mu^*$ both lie between $S^u(g, \overline{\nu_n})$ and $S^l(g, \overline{\nu_n})$, we conclude that

$$\left| S(g, \overline{\nu_n}) - \int g\, d\bar{\mu}^* \right| \leqslant \varepsilon.$$

Therefore, $S(g, \overline{\nu_n})$ with $n = \lceil \log ((\varepsilon/c)^{1/k}/|X|)/\log s \rceil$ is the required approximation and the complexity is $O(N^n)$.

ACKNOWLEDGMENTS

I thank Reinhold Heckmann for discussions on the Plotkin power domain. This work has been supported by EPSRC grants for "constructive approximation using domain theory" and "foundational structures in computer science" at Imperial College.

Received Spetember 9, 1994; final manuscript received September 29, 1995

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