A domain-theoretic approach to computability on the real line [1]

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Abstract

In recent years, there has been a considerable amount of work on using continuous domains in real analysis. Most notably are the development of the generalized Riemann integral with applications in fractal geometry, several extensions of the programming language PCF with a real number data type, and a framework and an implementation of a package for exact real number arithmetic.

Based on recursion theory we present here a precise and direct formulation of effective representation of real numbers by continuous domains, which is equivalent to the representation of real numbers by algebraic domains as in the work of Stoltenberg-Hansen and Tucker.

We use basic ingredients of an effective theory of continuous domains to spell out notions of computability for the reals and for functions on the real line. We prove directly that our approach is equivalent to the established Turing-machine based approach which dates back to Grzegorczyk and Lacombe, is used by Pour-El & Richards in their foundational work on computable analysis, and, moreover, is the standard notion of computability among physicists as in the work of Penrose. Our framework makes it possible to capture partial functions in an elegant way and it extends to the complex numbers and the n-dimensional Euclidean space.

1 Introduction

Computable analysis is traditionally approached from two different directions. On the one hand, we have the machine-oriented work, where computations are

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performed on a certain kind of abstract machine. Type 2 Theory of Effectivity (TTE) [20,40,41] falls into this class. In TTE, Turing machines operate on infinite tapes, the inscription of the tapes represent real numbers or other objects from analysis, for example subsets, functions or measures. The so-called Russian approach [5] is also of this type. The main difference with TTE lies in the restriction of input and output to computable elements. Although different in spirit, the recursive functions in the Blum-Shub-Smale model [4] can also be considered as machine-oriented. Real numbers are regarded as entities, but the computable functions are constructed from building blocks in a recursion-theoretic manner.

On the other hand, we have the analysis-oriented approach. Here concepts from classical analysis are effectively presented and used to develop a computability theory for real numbers. This approach to computable analysis originated from the work of Grzegorczyk [19] who classified Turing-machine computable real functions as those that map computable sequences to computable sequences and are effectively uniformly continuous. The work of Pour-El & Richards [29] is based on this definition and is now well-established and frequently cited in various communities including by physicists like Penrose [25].

The present paper is part of a programme to establish domain theory as a new approach to computable analysis. Domain theory was introduced independently by Dana Scott [32] for providing denotational semantics to functional programming languages and by Yuri Ershov [15] as a means to investigate partial computable functionals of finite type. The use of the so-called *algebraic domains* to model functional programming languages has become a well-established paradigm in computer science.

Various attempts have been made to use algebraic domains to represent classical spaces in mathematics. Weihrauch & Schreiber [43] constructed embeddings of Polish spaces (topologically complete separable metrizable spaces) into algebraic domains. Stoltenberg-Hansen and Tucker have shown how to represent complete local rings [35] and topological algebras, including locally compact Hausdorff spaces and the real line, by algebraic domains [36]. Di Gianantonio [6,7] has given an algebraic domain to model the real numbers. Blanck [3] has more recently shown how to embed complete metric spaces into algebraic domains.

In [36, Section 5.3], Stoltenberg-Hansen & Tucker use an algebraic domain to represent the real line and prove that the resulting notion of computable real function coincides with a slight strengthening of the approach by Pour-El & Richards. Also, the work in [36] allows them to generalise this result to $\mathbb{R}^n$ and $\mathbb{C}$ which is explicitly done by Blanck in [3, Theorem 2.27].

However, a more general class, that of so-called *continuous domains,* is more suitable to represent classical spaces. A continuous domain is a partially ordered set equipped with notions of *completeness* and *approximation.* The completeness axiom requires existence of least upper bounds for all directed subsets, approximation means that all elements arise as directed suprema of their essential parts or approximants. (All definitions are formally given in Section 2.) The particular case of *continuous lattices* [17] arises in many other branches of mathematics. The approximation axiom provides the link to the machine-based level of recursion theory or Turing machines: We will enumerate a convenient set of approximants and let the machine operate on this set.

The link to computable analysis on the real line is provided by the *interval domain,* the set of compact intervals of $\mathbb{R}$, partially ordered with reversed set inclusion. Already in [32], Scott suggested the idea of using the interval domain to construct a real number data type. The real line embeds as set of maximal elements in this continuous domain. Thus the above mentioned approximation by partial elements corresponds to describing a real number as the intersection of a sequence of shrinking nested intervals which is a standard way of defining a real number in computable analysis [31]. Thus domain theory also provides a link to the well-established field of interval analysis [23] and can lead to new insights in this subject.

There has recently been a considerable amount of work in domain theory which could be classified as part of the programme "Continuous domains in computable analysis". Most notably are the development of a domain theoretic framework for classical measure theory [11,9], the generalization of Riemann theory of integration [10] with applications in fractal geometry [12], several extensions of the programming language PCF with a real number data type [6,16,28], and a framework and an implementation of a package for exact real number computation [27,13]. This latter work is based on the one hand on continued fractions and linear fractional transformations as in [38,24] and on the other hand on the domain of intervals. These promising results suggest that a marriage of domain theory and computable analysis will indeed be fruitful for both subjects. The recent survey paper [8] gives an overview of these applications of continuous domains.

In this paper, we start a systematic exploration of the use of continuous domains for computable analysis. Here, we are concerned with analysis on the real line, the complex plane, and $\mathbb{R}^n$. A forthcoming paper [14] will deal with metric spaces and Banach spaces. The main results in the present paper are the following: The domain-theoretic notions for computable real numbers and for computable functions coincide with the well-established so-called Polish approach which dates back to Grzegorczyk and Lacombe [18,21] and is equivalent to the above mentioned TTE-approach and to the definitions of Pour-El & Richards.

It can be shown using some general properties of algebraic and continuous domains that computability on the reals in our sense coincides with computability via the algebraic approach. Hence, apart from the slight strengthening of Pour-El & Richards' definition in [36], our results can be obtained from those in *loc.cit.* and vice versa.

Compared to the continuous domain approach, however, any representation of the real line by an algebraic domain is much more involved for topological reasons. The domain considered in [36] is the ideal completion of the set of all intervals with rational endpoints, and the real line is recovered as a quotient of the set of so-called *total* elements. In contrast, the continuous domain for the real line considered in the present paper is based on quite familiar and well-established notions in elementary analysis and the real line is simply embedded as its set of maximal elements.

In the present paper, we intend to promote domain theory as a means of investigating computability aspects in traditional mathematics. Therefore, we choose the more accessible continuous domain approach to computability and present the framework and the proofs directly in a self-contained way.

1.1 Plan of the paper

This paper is divided in two parts. Part I deals with the mathematical tools and Part II investigates the computability structure for the real line and briefly covers the n-dimensional Euclidean space and the complex plane.

To be self-contained, Part I starts in Section 2 with a short introduction to continuous domains in general and the interval domain in particular. The link to actual computations is provided by *effective domain theory.* Although it dates back to the origins of domain theory, we have included a section on this topic. The existing sources either consider algebraic domains only (e.g. [26,34]), or, as [33], concentrate on certain subclasses of continuous domains which are useful in denotational semantics but too special for our purpose. The only exception is the unpublished set of lecture notes in German by Weihrauch and Deil [42], where domains are considered as computational models very much in the same spirit as in the present paper. Unfortunately, this source is rather hard to access. There is a short published note [39] which contains the most basic definitions but lacks the results we need for our work. So we provide all the definitions and results needed and give proofs in Section 3 of this paper.

Part II is the core of our work. In Section 4, we define the notions of computable real numbers and sequences by effectively presenting the interval domain. These are shown to coincide with the corresponding standard notions in classical computable analysis. Section 5 investigates the resulting notion of computable function on the real line. Again, the notion coincides with the standard one. As a corollary, we obtain a novel characterisation of computable functions: A function is computable if and only if it maps computable sequences of intervals to computable sequences of intervals. We conclude the paper by discussing real number representations within our framework.

1.2 Terminology

We will use the relevant notions from recursion theory as in the fairly standard language of [31]. The set $\{0, 1, 2, \ldots\}$ of natural numbers is denoted by $\mathbb{N}$. The nth partial recursive function is denoted by ϕ_n, the nth recursively enumerable (r.e.) set by W_n, so that $W_n = \mathsf{dom}(\phi_n)$. We will make use of a standard pairing function $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which could be defined as $\langle n, m \rangle = \frac{1}{2}(n^2 + 2nm + m^2 + 3n + m)$. The projections are denoted by π_1, π_2; they satisfy $\langle \pi_1(n), \pi_2(n) \rangle = n$ as well as $\pi_1(\langle n, m \rangle) = n$ and $\pi_2(\langle n, m \rangle) = m$ for all $n, m \in \mathbb{N}$. As usual, a relation $R \subseteq \mathbb{N} \times \mathbb{N}$ is said to be r.e. if the set $\{\langle n, m \rangle \mid (n, m) \in R\}$ is r.e. We will conveniently say that R is r.e. in n, m. Similarly for relations of higher arity.

Many of our results have the form *"There is f with property A iff there is g with property B".* Sometimes, we add the phrase *"This equivalence is effective."* This means that there are partial recursive function $\psi_1, \psi_2 : \mathbb{N} \to \mathbb{N}$ such that if ϕ_n has property A then $\psi_1(n)$ is defined and $\phi_{\psi_1(n)}$ has property B and, conversely, if ϕ_m has property B then $\psi_2(m)$ is defined and $\phi_{\psi_2(m)}$ has property A. Similarly for r.e. sets in place of functions. This is referred to as *uniformity* by Rogers in *loc.cit.*

We will employ the dovetailing principle in the form of the following construction: Every r.e. set $A \subseteq \mathbb{N}$ can be written as the union $A = \bigcup_{n \in \mathbb{N}} A_n$ of an increasing chain $A_0 \subseteq A_1 \subseteq A_2 \cdots$ such that the relation $i \in A_n$ is recursive in i, n. To see that this is true, we take a Turing machine based approach to recursion theory. If M is a machine which produces a list of the elements of A, then define A_n to contain those elements produced after n steps of computation by M. A consequence of effectiveness of this construction is the Selection Theorem [31, Theorem 5-XVIII]. It says that there is a partial recursive function $\psi : \mathbb{N} \to \mathbb{N}$ such that $\psi(n)$ is defined iff the set W_n is not empty in which case $\psi(n) \in W_n$.

Part I: Mathematical Tools

2 Continuous domains

Domain theory was introduced by Dana Scott in [32] as a mathematical theory of computation. See [1] for a detailed treatment of domain theory, in particular for topics which are not of interest here, but are important for denotational semantics, e.g. cartesian closed categories and solution of recursive domain equations. In this section, we give a short introduction to continuous domains as computational models, a topic which has become an active area of research in recent years.

2.1 Basic definitions

The first basic notion is that of a *partially ordered set* (poset) $(D, \sqsubseteq)$. This is a set D equipped with a binary relation $\sqsubseteq$ which is

reflexive: $x \sqsubseteq x$ for all $x \in D$,
transitive: $x \sqsubseteq y$ and $y \sqsubseteq z$ implies $x \sqsubseteq z$ for all $x, y, z \in D$,
antisymmetric: $x \sqsubseteq y$ and $y \sqsubseteq x$ implies $x = y$ for all $x, y \in D$.

The elements of D are thought of as descriptions of some objects. The order is referred to as *order of information.* Indeed, $x \sqsubseteq y$ is understood as "y carries more information than x". From this it is apparent that the set of maximal elements, denoted by $\mathsf{max}(D)$, will be of special interest. A non-empty subset $A \subseteq D$ is *directed,* if for all $x, y \in A$ there is $z \in A$ with $x \sqsubseteq z$ and $y \sqsubseteq z$. Important examples of directed sets are *increasing chains.* These are sets $A = \{a_0, a_1, a_2, \ldots\}$ such that $a_0 \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq \cdots$. We think of such a chain as a stepwise computation: in each step we gain more information about the computed entity. What would this entity be? An element containing precisely all the information gained during the computation, which is exactly the notion of *supremum* or *least upper bound* (lub): An element $x \in D$ is the lub of the subset $A \subseteq D$ if (1) it is an upper bound, i.e. $a \sqsubseteq x$ for all $a \in A$ and if (2) whenever $b \in D$ is any other upper bound then $x \sqsubseteq b$. We write $\bigvee^{\uparrow} A = x$ to denote that the set A is directed and has lub x. The first axiom for domains is closure under these computations: A *directed complete partial order* (commonly abbreviated as *dcpo*) is a partially ordered set such that suprema for all directed subsets exist. We call the dcpo *pointed,* if it contains a least element $\bot$ (pronounced "bottom").

If $(D, \sqsubseteq)$ is a dcpo and $x, y \in D$ then we say that x *approximates* y, and write $x \ll y$ if for every directed subset $A \subseteq D$ with $y \sqsubseteq \bigvee^{\uparrow} A$ there is some $a \in A$ with $x \sqsubseteq a$. If $x' \sqsubseteq x$, $y \sqsubseteq y'$, and $x \ll y$ then $x' \ll y'$. The intuitive meaning of $x \ll y$ is "the information content of x is essential for y". It is frequently referred to as "x is way-below y".

We now arrive at the definition of a continuous domain: We require that every element can be recovered from its essential ingredients.

Definition 1 *A* continuous domain *is a dcpo $(D, \sqsubseteq)$ such that for every element $x \in D$ the set*

$$\mathord{\downarrow\!\!\!\downarrow} x = \{y \in D \mid y \ll x\}$$

is directed and has x as its supremum:

$$\bigvee^{\uparrow} \mathord{\downarrow\!\!\!\downarrow} x = x.$$

We often refer to a continuous domain simply as a domain. The single most important property of the order of approximation $\ll$ on a continuous domain is the *interpolation property:* If $x \ll y$ then there is $z \in D$ such that $x \ll z$ and $z \ll y$. More generally: If $x_i \ll y$ for $i = 1, \ldots, n$ then there is $z \in D$ with $x_i \ll z$ for $i = 1, \ldots, n$ and $z \ll y$.

The unit interval $[0, 1]$ in its usual order serves as an example of a continuous domain. Here $x \ll y$ iff $x = 0$ or $x < y$. Similarly, the extended non-negative reals $[0, \infty]$ with the usual ordering form a continuous domain.

The *Scott-topology* of a dcpo $(D, \sqsubseteq)$ consists of all subsets O of D which are upwards closed ($x \in O,\ x \sqsubseteq y \Longrightarrow y \in O$) and inaccessible by directed suprema, i.e. if $\bigvee^{\uparrow} A \in O$ then $O \cap A \neq \emptyset$. This topology is always T_0 but typically not T_1. If D is a continuous domain, then, as a consequence of the interpolation property, the sets $\mathord{\uparrow\!\!\!\uparrow} x = \{y \in D \mid x \ll y\}$ for $x \in D$ are Scott-open. Moreover, they form a basis for the topology. A function $f\colon (D, \sqsubseteq) \to (E, \sqsubseteq)$ between dcpo's is continuous with respect to the Scott-topologies on D and E if and only if it is monotone and preserves suprema of directed subsets: $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$ and $f(\bigvee^{\uparrow} A) = \bigvee^{\uparrow} f(A)$. A function f between pointed dcpo's is called *strict* if $f(\bot) = \bot$. The collection of all Scott-continuous functions from D to E is denoted by $[D \to E]$. It is endowed with the *pointwise order,* i.e., $f \sqsubseteq g$ iff $f(x) \sqsubseteq g(x)$ for all $x \in D$, which makes $[D \to E]$ a dcpo.

The nontrivial Scott-open subsets in the above examples $([0, 1], \leq)$ and $([0, \infty], \leq)$ are of the form $(a, 1]$ and $(a, \infty]$, respectively. An endofunction on $[0, \infty]$ is Scott-continuous iff it is monotone and lower semicontinuous in the traditional sense. In general, a function $f\colon X \to [0, \infty]$ from any topological space X is continuous with respect to the Scott-topology on $[0, \infty]$ iff it is lower semicontinuous.

A subset B of a domain $(D, \sqsubseteq)$ is a *basis* if every element of D is the directed supremum of all basis elements approximating it:

$$x = \bigvee^{\uparrow}(\twoheaddownarrow x \cap B).$$

Every domain D has a basis, namely D itself. A domain is *ω-continuous,* if it has a countable basis. In each of the above examples the set of all rational numbers in the domain forms a basis; hence both $[0, 1]$ and $[0, \infty]$ are ω-continuous.

If D and D' are domains then so is their direct product, $D \times D'$. The order and order of approximation are coordinatewise, i.e.

$$(x, x') \sqsubseteq (y, y') \iff x \sqsubseteq y \ \& \ x' \sqsubseteq y'$$

and

$$(x, x') \ll (y, y') \iff x \ll y \ \& \ x' \ll y'.$$

The Scott-topology on the product coincides with the product topology of the Scott-topologies on the factors. Unlike the situation in general topology, it is true that a function $f: D \times D' \to E$ is Scott-continuous if and only if it is Scott-continuous in both variables separately. This is due to the fact that Scott-continuity can be characterised purely in order-theoretical terms.

2.2 The interval domain

The *interval domain* $\mathcal{I}$ gives the set $\mathbb{R}$ of real numbers a computational structure. It is the collection of all compact intervals, endowed with a least element:

$$\mathcal{I} = \{[a, b] \subseteq \mathbb{R} \mid a, b \in \mathbb{R},\ a \leq b\} \cup \{\bot\}$$

The order is reversed subset inclusion, i.e. $\bot \sqsubseteq I$ for all $I \in \mathcal{I}$ and $[a, b] \sqsubseteq [c, d]$ iff $a \leq c$ and $b \geq d$ in the usual ordering of real numbers. One can think of the least element $\bot$ as the set $\mathbb{R}$. Directed suprema are filtered intersections of intervals. The way-below relation is given by $I \ll J$ iff $\mathsf{int}(I) \supseteq J$, where $\mathsf{int}(I)$ denotes the interior of I. Thus $\bot \ll I$ for all $I \in \mathcal{I}$ and $[a, b] \ll [c, d]$ iff $a < c$ and $b > d$. The maximal elements are the intervals $[a, a]$, i.e. the singleton sets.

The interval domain can be thought of as a triangle as depicted in Figure 1. The upper edge of this triangle corresponds to the set of maximal elements,

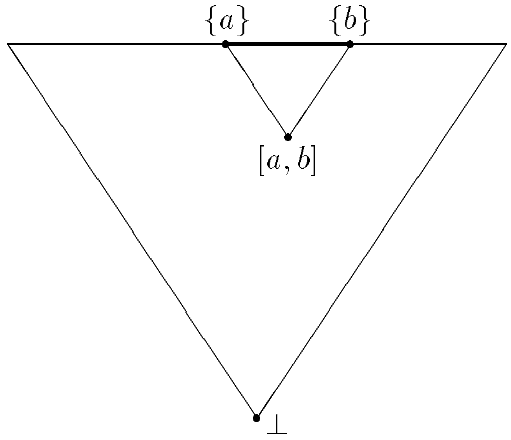

Fig. 1. The interval domain.

i.e. to the real line. Points in the interior correspond to closed intervals of non-zero length. As shown in the figure, the endpoints of such an interval can be determined by drawing parallel lines to the side edges of the triangle and intersecting these with the upper edge. The thick line segment in the picture denotes the set $\{\{x\} \mid a \le x \le b\}$ of maximal elements above the element $[a, b] \in \mathcal{I}$, which is mapped by $\{x\} \mapsto x$ to the interval $[a, b]$.

What is the Scott-topology on $\mathcal{I}$? A base is given by the sets $\Uparrow[a, b] = \{I \in \mathcal{I} \mid I \subseteq (a, b)\}$. So a base set for the relative Scott-topology on the set of maximal elements is of the form $\Uparrow[a, b] \cap \mathsf{max}(\mathcal{I}) = \{\{x\} \mid x \in (a, b)\}$. Under the canonical map $\{x\} \mapsto x \colon \mathsf{max}(\mathcal{I}) \to \mathbb{R}$ this is mapped to the open interval (a, b). Hence the set of maximal elements with the relative Scott-topology is homeomorphic to the real line via $\{x\} \mapsto x$.

Every continuous function $f \colon \mathbb{R} \to \mathbb{R}$ has a Scott-continuous extension to the interval domain. This means that there is a Scott-continuous function $g \in [\mathcal{I} \to \mathcal{I}]$ such that $g(\{x\}) = \{f(x)\}$ holds for all $x \in \mathbb{R}$. Moreover, among those extensions which are strict in the sense that $g(\perp) = \perp$ there is a largest one. It is explicitly given by

$$g(I) = \{f(x) \mid x \in I\}$$

for $I \neq \perp$. Since the non-bottom elements of the interval domain are exactly the compact connected subsets of $\mathbb{R}$ and as the operation of taking the direct image under a continuous function preserves both these properties, it is immediate that the function g is well-defined.

A convenient computational model for a compact interval $A \subseteq \mathbb{R}$ is defined in the same manner. We denote with $\mathcal{I}A$ the interval domain of A, consisting of all compact intervals contained in A:

$$\mathcal{I}A := \{[a, b] \subseteq A \mid a \le b\}.$$

The order is reversed subset inclusion as before. Note that A itself is a compact interval, so $A \in \mathcal{I}A$ and we do not need to add a least element. The above results for $\mathcal{I}$ concerning the Scott-topology and extensions of continuous functions do also hold for $\mathcal{I}A$. We can in addition consider continuous functions $f: A \to B$ for A and B different compact intervals or the real line; those extend to the interval domains as before. If A is a compact interval rather than the real line $\mathbb{R}$, then we can drop the assumption of strictness to find a largest extension.

3 An effective theory of continuous domains

The material covered in this section is rather well-known among domain theorists with interest in recursion theory. Unfortunately, there is no simply available source on the subject as mentioned in the introduction. We develop the theory along the lines of [42].

3.1 Effectively given domains

Definition 2 *Suppose $(D, \sqsubseteq)$ is an ω-continuous pointed domain with countable basis $D_0 = \{b_0, b_1, b_2, \cdots\}$. It is* effectively given *with respect to b if the relation $b_n \ll b_m$ is r.e. in n, m. An element $x \in D$ is* computable, *if the set $\{n \in \mathbb{N} \mid b_n \ll x\}$ is r.e. Without loss of generality, we will henceforth assume that $b_0 = \bot$.*

Remark. The reader might ask why we do not require the order of approximation $\ll$ or the predicate $b_n = \bot$ to be recursive (decidable). This is the approach in most other accounts of effective domain theory, e.g. [33] and the above mentioned Section 7 of [26]. These stronger assumptions (together with a restriction of the class of continuous domains) are needed in connection with the function space construction in order to yield a cartesian closed category of effective domains. As we are not interested in this topic here, we keep the definition as general as possible.

The computability structure on two effectively given domains induces an effective structure on their product as follows: If $(D, \sqsubseteq)$ and $(D', \sqsubseteq)$ are effectively given with respect to $(b_n)_{n\in\mathbb{N}}$ and $(b'_n)_{n\in\mathbb{N}}$, respectively, then the product $D \times D'$ has the effective basis $(b^*_n)_{n\in\mathbb{N}}$ with $b^*_{\langle n,m\rangle} = (b_n, b'_m)$.

Proposition 3 *An element $x \in D$ is computable iff it is the lub of an effective chain in D_0, i.e. iff there is $f: \mathbb{N} \to \mathbb{N}$ total recursive such that $b_{f(0)} \sqsubseteq b_{f(1)} \sqsubseteq b_{f(2)} \sqsubseteq \cdots$ and $x = \bigvee^{\uparrow}_{n \in \mathbb{N}} b_{f(n)}$. This equivalence is effective. Moreover, the chain can be chosen to be a $\ll$-chain, i.e. such that $b_{f(0)} \ll b_{f(1)} \ll b_{f(2)} \ll \cdots$.*

PROOF. Suppose $\mathord{\downarrow\!\!\downarrow} x \cap D_0 = \{b_{g(n)} \mid n \in \mathbb{N}\}$ for some total recursive $g: \mathbb{N} \to \mathbb{N}$. The function $f: \mathbb{N} \to \mathbb{N}$ is defined inductively. First we put $f(0) = g(0)$. To define $f(n+1)$ consider the set

$$A_n := \{m \in \mathbb{N} \mid b_{f(n)} \ll b_{g(m)} \ \& \ b_{g(n)} \ll b_{g(m)}\}.$$

This set is r.e. as g is recursive. Furthermore, it is non-empty by the interpolation property of continuous domains. So we can apply the Selection Theorem to get an element $a_n \in A_n$. Then $f(n+1) := g(a_n)$. The resulting function f gives an effective $\ll$-chain. We claim that its lub is x. To verify the claim, observe that $f(n) \in g(\mathbb{N})$ for every $n \in \mathbb{N}$. Hence $b_{f(n)} \ll x$ and so $\bigvee^{\uparrow}_{n \in \mathbb{N}} b_{f(n)} \sqsubseteq x$. On the other hand, we have $b_{g(n)} \sqsubseteq b_{f(n+1)}$ for every $n \in \mathbb{N}$. Thus $x = \bigvee^{\uparrow}_{n \in \mathbb{N}} b_{g(n)} \sqsubseteq \bigvee^{\uparrow}_{n \in \mathbb{N}} b_{f(n)}$. This proves the claim, so the (only if) part of the proposition holds.

For the (if) part, note that if $x = \bigvee^{\uparrow}_{n \in \mathbb{N}} b_{f(n)}$ then

$$b_m \ll x \iff \exists n \in \mathbb{N}.\ b_m \ll b_{f(n)}.$$

This enables us to effectively obtain an index for $\mathord{\downarrow\!\!\downarrow} x \cap D_0$ from f. $\square$

We now define an enumeration ξ of the set D_c of all computable elements. This is done in the following manner:

(1) Start with a natural number n.
(2) This describes a partial recursive function $\phi_n: \mathbb{N} \to \mathbb{N}$.
(3) Effectively construct an index n' of a *total* recursive function $\phi_{n'}$ such that $\mathsf{range}(\phi_{n'}) = \mathsf{range}(\phi_n) \cup \{0\}$. (Recall that $b_0 = \perp$.)
(4) Effectively get an index n'' of the total function $\phi_{n''}$ which is recursively defined by putting $\phi_{n''}(0) = \phi_{n'}(0)$ and the following procedure.
    (a) Start with $i = 0$ and $k = 1$.
    (b) The set $A = \{\ell \geq k \mid b_{\phi_{n''}(i)} \ll b_{\phi_{n'}(\ell)}\} \subseteq \mathbb{N}$ is r.e.
    (c) Write $A = \bigcup_{m \in \mathbb{N}} A_m$, where the test $\ell \in A_m$ is recursive in ℓ, m and where $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$.
    (d) Now, starting with $m = 1$, each of these sets is tested for the existence of $\ell \leq m$ with $\ell \in A_m$. Whenever no such element is found, we put $\phi_{n''}(i+m) = \phi_{n''}(i)$ and check for the next value of m.

(e) If, at some stage, there is $\ell \leq m$ with $\ell \in A_m$ then let $\phi_{n''}(i+m) = \phi_{n'}(\ell)$, increment i by m, set $k = \ell + 1$, and go to step (4b).

Note that $\phi_{n''}$ defines an effective chain in $\mathsf{range}(\phi_{n'})$: for each $m \in \mathbb{N}$ we have either $b_{\phi_{n''}(m)} \ll b_{\phi_{n''}(m+1)}$ or $\phi_{n''}(m) = \phi_{n''}(m+1)$. Moreover, if $\mathsf{range}(\phi_{n'})$ happened to be a chain already, then for each $i \in \mathbb{N}$ there is $j \in \mathbb{N}$ such that $b_{\phi_{n'}(i)} \ll b_{\phi_{n''}(j)}$ so that $\bigvee^{\uparrow}_{i \in \mathbb{N}} b_{\phi_{n''}(i)} = \bigvee^{\uparrow} \mathsf{range}(\phi_{n'})$.

(5) Now define $\xi(n) := \bigvee^{\uparrow}_{i \in \mathbb{N}} b_{\phi_{n''}(i)}$.

Remark. Although the chain given by $\phi_{n''}$ as constructed is not a $\ll$-chain, it is possible to effectively obtain an effective $\ll$-chain from this by Proposition 3.

Proposition 4 *Every element from D_0 is computable. Moreover, there is a total recursive function $k: \mathbb{N} \to \mathbb{N}$ such that $\xi \circ k = \iota \circ b$, where $\iota: D_0 \hookrightarrow D_c$ is the inclusion map.*

PROOF. Clearly, for each $n \in \mathbb{N}$ the set $\{m \in \mathbb{N} \mid b_m \ll b_n\}$ is r.e. and an index for it is effectively obtainable from n. As the equivalence in Proposition 3 is effective, we can construct the function k from this. $\Box$

Lemma 5 *The relation $b_n \ll \xi(m)$ is r.e. in n and m.*

PROOF. This is an immediate consequence of the fact that $b_n \ll \bigvee^{\uparrow}_{i \in \mathbb{N}} b_{f(i)}$ iff there is $i \in \mathbb{N}$ with $b_n \ll b_{f(i)}$. $\Box$

Proposition 6 *Least upper bounds of effective chains of computable elements are computable with effectively obtainable index.*

PROOF. The set of basis elements way-below the supremum is the union of the sets of elements way-below the individual elements. This can be obtained effectively. $\Box$

Remark. It is evident that Proposition 6 holds for effective *directed* sets in place of chains, too. Enumerations of D_c satisfying Lemma 5 and Proposition 6 are called *admissible* in [42]. In *loc.cit.*, it is shown that all admissible enumerations are isomorphic, i.e. if $\xi, \xi': \mathbb{N} \to D_c$ are both admissible then there is a recursive bijection $f: \mathbb{N} \to \mathbb{N}$ with $\xi \circ f = \xi'$.

Definition 7 *Suppose that the domains $(D, \sqsubseteq)$ and $(D', \sqsubseteq)$ are effectively given with respect to b and b', respectively. A continuous function $f: D \to D'$ is* computable, *if the relation*

$$b'_m \ll f(b_n)$$

is r.e. in n, m.

Proposition 8 *If $f: D \to D'$ between effectively given domains is computable then the relation*

$$b'_m \ll f(\xi_D(n))$$

is r.e. in n, m.

PROOF. Note that

$$\begin{aligned} f(\xi_D(n)) &= f\Big(\bigvee^{\uparrow} \{y \mid y \ll \xi_D(n)\}\Big) \\ &= \bigvee^{\uparrow} \{f(y) \mid y \ll \xi_D(n)\} \\ &= \bigvee^{\uparrow} \{x' \in D \mid \exists y \ll \xi_D(n).\ x' \ll f(y)\} \\ &= \bigvee^{\uparrow} \{b'_m \mid \exists i \in \mathbb{N}.\ b_i \ll \xi_D(n) \ \&\ b'_m \ll f(b_i)\} \end{aligned}$$

by continuity of f. Hence

$$b'_m \ll f(\xi_D(n)) \iff \exists i \in \mathbb{N}.\ (b'_m \ll f(b_i) \ \&\ b_i \ll \xi_D(n)).$$

This is r.e. in n and m by Lemma 5 and the fact that f is computable. $\square$

Theorem 9 *A continuous function $f: D \to D'$ between effectively given domains is computable iff there is a total recursive function $\psi: \mathbb{N} \to \mathbb{N}$ such that $f \circ \xi_D = \xi_{D'} \circ \psi$.*

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\psi"] \arrow[d, "\xi_D"'] & \mathbb{N} \arrow[d, "\xi_{D'}"] \\ D \arrow[r, "f"] & D' \end{tikzcd}$$

PROOF. (only if) Suppose that $\xi_D(n_0) = x$. By Proposition 8, the relation $b'_m \ll f(\xi_D(n))$ is r.e. in n, m. So an index for

$$\{m \in \mathbb{N} \mid b'_m \ll f(\xi_D(n_0))\}$$

is effectively obtainable from n_0. This gives the function ψ.

(if) Suppose $f \circ \xi_D = \xi_{D'} \circ \psi$. With k from Proposition 4 we have

$$\begin{aligned} b'_m \ll f(b_n) &\iff b'_m \ll f(\xi_D(k(n))) \\ &\iff b'_m \ll \xi_{D'}(\psi(k(n))). \end{aligned}$$

This set is r.e. by Lemma 5 and recursiveness of $\psi \circ k$. $\square$

Remark. Theorem 9 is part of the *Myhill-Shepherdson-Theorem* in this setting. What is missing is the fact that, roughly speaking, computability implies continuity. This means that if a recursive function $\psi: \mathbb{N} \to \mathbb{N}$ defines a function $f: D_c \to D'_c$ on the computable elements via $\xi_{D'} \circ \psi = f \circ \xi_D$, then this function f is necessarily Scott-continuous in the sense that it preserves all existing suprema of directed subsets. As we do not need this result here, we refer the reader to Satz 7 in [42] for the proof. Alternatively, the proof of Theorem 3.6.16 of [40]) which contains the result for the algebraic case can be translated to the continuous setting.

Using the concept of computable sequences, Theorem 9 can be put in a very appealing form.

Definition 10 *A sequence $(x_n)_{n\in\mathbb{N}}$ in D is* computable, *if there is a recursive function $h: \mathbb{N} \to \mathbb{N}$ such that $x_n = \xi_D(h(n))$.*

Corollary 11 *A continuous function $f: D \to D'$ between effectively given domains is computable iff it maps computable sequences to computable sequences.*

PROOF. (if) As the identity on $\mathbb{N}$ is recursive, the sequence $(\xi_D(n))_{n\in\mathbb{N}}$ is computable. Computability of the image sequence $(f(\xi_D(n)))_{n\in\mathbb{N}}$ ensures computability of f by Theorem 9.

(only if) Assume that f is computable and pick $\psi: \mathbb{N} \to \mathbb{N}$ with $f \circ \xi_D = \xi_{D'} \circ \psi$ (Theorem 9). If $(x_n)_{n\in\mathbb{N}}$ is a computable sequence then there is $h: \mathbb{N} \to \mathbb{N}$ recursive such that $x_n = \xi_D(h(n))$. Then $f(x_n) = f(\xi_D(h(n))) = \xi_{D'}(\psi(h(n)))$ so the sequence $(f(x_n))_{n\in\mathbb{N}}$ is computable since the function $\psi \circ h$ is recursive. $\square$

Remark. It is possible to have a unified framework for effectively presenting domains as suggested to us by Dana Scott. If one restricts to ω-continuous *bounded-complete* domains, i.e. domains where every subset which is bounded above has a supremum, then there is a *universal domain* U. It has the property that every such domain is isomorphic to the image of a retraction on U. Thus an effective structure on U, which can be concretely constructed as the set of all non-empty closed subsets of the Cantor space under reverse inclusion, gives rise to effective structures on all domains which are computable retracts of U. Computable functions between such domains can be treated in a similar fashion. However, we do not take this approach here, firstly because it requires significantly more domain theory and secondly because, in the sequel [14] to this paper, we will apply the framework to Banach spaces and employ domains which are not bounded-complete.

Part II: Computability via Domain Theory

4 Computability on the real line

4.1 The effective interval domain

The interval domain $\mathcal{I}$ is ω-continuous. An example for a countable basis is the collection $\mathcal{I}_0$ of all intervals with rational endpoints together with the least element $\perp$. We will use this domain to endow the real numbers with a computable structure.

In order to proceed we first have to say how to deal with the set $\mathbb{Q}$ of rational numbers. We denote by $q_0, q_1, q_2, \ldots$ a standard numeration of the rationals, e.g. $q_{\langle n,\langle m,k\rangle\rangle} = \frac{n-m}{k+1}$. The arithmetic operations $+, -, \cdot, /$ as well as the comparisons $<, \leq, =$ and the absolute value function $|\cdot|$ on rationals are recursive in their indices.

Now we are ready to define an effective structure for the interval domain. We set

$$I_0 = \perp$$

and

$$I_{\langle n,m\rangle+1} = [q_n - |q_m|, q_n + |q_m|].$$

Clearly, this enumerates the basis $\mathcal{I}_0$. Let us check that the way-below relation is r.e. We have $I_n \ll I_m$ iff $I_m \subseteq \mathsf{int}(I_n)$ iff

$$n = 0 \text{ or } \Big(m, n > 0 \text{ and } |q_{\pi_1(n-1)} - q_{\pi_1(m-1)}| + |q_{\pi_2(m-1)}| < |q_{\pi_2(n-1)}|\Big),$$

so this relation is even recursive. It should be remarked that the particular choice of the basis and the enumeration for the basis is not essential for the theory as long as one can pass effectively back and forth between the bases. We picked the given enumeration as it makes the characterization of $\ll$ particularly easy. The resulting enumeration of the computable elements of $\mathcal{I}$ is denoted by $\xi_{\mathcal{I}}: \mathbb{N} \to \mathcal{I}$.

4.2 Computable numbers and sequences

A real number $x \in \mathbb{R}$ is called *left computable,* if the set $\{n \in \mathbb{N} \mid q_n < x\}$ is r.e. Right computability is defined in an analogous way.

Proposition 12 *An interval $[x, y] \in \mathcal{I}$ is computable iff x is left-computable and y is right-computable.*

PROOF. The interval is computable iff $\{n \mid I_n \ll [x, y]\}$ is r.e. Now $q_n < x$ iff there are $m, k \in \mathbb{N}$ such that $q_n = q_m - |q_k|$ and $I_{\langle m,k\rangle+1} \ll [x, y]$. The relation $q_n > y$ can be characterised similarly, hence the proposition follows. $\square$

One possible definition of computability for a real number x is "x is both left- and right-computable." From this it is an immediate consequence that a real number x is computable iff the set $\{x\}$ is a computable element of the interval domain. We will obtain an effective version of this result in Corollary 19 below, using the approach via fast converging Cauchy-sequences of rationals to formulate computability for real numbers.

Definition 13 *A real number x is* computable, *if there is a total recursive function $h: \mathbb{N} \to \mathbb{N}$ such that*

$$|q_{h(n)} - x| \leq \frac{1}{2^n}$$

for all $n \in \mathbb{N}$.

Computable real numbers were first investigated by Turing [37]. Our definition is used by many authors (for early sources see e.g. [30,18]), is widely accepted, and has many different equivalent characterizations. It is also used for constructive analysis in [2]. Before we proceed to the above mentioned result, we first turn our attention to the width of intervals. For $I = [a, b]$ we set $|I| = b - a$, for $I = \perp = \mathbb{R}$ we define $|I| = \infty$.

Lemma 14 *1) The relation $|I_n| \leq q_m$ is recursive in n, m.*
2) The relation $|\xi_{\mathcal{I}}(n)| < q_m$ is r.e. in n, m.

PROOF. 1) We have $|I_n| = \infty$ for $n = 0$ and $|I_n| = 2|q_{\pi_2(n-1)}|$ otherwise. So (1) clearly holds.
2) As $\xi_{\mathcal{I}}(n) = \bigcap\{I_k \mid I_k \ll \xi_{\mathcal{I}}(n)\}$ it is true that $|\xi_{\mathcal{I}}(n)| < q_m$ holds iff there is $k \in \mathbb{N}$ with $I_k \ll \xi_{\mathcal{I}}(n)$ and $|I_k| < q_m$. These two relations are r.e. by Lemma 5 and part (1), respectively. □

It is well-known that it is not possible to effectively enumerate all computable real numbers (see, e.g. [40, Lemma 3.8.9]). The set of computable intervals, i.e. computable elements of the interval domain, however, can be enumerated with $\xi_{\mathcal{I}}$. This gives rise to a *partial* enumeration of the set of computable real numbers.

Theorem 15 *There is a partial recursive function $h\colon \mathbb{N} \to \mathbb{N}$ such that $h(n)$ is defined whenever $\xi_{\mathcal{I}}(n)$ is maximal in $\mathcal{I}$ and such that in this case $\phi_{h(n)}$ is a total function which defines a fast converging sequence of rationals with limit x_n, where $\xi_{\mathcal{I}}(n) = \{x_n\}$.*

PROOF. The relation R with

$$\langle n, k, i\rangle \in R \iff I_i \ll \xi_{\mathcal{I}}(n) \ \&\ |I_i| \leq \frac{1}{2^k} \tag{1}$$

is r.e. in n, k, i by Lemmata 5 and 14(1). Moreover, it is true that if $\xi_{\mathcal{I}}(n)$ is maximal and $k \in \mathbb{N}$ there is $i \in \mathbb{N}$ with $\langle n, k, i\rangle \in R$. By the Selection Theorem, there exists a partial recursive function $g\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that

$$\langle n, k, g(n, k)\rangle \in R \tag{2}$$

holds for all k whenever $\xi_{\mathcal{I}}(n)$ is maximal. Define $j\colon \mathbb{N}^2 \to \mathbb{N}$ by $j(n, k) = \pi_1(g(n, k) - 1)$. Then $q_{j(n,k)}$ is the middle point of the interval $I_{g(n,k)}$. Now $(q_{j(n,k)})_{k\in\mathbb{N}}$ is a fast converging sequence with limit x_n: By (2) and (1) we have in particular $x_n \in I_{g(n,k)}$. Now $q_{j(n,k)} \in I_{g(n,k)}$ and $|I_{g(n,k)}| \leq 2^{-k}$, so $|x_n - q_{j(n,k)}| \leq 2^{-k}$ as required. Finally, we define the function h to assign to a number $n \in \mathbb{N}$ the derived index for the function sending k to $j(n, k)$. □

The converse of this is also true: If a sequence of rationals effectively converges, then an index for the limit is effectively obtainable from an index for the sequence. As we are going to prove this result for sequences of reals, we need some preparatory definitions.

Definition 16 *A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is* computable, *if the sequence $(\{x_n\})_{n\in\mathbb{N}}$ is computable in $\mathcal{I}$. (In other words, if there exists a a total recursive function $f: \mathbb{N} \to \mathbb{N}$ such that $\{x_n\} = \xi_{\mathcal{I}}(f(n))$.)*

We say that the sequence converges effectively *to $x \in \mathbb{R}$, if there is $g: \mathbb{N} \to \mathbb{N}$ recursive (the* modulus of convergence*) such that $k \geq g(n)$ implies $|x_k - x| \leq 2^{-n}$.*

Lemma 17 *The function $(q, [a,b]) \mapsto [a-|q|, b+|q|]: \mathbb{Q}\times\mathcal{I} \to \mathcal{I}$ is computable.*

Now we are able to prove the second half of the characterization of computable real numbers as promised above: limits of effectively convergent sequences are computable.

Theorem 18 *If $(x_n)_{n\in\mathbb{N}}$ is an effectively convergent computable sequence, then its limit x is computable. Moreover, an index for $\{x\}$ can be obtained effectively from the indices for the sequence and the modulus of convergence.*

PROOF. Assume $g: \mathbb{N} \to \mathbb{N}$ is such that $k \geq g(n)$ implies $|x_k - x| \leq 2^{-n}$. Using Lemma 17 and effectiveness of the sequence $(x_n)_{n\in\mathbb{N}}$, we see that there is $h: \mathbb{N} \to \mathbb{N}$ recursive such that

$$\xi_{\mathcal{I}}(h(n)) = [x_{g(n)} - \frac{1}{2^{n-1}}, x_{g(n)} + \frac{1}{2^{n-1}}]. \tag{3}$$

Then $x \in \mathsf{int}(\xi_{\mathcal{I}}(h(n)))$ and $\{x\} = \bigcap_{n\in\mathbb{N}} \xi_{\mathcal{I}}(h(n))$. The interval sequence $(\xi_{\mathcal{I}}(h(n)))_{n\in\mathbb{N}}$ need not be shrinking, but the sequence $(\xi_{\mathcal{I}}(h(2n)))_{n\in\mathbb{N}}$ is. To see this, suppose $y \in \xi_{\mathcal{I}}(h(2n+2))$. Then $|y - x_{g(2n+2)}| \leq 2^{-(2n+1)}$. But $|x_{g(2n+2)} - x| \leq 2^{-(2n+2)}$ and $|x - x_{g(2n)}| \leq 2^{-2n}$. So $|y - x_{g(2n)}| \leq 2^{-(2n+1)} + 2^{-(2n+2)} + 2^{-2n} < 2^{-(2n-1)}$. Thus $y \in \mathsf{int}(\xi_{\mathcal{I}}(h(2n)))$.

Hence, via Proposition 6, an index for the function sending n to $h(2n)$ is an index for $\{x\}$. □

In particular, this result allows us to conclude that our notion of computable number coincides with the classical one.

Corollary 19 *A real number x is computable if and only if the set $\{x\}$ is a computable element of the interval domain.*

PROOF. Theorem 15 yields one direction and Theorem 18 together with the fact that every computable sequence of rational numbers is a computable sequence of real numbers the other. $\square$

The effectivity of Theorems 15 and 18 enables us to show that our notion of computable sequence coincides with the notion introduced in [29].

Theorem 20 *A sequence $(x_n)_{n\in\mathbb{N}}$ is computable if and only if there is a recursive function $r\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that $|q_{r(n,k)} - x_n| \leq 2^{-k}$ for all $n, k \in \mathbb{N}$.*

PROOF. Assume that $(x_n)_{n\in\mathbb{N}}$ is a sequence of reals and that $r\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ is recursive such that $|q_{r(n,k)} - x_n| \leq 2^{-k}$ for all $n, k \in \mathbb{N}$. This means that $(q_{r(n,k)})_{k\in\mathbb{N}}$ is a computable sequence effectively convergent to x_n with the identity function as modulus of convergence. So we can apply Theorem 18 to effectively get an index for x_n from r and n. This means that the sequence x_n is computable.

Now assume that there is $f\colon \mathbb{N} \to \mathbb{N}$ with $\xi_{\mathcal{I}}(f(n)) = \{x_n\}$. By Theorem 15 there is a recursive function $h\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that the sequence $(q_{h(f(n),k)})_{k\in\mathbb{N}}$ is fast converging with limit x_n. Thus we have a double sequence as required. $\square$

It is another immediate consequence of effectivity of Theorem 18 that Theorem 20 generalizes to double sequences of *real* numbers. A double sequence $(x_{n,k})_{n,k\in\mathbb{N}}$ of real numbers is said to be computable, if there is $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ recursive such that $\{x_{n,k}\} = \xi_{\mathcal{I}}(h(n,k))$ for all $n, k \in \mathbb{N}$.

Corollary 21 (Proposition 0-1 of [29]) *Suppose $(y_n)_{n\in\mathbb{N}}$ is a sequence of real numbers. If there is a computable double sequence $(x_{n,k})_{n,k\in\mathbb{N}}$ such that $|x_{n,k} - y_n| \leq 2^{-k}$ for all $n, k \in \mathbb{N}$, then the sequence $(y_n)_{n\in\mathbb{N}}$ is computable.*

4.3 Computability on $\mathbb{R}^n$ and $\mathbb{C}$

It is clear that the nth power $\mathcal{I}^n$ of the interval domain may serve as computational model for the Euclidean space $\mathbb{R}^n$. The elements of $\mathcal{I}^n$ are n-tuples of intervals. The function

$$(A_1, \ldots, A_n) \mapsto A_1 \times \cdots \times A_n$$

gives an isomorphism between $\mathcal{I}^n$ and the set

$$\{A_1 \times \cdots \times A_n \mid A_i \in \mathcal{I}\}$$

containing all n-dimensional rectangles whose edges are either compact intervals or the entire real line $\mathbb{R}$. In this setting, again, the order $\sqsubseteq$ is reversed subset inclusion and $\ll$ is reversed inclusion-in-the-interior. The set of rectangles with rational corner points serves as a basis for the domain. The enumeration $J_0, J_1, J_2, \ldots$ of this basis is defined via the n-tupling function

$$(i_1, \cdots, i_n) \mapsto \langle i_1, \ldots, i_n \rangle : \mathbb{N}^n \to \mathbb{N}$$

with corresponding projections $\pi_i^n : \mathbb{N} \to \mathbb{N}$. We set

$$J_{\langle i_1, \ldots, i_n \rangle} = I_{i_1} \times \cdots \times I_{i_n}.$$

All the results from Section 4.2 readily generalize to the n-dimensional case. Occurrences of the absolute value $|x|$ have to be replaced (where appropriate) by, for example, the ∞-norm

$$\|(x_1, \ldots, x_n)\| = \max(x_1, \ldots, x_n)$$

which is equivalent to the Euclidean norm. Width of intervals has to be replaced by the maximum width of the sides of the rectangles.

Having dealt with $\mathbb{R}^n$ we of course get immediately a computability theory for the complex plane $\mathbb{C}$, via the identification $\mathbb{C} = \mathbb{R}^2$.

5 Computable real functions

The domain theoretic notion of computable function gives rise to a natural definition of computable function on the reals.

Definition 22 *A function $f: \mathbb{R} \to \mathbb{R}$ is* computable *iff there is an extension $g: \mathcal{I} \to \mathcal{I}$ (i.e. $g(\{x\}) = \{f(x)\}$ for all $x \in \mathbb{R}$) which is computable in the sense of Section 3.*

Employing Corollary 11 and the fact that the restriction of the Scott-topology on $\mathcal{I}$ to the set of maximal elements coincides with the usual topology of $\mathbb{R}$, we immediately get:

Proposition 23 *Every computable function $\mathbb{R} \to \mathbb{R}$ is continuous. A continuous function is computable if and only if it maps computable sequences of intervals to computable sequences of intervals.*

We will show that this notion of computable function coincides with the classical notion of Grzegorczyk and Lacombe [18,21].

5.1 Equivalence with the classical notion

For the classical definition of computable real function, we use the re-formulation of [19] which Pour-El and Richards employ as a starting point for a detailed treatment of computable analysis [29].

Definition 24 *A function $f: \mathbb{R} \to \mathbb{R}$ is* PR-computable *iff (1) it maps computable sequences to computable sequences and (2) it is effectively uniformly continuous on intervals $[-n, n]$, i.e. there is a recursive function $h: \mathbb{N}^2 \to \mathbb{N}$ such that $|x - y| \le \frac{1}{2^{h(n,k)}}$ & $x, y \in [-n, n]$ implies $|f(x) - f(y)| \le \frac{1}{2^k}$ for all $n, k \in \mathbb{N}$ and $x, y \in \mathbb{R}$.*

Proposition 25 *Suppose $A \subseteq \mathbb{R}$ is a compact interval and $f: A \to \mathbb{R}$ is continuous. Then every Scott-continuous extension $g: \mathcal{I}A \to \mathcal{I}$ of f satisfies the following property: For every $\varepsilon > 0$ there is $\delta > 0$ such that $|B| < \delta$ implies $|g(B)| < \varepsilon$ for all compact intervals $B \subseteq A$. Moreover, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$ with ε and δ as above.*

PROOF. Suppose $\varepsilon > 0$. For each $x \in A$, the function g is continuous at $\{x\}$. This means that there is $\delta_x > 0$ such that $B \subseteq (x - \delta_x, x + \delta_x)$ implies $g(B) \subseteq (f(x) - \frac{\varepsilon}{2}, f(x) + \frac{\varepsilon}{2})$ for all $B \in \mathcal{I}A$. Now the interval A is entirely contained in the infinite union $\bigcup_{x \in A}(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$. So, by compactness, there is a finite set $\{x_1, x_2, \ldots, x_n\}$ such that

$$A \subseteq (x_1 - \frac{\delta_{x_1}}{2}, x_1 + \frac{\delta_{x_1}}{2}) \cup (x_2 - \frac{\delta_{x_2}}{2}, x_2 + \frac{\delta_{x_2}}{2}) \cup \cdots \cup (x_n - \frac{\delta_{x_n}}{2}, x_n + \frac{\delta_{x_n}}{2}).$$

Let $\delta := \min(\frac{\delta_{x_1}}{2}, \frac{\delta_{x_2}}{2}, \ldots, \frac{\delta_{x_n}}{2})$. Suppose $|B| < \delta$. Then there is $i \in \{1, 2, \ldots, n\}$ with $B \subseteq (x_i - \delta_{x_i}, x_i + \delta_{x_i})$. Hence $g(B) \subseteq (f(x_i) - \frac{\varepsilon}{2}, f(x_i) + \frac{\varepsilon}{2})$. In particular $|g(B)| < \varepsilon$, so the first part of the proposition is proved. For the second part, it suffices to remark that $f([a, b]) \subseteq g([a, b])$ holds for all $a, b \in A$. $\square$

Theorem 26 *Every computable function is PR-computable.*

PROOF. Suppose $f: \mathbb{R} \to \mathbb{R}$ is computable with witnessing continuous computable extension $g: \mathcal{I} \to \mathcal{I}$. By Proposition 23 and Theorem 20, the function f maps computable sequences to computable sequences. We will obtain an effective modulus of continuity for f by using Proposition 25. Suppose $n \in \mathbb{N}$. Effectively construct a recursive function $j_n: \mathbb{N} \to \mathbb{N}$ such that

$$q_{\pi_1(j_n(2^k+\ell)-1)} = -n + 2n \cdot \frac{\ell}{2^k} \text{ and } q_{\pi_2(j_n(2^k+\ell)-1)} = \frac{2n}{2^k},$$

i.e. such that

$$I_{j_n(2^k+\ell)} = \left[-n + 2n \cdot \frac{\ell-1}{2^k}, -n + 2n \cdot \frac{\ell+1}{2^k}\right]$$

for all $k \in \mathbb{N}$ and $\ell = 1, \dots, 2^k - 1$. Then define the relation R to consist of all triples $\langle n, k, i \rangle$ such that

$$\forall \ell \in \{1, \dots, 2^i - 1\} \exists \ell' \in \mathbb{N}.\ I_{\ell'} \ll g(I_{j_n(2^i+\ell)}) \;\&\; |I_{\ell'}| < \frac{1}{2^k}. \tag{4}$$

This relation is r.e. as the set over which ℓ ranges is finite. We now claim that

$$\begin{aligned} &\langle n, k, i \rangle \in R \\ &\Longrightarrow \left(\forall x, y \in [-n, n].\ |x - y| \le \frac{2n}{2^i} \Longrightarrow |g([x, y])| < \frac{1}{2^k}\right) && (5) \\ &\Longrightarrow \langle n, k, i+1 \rangle \in R. && (6) \end{aligned}$$

To see the first implication, assume $\langle n, k, i \rangle \in R$ and $x, y \in [-n, n]$ with $|x - y| \le \frac{2n}{2^i}$. The intervals $I_{j_n(2^i+\ell)}$ for $\ell = 1, \cdots, 2^i$ cover $[-n, n]$ and have an overlap of width $\frac{n}{2^{i-1}}$. Thus there is ℓ with $x, y \in I_{j_n(2^i+\ell)}$. By definition of R, there is ℓ' such that $I_{\ell'} \ll g(I_{j_n(2^i+\ell)})$ and $|I_{\ell'}| < \frac{1}{2^k}$. In particular, this implies $|g([x, y])| < \frac{1}{2^k}$ as required. For the second implication, assume that the formula in (5) holds. The interval $I_{j_n(2^{i+1}+\ell)}$ has width $\frac{2n}{2^i}$ hence its image under g has width smaller than $\frac{1}{2^k}$. This implies that there is $\ell' \in \mathbb{N}$ with $|I_{\ell'}| < \frac{1}{2^k}$ and such that $g(I_{j_n(2^{i+1}+\ell)})$ is contained in the interior of $I_{\ell'}$. Hence $\langle n, k, i+1 \rangle \in R$.

By the implication in (6), continuity of f on intervals $[-n, n]$, and Proposition 25, for all $n, k \in \mathbb{N}$ there is $i \in \mathbb{N}$ such that $\langle n, k, i \rangle \in R$. Hence, by the Selection Theorem, there is $h: \mathbb{N}^2 \to \mathbb{N}$ total recursive such that $\langle n, k, h(n, k) \rangle \in R$ for all $n, k \in \mathbb{N}$. This function is an effective modulus of continuity for f by the implication in (5) and Proposition 25. □

We now consider maxima of PR-computable functions. Pour-El and Richards show that the sequence of maxima of a computable sequence of functions on a fixed interval is a computable sequence of real numbers [29, Theorem 0.7]. Their proof can be adopted to a similar situation: the sequence of maxima of a fixed function on a sequence of intervals.

Proposition 27 *If $f: \mathbb{R} \to \mathbb{R}$ is PR-computable and $h: \mathbb{N} \to \mathbb{N} \setminus \{0\}$ is recursive, then the sequence $(\max(f(I_{h(n)})))_{n\in\mathbb{N}}$ is a computable sequence of real numbers.*

PROOF. As $h(n) > 0$ we have $I_{h(n)} \neq \mathbb{R}$ for all $n \in \mathbb{N}$; so the sequence is well-defined. We assume that $g: \mathbb{N}^2 \to \mathbb{N}$ is the recursive modulus of continuity for f, i.e. that

$$|x - y| \leq \frac{1}{2^{g(n,k)}} \ \&\ x, y \in [-n, n] \Longrightarrow |f(x) - f(y)| \leq \frac{1}{2^k} \tag{7}$$

for all $n, k \in \mathbb{N}$. There is $j: \mathbb{N} \to \mathbb{N}$ total recursive such that

$$I_{h(n)} \subseteq [-j(n), j(n)] \tag{8}$$

for all $n \in \mathbb{N}$. Define $\alpha: \mathbb{N}^2 \to \mathbb{N}$ total recursive such that

$$\alpha(n, k) \geq 2^{g(j(n),k)} \cdot 2|q_{\pi_2(h(n)-1)}|,$$

then

$$\frac{|I_{h(n)}|}{\alpha(n, k)} \leq \frac{1}{2^{g(j(n),k)}}. \tag{9}$$

Now set

$$s_{n,k} = \max\{f(\underbrace{q_{\pi_1(h(n)-1)} + \frac{i}{\alpha(n, k)}|I_{h(n)}|}_{:=x_{n,k,i}}) \mid 1 \leq i \leq \alpha(n, k)\}.$$

Then $(s_{n,k})_{n,k\in\mathbb{N}}$ is a computable double sequence of real numbers. Suppose f attends its maximum on $I_{h(n)}$ at $\hat{x} \in I_{h(n)}$. There is $i \in \{1, \ldots, \alpha(n, k)\}$ such that $|\hat{x} - x_{n,k,i}| \leq \frac{|I_{h(n)}|}{\alpha(n,k)}$. Hence

$$|\hat{x} - x_{n,k,i}| \leq \frac{1}{2^{g(j(n),k)}} \tag{10}$$

by (9). Combining (7) with (8) and (10) yields $|f(\hat{x}) - f(x_{n,k,i})| \leq \frac{1}{2^k}$. But $f(x_{n,k,i}) \leq s_{n,k} \leq f(\hat{x})$ by definition of $s_{n,k}$ and $\hat{x}$. Thus $|\max(f(I_{h(n)})) - s_{n,k}| \leq \frac{1}{2^k}$ and the sequence is computable by Corollary 21. $\square$

Lemma 28 *If $(x_n)_{n\in\mathbb{N}}$ is a computable sequence of real numbers then the predicate*

$$x_n < q_m$$

is r.e. in n and m.

PROOF. There is a recursive function $c: \mathbb{N}^2 \to \mathbb{N}$ such that

$$|q_{c(n,i)} - x_n| \leq \frac{1}{2^i} \text{ for all } i \in \mathbb{N}. \tag{11}$$

Now $x_n < q_m$ holds iff there is $i \in \mathbb{N}$ with

$$x_n + \frac{1}{2^i} < q_m. \tag{12}$$

Using (11), we see that (12) implies $q_{c(n,i+1)} + \frac{1}{2^{i+1}} < q_m$. On the other hand, $q_{c(n,i)} + \frac{1}{2^i} < q_m$ certainly implies $x_n < q_m$. So we have

$$x_n < q_m \iff \exists i \in \mathbb{N}.\ q_{c(n,i)} + \frac{1}{2^i} < q_m$$

and the latter predicate is r.e. $\square$

Theorem 29 *Every PR-computable function is computable.*

PROOF. Suppose $f: \mathbb{R} \to \mathbb{R}$ is PR-computable. Denote by $\bar{f}: \mathcal{I} \to \mathcal{I}$ the largest strict extension of f to the interval domain, i.e. $\bar{f}(\bot) = \bot$ and $\bar{f}(J) = f(J)$ is the forward image of J under f for all $J \in \mathcal{I}$. Now

$$\begin{aligned} I_m \ll \bar{f}(I_n) &\iff m = 0 \text{ or } (n > 0 \;\&\; [\min f(I_n), \max f(I_n)] \subseteq \mathsf{int}(I_m)) \\ &\iff m = 0 \text{ or } \Big(n, m > 0 \quad \& \\ &\qquad\qquad\qquad \min f(I_n) > q_{\pi_1(m-1)} - |q_{\pi_2(m-1)}| \quad \& \\ &\qquad\qquad\qquad \max f(I_n) < q_{\pi_1(m-1)} + |q_{\pi_2(m-1)}|\Big). \end{aligned}$$

This relation is r.e. by Proposition 27 and Lemma 28. Hence f is computable. $\square$

Theorems 26 and 29 show that our approach is equivalent to the one by Pour-El & Richards and therefore to that of TTE and Grzegorczyk's definition:

Corollary 30 *A function is computable if and only if it is PR-computable.*

In conjunction with Proposition 23, this yields a novel characterisation of PR-computability which sheds new light on the difference with the computability notion of Banach-Mazur [22], also known as sequential computability (A function is computable in this sense iff it preserves computability of sequences of real numbers.)

Corollary 31 *A continuous function is PR-computable if and only if it maps computable sequences of intervals to computable sequences of intervals.*

Corollary 32 *If a function $f : \mathcal{I} \to \mathcal{I}$ is computable, strict, and maps maximal elements to maximal elements then the largest strict continuous function above f is computable.*

5.2 Partial functions

For $f : \mathcal{I} \to \mathcal{I}$ we let

$$D_f = \{x \in \mathbb{R} \mid f(\{x\}) \in \mathsf{max}(\mathcal{I})\}$$

be the domain of the associated partial function on the real line. Our aim is to characterize the partial computable functions of reals in the spirit of [29]. As a starting point, we derive from Corollary 11

Proposition 33 *If $f : \mathcal{I} \to \mathcal{I}$ is computable and $(x_n)_{n\in\mathbb{N}}$ is a computable sequence of reals in D_f then the sequence $(f(x_n))_{n\in\mathbb{N}}$ is computable.*

Proposition 34 *Suppose $f : \mathcal{I} \to \mathcal{I}$ is computable. Then there is a recursive function ψ such that $\psi(n)$, for $n \geq 1$, is the index for a modulus of continuity on I_n if and only if $I_n \subseteq D_f$.*

PROOF. This is essentially the same as the second part of the proof of Theorem 26. It is a little harder as the boundaries are more complicated. With the abbreviations $a_n = q_{\pi_1(n)-1} - |q_{\pi_2(n)-1}|$ (i.e. $I_n = [a_n, a_n + 2|q_{\pi_2(n)-1}|]$), we define $j_n : \mathbb{N} \to \mathbb{N}$ for $n > 0$ in such a way that

$$q_{\pi_1(j_n(2^k+\ell)-1)} = a_n + \ell\frac{|I_n|}{2^k} \text{ and } q_{\pi_2(j_n(2^k+\ell)-1)} = \frac{|I_n|}{2^k},$$

i.e. such that

$$I_{j_n(2^k+\ell)} = \Big[a_n + (\ell - 1)\frac{|I_n|}{2^k}, a_n + (\ell + 1)\frac{|I_n|}{2^k}\Big]$$

for all $k \in \mathbb{N}$ and $\ell = 1, \ldots, 2^k - 1$. The resulting function $h(n,k)$ will be total for fixed n if and only if $I_n \subseteq D_f$. $\square$

Almost the same is possible for computable intervals. We only have to take care to exclude all indices for the least element of $\mathcal{I}$.

Proposition 35 *Suppose $f : \mathcal{I} \to \mathcal{I}$ is computable, Then there is a recursive function ψ such that whenever $\xi_{\mathcal{I}}(n) \neq \bot$ the number $\psi(n)$ is the index for a modulus of continuity on $\xi_{\mathcal{I}}(n)$ if and only if $\xi_{\mathcal{I}}(n) \subseteq D_f$.*

PROOF. Again, the proof is essentially the same as before. We employ Proposition 8 to prove that the relation in (4) is r.e. $\square$

This treatment of partial functions leaves a number of open problems, e.g. the following:

- Is the domain of a partial computable real function a union of computable intervals? Is this union effective?
- The converse: Suppose f is a partial real function with an effective union of computable intervals as domain, effectively uniformly continuous on these intervals and such that f maps computable sequences to computable sequences. Is there a computable extension of f to the interval domain? Is there a largest strict extension? Is this extension computable?

5.3 Computable functions on $\mathbb{R}^n$

A characterization of computable functions $\mathbb{R}^n \to \mathbb{R}$ similar to Corollary 30 holds. In particular this implies that the notion coincides with Pour-El & Richards' definition [29]. The proof becomes slightly more complicated: In Theorem 26 we constructed an effective modulus of continuity by covering the interval $[-m, m]$ with several "layers" of intervals. The kth layer consisted of $2^{k+1} - 1$ intervals which had width $\frac{2m}{2^k}$ and overlap $\frac{m}{2^k}$. In n dimensions, we have to cover $[-m, m]^n$ by layers of n-dimensional rectangles. The edges are taken from the set of intervals from the 1-dimensional case, so there will be $(2^{k+1} - 1)^n$ rectangles. Width and overlap are the same as in the 1-dimensional case. In this way, we obtain an effective construction which is more involved than for the real line.

Likewise, functions $\mathbb{C} \to \mathbb{R}$ can be dealt with. Functions $\mathbb{C} \to \mathbb{C}$ are split into real and imaginary part which are investigated separately.

6 Real number representations

By Corollary 19, a real number x is computable iff the set $\{x\}$ is the lub of an effective ascending chain in $\mathcal{I}$, that is iff it is the intersection of an effective chain of shrinking nested intervals. We now give an example of how to obtain such shrinking interval sequences.

Let us, for simplicity, consider a computable interval A. An *iterated function system (IFS)* on A is a finite set of computable functions $f_i\colon A \to A$, $i \in K$, where K is some finite indexing set. We will assume the functions to be contracting (so that the IFS is *hyperbolic*) and computable. Then for every total recursive function $h\colon \mathbb{N} \to K$ the sequence

$$\begin{aligned}
J_0 &:= A \\
J_1 &:= f_{h(0)}(A) \\
J_2 &:= f_{h(0)}(f_{h(1)}(A)) \\
&\vdots \\
J_n &:= f_{h(0)}(f_{h(1)}(\cdots f_{h(n)}(A)\cdots)) \\
&\vdots
\end{aligned}$$

is a shrinking sequence of intervals and defines a computable element x_h of A with $\{x_h\} = \bigcap_{n\in\mathbb{N}} J_n$. The interval J_n contains x_h and is regarded as an approximation to this point.

If we further specialise this example to the IFS with $A = [0,1]$, $K = \{0,\ldots,9\}$ and

$$f_i(x) = \frac{x+i}{10},\ i = 0,\ldots,9,$$

then the sequence $h(0), h(1), \ldots$ is exactly the decimal expansion of the real number x_h in the intersection of intervals. As a concrete example, take h such that $h(i)$ is the ith decimal place of the number π. Then

$$J_0 := [0,1]$$

$$\begin{aligned}
J_1 &:= f_3([0,1]) = [0.3, 0.4] \\
J_2 &:= f_3(f_1([0,1])) = f_3([0.1, 0.2]) = [0.31, 0.32] \\
J_3 &:= f_3(f_1(f_4([0,1]))) = f_3(f_1([0.4, 0.5])) = f_3([0.14, 0.15]) = [0.314, 0.315] \\
&\ \vdots
\end{aligned}$$

Note that J_{n+1} is *not* obtained by applying $f_{h(n)}$ to J_n.

Similarly, expansions with respect to bases different from 10 can be expressed via IFS's. Also, one can choose the f_i to have overlapping ranges, yielding various signed digit representations. For example $K = \{-1, 0, 1\}$ and

$$f_i(x) = \frac{x+i}{2}, \; i = -1, 0, 1$$

for the interval $A = [-1, 1]$ gives the signed binary expansion, where $h(i) = k$ corresponds to digit k at the ith place.

This IFS framework can be used to represent more sophisticated number systems such as the redundant base 2 fixed-point digit-serial numbers [24], where real numbers are generated by three functions, and exact floating point [27,13] where extended real numbers are generated by the composition of one of four *sign functions* followed by an infinite product of *digit functions* chosen from a finite set of maps.

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