Domain-theoretic Solution of Differential Equations (Scalar Fields)

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Abstract

We provide an algorithmic formalization of ordinary differential equations in the framework of domain theory. Given a Scott continuous, interval-valued and time-dependent scalar field and a Scott continuous initial function consistent with the scalar field, the domain-theoretic analogue of the classical Picard operator, whose fix-points give the solutions of the differential equation, acts on the domain of continuously differentiable functions by successively updating the information about the solution and the information about its derivative. We present a linear and a quadratic algorithm respectively for updating the function information and the derivative information on the basis elements of the domain. In the generic case of a classical initial value problem with a continuous scalar field, which is Lipschitz in the space component, this provides a novel technique for computing the unique solution of the differential equation up to any desired accuracy, such that at each stage of computation one obtains two continuous piecewise linear maps which bound the solution from below and above, thus giving the precise error. When the scalar field is continuous and computable but not Lipschitz, it is known that no computable classical solution may exist. We show that in this case the interval-valued domain-theoretic solution is computable and contains all classical solutions. This framework also allows us to compute an interval-valued solution to a differential equation when the initial value and/or the scalar field are interval-valued, i.e. imprecise.

1. Introduction

We use domain theory [17, 2] and in particular the domain-theoretic model for differential calculus presented in [7], to synthesize Differential Equations, introduced by Newton and Leibnitz in the 17th century, and the modern science of Computability and the Theory of Algorithms and Data Structures developed in the 20th century.

The question of computability of the solutions of differential equations has been generally studied in the school of computable analysis pioneered by Grzegorczyk [9, 10, 1, 5, 12, 16]. As far as the general theoretical issues of computability are concerned, the domain-theoretic approach is equivalent to this traditional one [18]. We will however use the domain-theoretic model to develop an algorithmic formalization of differential equations, i.e. to provide proper data structures which support tractable and robust algorithms for solving differential equations. The established numerical techniques for solving ordinary differential equations, such as the Euler and the Runge-Kutta methods, all suffer from the major problem that their error estimation is too conservative to be of any practical use [13, Section 3.5 and page 127] and [11, page 7]. We show in this paper that, based on the domain-theoretic model, we can get realistic lower and upper bounds for the solution at each stage of computation.

The classical initial value problem for a scalar field is of the form,

$$\dot{x} = v(t,x) \ , \qquad x(t_0) = x_0,$$

where $\dot{x} = \frac{dx}{dt}$ and $v : O \to \mathbb{R}$ is a continuous, time-dependent scalar field in a neighbourhood $O \subset \mathbb{R}^2$ with $(t_0, x_0) \in O$. If v is Lipschitz in its second argument uniformly in the first argument, then Picard's theorem establishes that there exists a unique solution $h : T \to \mathbb{R}$ to the initial value problem, satisfying $h(t_0) = x_0$, in a neighbourhood $T = [t_0 - \delta, t_0 + \delta]$ of t_0 for some $\delta > 0$. The unique solution will be the unique fixed point of the Picard functional $P : C^0(T) \to C^0(T)$ [1] defined by $P : f \mapsto \lambda t.x_0 + \int_{t_0}^{t} v(u, f(u))du$. The operator P was reformulated in [7] as the composition of two operators $U, A_v : (C^0(T))^2 \to (C^0(T))^2$ on pairs (f, g), where f gives approximation to the solution and g gives approximation to the derivative of the solution:

$$U(f,g) = (\lambda t.(x_0 + \int_{t_0}^{t} g(u)\, du), g) \ ,$$

$$A_v(f,g) = (f, \lambda t.v(t, f(t))).$$

The map A_v updates the information on the derivative using the information on the function and U updates the information on the function itself using the derivative information. We have $P(f) = \pi_0(U \circ A_v(f,g))$, for any g, where π_0 is projection to the first component. The unique fix-point (h, g) of $U \circ A_v$ will satisfy: $h' = g = \lambda t.v(t, h(t))$, where h' is the derivative of h.

We consider Scott continuous, interval-valued and time dependent scalar fields of the form $v : [0,1] \times \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$, where $\mathbf{I}\mathbb{R}$ is the domain of non-empty compact intervals of $\mathbb{R}$, ordered by reverse inclusion and equipped with a bottom element [2]. Such set-valued scalar fields have also been studied under the name of upper semi-continuous, compact and convex set-valued vector fields in the theory of differential inclusions and viability theory [3], which have become an established subject in the past decades with applications in control theory [4]. Our work also aims to bridge differential equations and computer science by connecting differential inclusions with domain theory. It can also be considered as a new direction in interval analysis [14].

In [7], three ingredients that are fundamental bases of this paper were presented: (i) a domain for continuously differentiable functions, (ii) a Picard-like operator acting on the domain for continuously differentiable functions, which is composed of two operators as in the classical case above one for function updating and one for derivative updating, and finally (iii) a domain-theoretic version of Picard's theorem.

Here, a complete algorithmic framework for solving general initial value problems will be constructed. We will develop explicit domain-theoretic operations and algorithms for function updating and derivative updating and seek the least fixed point of the composition of these two operations, which refines a given Scott continuous *initial function* consistent with the vector field. We show that this least fixed point is computable when the initial function is computable. The classical initial value problem can be solved in this framework by working with the canonical extension of the scalar field to the domain of intervals and a canonical domain-theoretic initial value. This gives a novel technique for solving the classical initial value problem. It is distinguished by the property that, at each stage of iteration, the approximation is bounded from below and above by continuous piecewise linear functions, which gives a precise error estimate. The framework also enables us to solve differential equations with imprecise (partial) initial condition or scalar field.

When the scalar field is continuous and computable but not Lipschitz, no computable classical solution may exist [15]. We show that in this case the interval-valued domain-theoretic solution is computable and contains all classical solutions.

2. Background

We will first outline the main results from [7] that we require in this paper. Consider the function space $D^0[0,1] = ([0,1] \to \mathbf{I}\mathbb{R})$ of interval-valued function on $[0,1]$ that are continuous with respect to the Euclidean topology on $[0,1]$ and the Scott topology of $\mathbf{I}\mathbb{R}$. [3] We often write D^0 for $D^0[0,1]$. With the ordering induced by $\mathbf{I}\mathbb{R}$, D^0 is a continuous Scott domain. For $f \in D^0$ the lower semi-continuous function $f^- : [0,1] \to \mathbb{R}$ and the upper semi-continuous function $f^+ : [0,1] \to \mathbb{R}$ are given by $f(x) = [f^-(x), f^+(x)]$ for all $x \in [0,1]$. We denote the set of real-valued continuous function on $[0,1]$ with the sup norm by $C^0[0,1]$ or simply C^0. The topological embedding $\Gamma^0 : C^0 \to D^0$ given by $\Gamma(f)(x) = \{f(x)\}$ allows us to identify the map $f \in C^0$ with the map $x \mapsto \{f(x)\}$ in D^0. For an open subset $O \subset [0,1]$ and a non-empty compact interval $b \in \mathbf{I}\mathbb{R}$, the single-step function $O \searrow b : [0,1] \to \mathbf{I}\mathbb{R}$ is given by:

$$(O \searrow b)(x) = \begin{cases} b & x \in O \\ \perp & x \notin O \end{cases}$$

Given two constant (respectively, linear) functions $f^-, f^+ : a \to \mathbb{R}$ with $f^- \leq f^+$ on a, the *standard (respectively, linear) single-step function* $a \searrow [f^-, f^+] : [0,1] \to \mathbf{I}\mathbb{R}$ is defined, for a compact interval $a \subseteq [0,1]$, by

$$(a \searrow [f^-, f^+])(x) = \begin{cases} [f^-(x), f^+(x)] & x \in a^\circ \\ \perp & x \notin a^\circ \end{cases}$$

The collection of lubs of finite and consistent standard (respectively, linear) single-step functions as such,

[1] Here, $C^0(T) = T \to \mathbb{R}$ is the set of real-value continuous functions on T with the sup norm.

[2] This problem is equivalent to the case when the scalar field is of type $v : [0,1] \times \mathbf{I}O \to \mathbf{I}\mathbb{R}$.

[3] Note that in [7], the following different notations were used $D^0[0,1] := \mathbf{I}[0,1] \to \mathbf{I}\mathbb{R}$ and $D^0_r[0,1] := [0,1] \to \mathbf{I}\mathbb{R}$.

when a is a rational compact interval and f^- and f^+ are rational constant (respectively, linear) maps, forms a basis for D^0, which we call the *standard (respectively, the linear)* basis. Sometimes, we work with the *semi-rational polynomial basis* which is obtained as above when f^-, f^+ are polynomials with rational coefficients except possibly the constant term which is assumed to be algebraic. We denote the number of single-step functions in a step function f by $\mathcal{N}_f$. Each standard (respectively, linear or polynomial) step function $g \in D^0$ induces a partition of $[0,1]$ such that g is constant (respectively, linear or polynomial) in the interior of each subinterval of the partition; we call it the partition induced by g. If g_1 and g_2 are step functions then we call the common refinement of the partitions induced by g_1 and g_2 simply the partition induced by g_1 and g_2.

The *indefinite* integral map $\int : D^0 \to (\mathsf{P}(D^0), \supseteq)$, where $\mathsf{P}(D^0)$ is the power set of D^0, is defined on a single-step function by $\int a \searrow b = \delta(a,b)$ where

$$\delta(a,b) =$$
$$\{f \in D^0 \mid \forall x, y \in a^\circ.\ b(x-y) \sqsubseteq f(x) - f(y)\}$$

and is extended by continuity to any Scott continuous function given as the lub of a bounded set of single-step functions:

$$\int \bigsqcup_{i \in I} a_i \searrow b_i = \bigcap_{i \in I} \delta(a_i, b_i).$$

The derivative of $f \in D^0$ is the Scott continuous function $\frac{df}{dx} \in D^0$ defined as

$$\frac{df}{dx} = \bigsqcup_{f \in \delta(a,b)} a \searrow b : [0,1] \to \mathbf{I}\mathbb{R}.$$

The indefinite integral and the derivative are related by the relation

$$h \in \int g \iff g \sqsubseteq \frac{dh}{dx}.$$

The consistency relation $\mathsf{Cons} \subset D^0 \times D^0$ is defined by $(f,g) \in \mathsf{Cons}$ if $\uparrow f \cap \int g \neq \emptyset$. We have $(f,g) \in \mathsf{Cons}$ iff $\exists h \in D^0.\ f \sqsubseteq h$ and $g \sqsubseteq \frac{dh}{dx}$. The continuous Scott domain $D^1[0,1]$ of continuously differentiable functions is now defined as the subdomain of the consistent pairs in $D^0 \times D^0$:

$$D^1 = \{(f,g) \in D^0 \times D^0 \mid (f,g) \in \mathsf{Cons}\}\ .$$

3. Function Updating

In analogy with the map U presented in Section 1 for the classical reformulation of the Picard's technique, we have a domain-theoretic, function updating map as introduced in [7].

Let $L[0,1] := [0,1] \to \overline{\mathbb{R}}$, with $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, be the collection of partial extended real-valued functions on $[0,1]$. The functions $s : D^0 \times D^0 \to (L[0,1], \leq)$ and $t : D^0 \times D^0 \to (L[0,1], \geq)$ are defined as

$$s(f,g) = \inf\{h : \mathrm{dom}(g) \to \mathbb{R} \,|\, h \in \int g,\ h \geq f^-\}$$

$$t(f,g) = \sup\{h : \mathrm{dom}(g) \to \mathbb{R} \,|\, h \in \int g,\ h \leq f^+\}\ .$$

If $s(f,g)$ is real-valued then it is continuous and $s(f,g) \in \int g$; similarly for $t(f,g)$. We have $(f,g) \in \mathsf{Cons}$ iff $s(f,g) \leq t(f,g)$; see [7]. Figure 1 shows a consistent pair of functions. The *function updating* map $\mathrm{Up} : D^1 \to D^1$ is defined by $\mathrm{Up}(f,g) = ([s(f,g), t(f,g)], g)$ and we put $\mathrm{Up}_1(f,g) = [s(f,g), t(f,g)]$ as in [7].

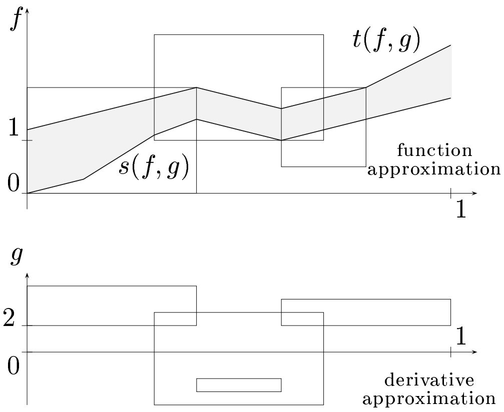

Figure 1. A pair of consistent step functions

We here derive explicit expressions for $s(f,g)$ and $t(f,g)$ on the one hand and the function updating map on the other. Let $K^{+-} : D^0 \to ([0,1]^2 \to \overline{\mathbb{R}}, \leq)$ with

$$K^{+-}(g)(x,y) = \begin{cases} \int_y^x g^-(u)\,du & x \geq y \\ -\int_x^y g^+(u)\,du & x < y \end{cases},$$

and put $S : D^0 \times D^0 \to ([0,1]^2 \to \overline{\mathbb{R}}, \leq)$ with $S(f,g)(x,y) = f^-(y) + K^{+-}(g)(x,y)$. For $h \in D^0$ we here use the convention that $h^\pm(u) = \pm\infty$ when $h(u) = \perp$. Similarly, let $K^{-+} : D^0 \to ([0,1]^2 \to \overline{\mathbb{R}}, \leq)$ with

$$K^{-+}(g)(x,y) = \begin{cases} \int_y^x g^+(u)\,du & x \geq y \\ -\int_x^y g^-(u)\,du & x < y \end{cases},$$

and put $T : D^0 \times D^0 \to ([0,1]^2 \to \overline{\mathbb{R}}, \geq)$ with $T(f,g)(x,y) = f^+(y) + K^{-+}(g)(x,y)$.

Then K^{+-}, K^{-+}, S and T are Scott continuous. In words, for a given $y \in \mathrm{dom}(g)$, the map $\lambda x.S(f,g)(x,y)$ is the least function $h : [0,1] \to \mathbb{R}$ such that $h \in \int g$ and $h(y) \geq f^-(y)$. It follows that

$$s(f,g) = \lambda x. \sup_{y \in \mathrm{dom}(g)} S(f,g)(x,y). \qquad (1)$$

Similarly,

$$t(f,g) = \lambda x. \inf_{y \in \mathrm{dom}(g)} T(f,g)(x,y). \qquad (2)$$

Theorem 3.1 *Let $f = [f^-, f^+] : [0,1] \to \mathbf{I}\mathbb{R}$ be a linear step function and $g = [g^-, g^+] : [0,1] \to \mathbf{I}\mathbb{R}$ a standard step function. Then $[s(f,g), t(f,g)]$ is a linear step function, which can be computed in finite time when f and g are basis elements.*

We first need a lemma; assume the conditions of Theorem 3.1.

Lemma 3.2 *Let O be a connected component of dom(g) and $J = \{y_i \,|\, 0 \leq i \leq n\}$ be the partition of O induced by f and g with $\underline{O} = y_0 < y_1 < \cdots < y_n = \overline{O}$. Then, for every $x \in O$ the following hold:*

$$s(f,g)(x) = \max_{y_k \in J \cap dom(f)} \{f^-(x)\} \cup \{\overline{\lim}_{y \to y_k} S(f,g)(x,y)\},$$

$$t(f,g)(x) = \min_{y_k \in J \cap dom(f)} \{f^+(x)\} \cup \{\underline{\lim}_{y \to y_k} T(f,g)(x,y)\}.$$

Proof For convenience in this proof, we will write $S(f,g)(x,y)$ simply as $S(x,y)$. Let $y \in O \cap \mathrm{dom}(f)$ and $0 \leq k \leq n$ such that $y_{k-1} \leq y \leq y_k$. We claim that for any $x \in O$ we have $S(x,y) \leq \max\{f^-(x), \overline{\lim}_{y \to y_{k-1}} S(x,y), \overline{\lim}_{y \to y_k} S(x,y)\}$, from which the result follows. Let f_k^-, g_k^-, g_k^+ denote, respectively, the restrictions of f^-, g^-, g^+ to the interval (y_{k-1}, y_k). As y_{k-1}, y_k are successive elements of the partition J, the maps g_k^- and g_k^+ are constant say with values e_k^- and e_k^+ respectively, while for some $a_k \in \mathbb{R}$ we have $f_k^-(y) = f^-(y_{k-1}^+) + a_k(y - y_{k-1})$, where $f^-(y_{k-1}^+) = \lim_{y \to y_{k-1}^+} f^-(y)$. Note that $\overline{\lim}\, f^-(y_{k-1}) \geq f^-(y_{k-1}^+)$. Thus our claim is certainly valid for $y = x$ or $y \in J$. Otherwise, we have four cases to consider:

Case 1. $(y_{k-1} < y < y_k \leq x)$ We have:

$$S(x,y) = f_k^-(y) + K^{+-}(g)(x,y) = f^-(y_{k-1}^+) + a_k(y - y_{k-1}) + \int_{y_k}^{x} g^-(u)\,du + e_k^-(y_k - y),$$

and thus:

$$\overline{\lim}_{y \to y_{k-1}} S(x,y) \geq f^-(y_{k-1}^+) + \int_{y_k}^{x} g^-(u)\,du + e_k^-(y_k - y_{k-1}),$$

$$\overline{\lim}_{y \to y_k} S(x,y) \geq f^-(y_{k-1}^+) + a_k(y_k - y_{k-1}) + \int_{y_k}^{x} g^-(u)\,du.$$

If $a_k < e_k^-$, then

$$a_k(y - y_{k-1}) + e_k^-(y_k - y) < e_k^-(y_k - y_{k-1}),$$

i.e., $S(x,y) < \overline{\lim}_{y \to y_{k-1}} S(x,y)$. On the other hand, if $e_k^- \leq a_k$, then

$$a_k(y - y_{k-1}) + e_k^-(y_k - y) \leq a_k(y_k - y_{k-1}),$$

i.e., $S(x,y) \leq \overline{\lim}_{y \to y_k} S(x,y)$.

Case 2. $(x \leq y_{k-1} < y < y_k)$ Similarly as in the previous case, but this time comparing a_k with e_k^+, we get the following: if $a_k < e_k^+$ then $S(x,y) < \overline{\lim}_{y \to y_{k-1}} S(x,y)$. If $e_k^+ \leq a_k$ then $S(x,y) \leq \overline{\lim}_{y \to y_k} S(x,y)$.

Case 3. $(y_{k-1} < y < x \leq y_k)$ We have:

$$S(x,y) = f^-(y_{k-1}^+) + a_k(y - y_{k-1}) + e_k^-(x - y),$$

$$\overline{\lim}_{y \to y_{k-1}} S(x,y) \geq f^-(y_{k-1}^+) + e_k^-(x - y_{k-1}),$$

$$f^-(x) = f^-(y_{k-1}^+) + a_k(x - y_{k-1}) \ .$$

If $a_k < e_k^-$ then

$$a_k(y - y_{k-1}) + e_k^-(x - y) < e_k^-(x - y_{k-1}) \ ,$$

i.e., $S(x,y) \leq \overline{\lim}_{y \to y_{k-1}} S(x,y)$. If, on the other hand, $e_k^- \leq a_k$ then

$$a_k(y - y_{k-1}) + e_k^-(x - y) \leq a_k(x - y_{k-1}),$$

which implies $S(x,y) \leq f^-(x)$.

Case 4. $(y_{k-1} \leq x < y < y_k)$ Finally, in this case we have:

$$S(x,y) = f^-(y_{k-1}^+) + a_k(y - y_{k-1}) + e_k^+(x - y),$$

$$\overline{\lim}_{y \to y_k} S(x,y) \geq f^-(y_{k-1}^+) + a_k(y_k - y_{k-1}) + e_k^+(x - y_k) \ ,$$

$$f^-(x) = f^-(y_{k-1}^+) + a_k(x - y_{k-1}) \ .$$

If $a_k < e_k^+$, then $a_k(y - y_{k-1}) + e_k^+(x - y) = a_k(x - y_{k-1}) + (a_k - e_k^+)(y - x) < a_k(x - y_{k-1})$, i.e., $S(x,y) \leq f^-(x)$. However, if $e_k^+ \leq a_k$ then $(y - x)(a_k - e_k^+) \leq (y_k - x)(a_k - e_k^+)$, which after rewriting and addition of $a_k(x - y_{k-1})$ implies $S(x,y) \leq \overline{\lim}_{y \to y_k} S(x,y)$. This concludes the proof for $s(f,g)$. The proof for $t(f,g)$ is similar. $\square$

Proof of Theorem 3.1. We note that, for each fixed $y \in \mathrm{dom}(g)$, the map $x \mapsto S(f,g)(x,y)$ is a piecewise linear map since g is a step function. Thus, by Lemma 3.2, $s(f,g)$ is the maximum of at most $(n+2)$ piecewise linear functions, namely, f^- and $\lambda x.\overline{\lim}_{y \to y_k} S(f,g)(x,y)$ for $y_k \in \mathrm{dom}(f)$. Finally, note that the maximum of a finite number of piecewise linear functions is piecewise linear and can be computed in finite time. □

We present a linear time algorithm for computing the function update $s(f,g)$ of a pair $(f,g) \in D^1$, where f is a linear basis element and g is a standard basis element. A similar algorithm computes $t(f,g)$.

Algorithm 3.3 *The function updating algorithm will consist of an initialisation and two main steps; see Figure 2. The initialisation process is used to get the induced partition points $\{y_0, \cdots, y_n\}$ of (f,g). Recall that on each interval (y_{k-1}, y_k), the functions g^- and g^+ are constant, with $g^-|_{(y_{k-1},y_k)} = \lambda t.e_k^-$ and $g^+|_{(y_{k-1},y_k)} = \lambda t.e_k^+$, where $e_k^-, e_k^+ \in \mathbb{R}$. Furthermore, on each interval (y_{k-1}, y_k), the map f^- has a constant slope, a_k say, i.e., $f^-|_{(y_{k-1},y_k)} = f_k^-$, with $f_k^-(x) = a_k x + b_k$.*

Input: $f, g : [0,1] \to \mathbf{I}\mathbb{R}$ *where f is a linear step function and g is a step function.*

Output: *Continuous function $s(f,g) : [0,1] \to \mathbf{I}\mathbb{R}$ which represents the least function consistent with the information from f and g.*

\# INITIALISATION
$\{y_0, \cdots, y_n\} :=$ `induced-partition-of` (f,g)
\# PART 1: LEFT TO RIGHT
$u(y_0) := f^-(y_0^+)$
for $k = 1 \ldots n$ and $\forall x \in [y_{k-1}, y_k)$
  $u(x) := \max\{\, f^-(x),\ u(y_{k-1}) + (x - y_{k-1})e_k^- \}$
  $u(y_k) := \max\{\, \overline{\lim}\, f^-(y_k),\ u(y_{k-1}) + (y_k - y_{k-1})e_k^- \}$
\# PART 2: RIGHT TO LEFT
$s(y_n) := u(y_n)$
for $k = n \ldots 1$ and $\forall x \in [y_{k-1}, y_k)$
  $s(f,g)(x) := \max\{\, u(x),\ s(y_k) + (x - y_k)e_k^+ \}$

Correctness: *In the first stage, we compute: $u(x) = \max_{y_k \le x}\{\, \underline{f^-}(x),\ \overline{\lim}\, S(x, y_k)\,\}$. Let $v(x) = \max_{y_k \ge x}\{\, f^-(x),\ \overline{\lim}\, S(x, y_k)\,\}$. By Lemma 3.2, it follows that: $s(f,g)(x) = \max\{u(x), v(x)\}$, which is precisely the output of the second stage.*

Complexity: *Computing $\overline{\lim}\, f^-(y_k)$ consists of calculating linear functions f_{k-1} and f_k at y_k. Determining $\max\{\, f^-(x),\ u(y_{k-1}) + (x - y_{k-1})e_k^- \}$ and $\max\{\, u(x),\ s(y_k) + (x - y_k)e_k^+ \}$, is simply finding the maximum of two linear functions. Therefore, the algorithm is linear in the number of induced partition points of (f,g), thus linear in $O(\mathcal{N}_f + \mathcal{N}_g)$.*

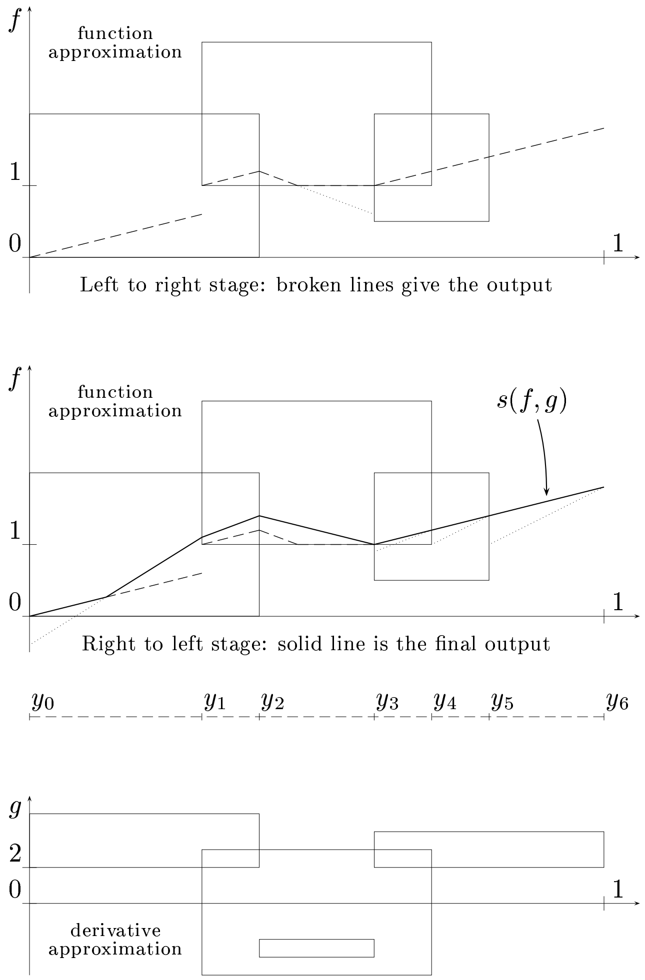

Figure 2. The function updating algorithm

4. Derivative Updating

We now consider a Scott continuous, time-dependent and interval-valued scalar field $v : [0,1] \times \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$. In analogy with the classical map A_v presented in Section 1 for the classical reformulation of the Picard's technique, we define the Scott continuous map $A : ([0,1] \times \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}) \times D^0 \to D^0$ with $A(v,f) = \lambda t.\, v(t, f(t))$ and put $A_v : D^0 \to D^0$ with $A_v(f) := A(v,f)$. The *derivative updating* map for v is now defined as the Scott continuous function

$$\mathsf{Ap} : ([0,1] \times \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}) \times D^0 \times D^0 \to D^0 \times D^0$$

with $\mathsf{Ap}(v,(f,g)) = (f, A(v,f))$ and we put

$$\mathsf{Ap}_v : D^0 \times D^0 \to D^0 \times D^0$$

with $\mathsf{Ap}_v(f,g) = \mathsf{Ap}(v,(f,g))$. The map Ap *applies* the vector field to the function approximation in order to update the derivative approximation.

Note that for a step function $(f,g) \in D^1$, the function update $\mathsf{Up}_1(f,g)$ is a linear step function. Thus, in order to compute $\mathsf{Ap}_v \circ \mathsf{Up}(f,g)$ we need to compute A_v on linear step functions. We obtain an explicit expression for $A_v(f)$ when v is given as the lub of a collection of single-step functions and f is the lub of a collection of linear single-step functions, which includes the case of standard step functions as well. Given $g = [g^-, g^+] \in D^0$ and $b = [\underline{b}, \overline{b}] \in \mathbf{I}\mathbb{R}$, we write $b \prec g$ if there exists $x \in [0,1]$ such that $b \ll g(x)$, i.e., if $g^{-1}(\Uparrow b) \neq \emptyset$. In that case, $b \ll g(x)$ for $x \in ((g^-)^{-1}(\underline{b}, \infty)) \cap ((g^+)^{-1}(-\infty, \overline{b}))$. Let $v = \bigsqcup_{i\in I} a_i \times b_i \searrow c_i$ and assume $f = \bigsqcup_{j \in J} d_j \searrow [g_j^-, g_j^+]$ is the lub of linear step functions. If $b_i \prec g_j$, then we denote by $d_{ji} \subset d_j$ the closed interval with $d_{ji}^\circ = ((g_j^-)^{-1}(\underline{b_i}, \infty)) \cap ((g_j^+)^{-1}(-\infty, \overline{b_i}))$. Thus, $b_i \ll g_j(x) \iff x \in d_{ji}^\circ$. The following result follows immediately. We write $a \Uparrow b$ if a and b are bounded above with respect to the way-below relation.

Proposition 4.1

$$\lambda t.\, v(t, f(t)) = \bigsqcup \{a_i \sqcup d_{ji} \searrow c_i \mid b_i \prec g_j, a_i \Uparrow d_{ji}\}\ . \ \square$$

Corollary 4.2 *If for $g_j = [g_j^-, g_j^+]$ the maps g_j^- and g_j^+ are constant for all $j \in J$, then denoting the constants by $e_j^-, e_j^+ \in \mathbb{R}$, we have: $\lambda t.\, v(t, f(t)) = \bigsqcup\{a_i \sqcup d_j \searrow c_i \mid b_i \ll e_j, a_i \Uparrow d_j\}$, with $e_j = [e_j^-, e_j^+]$.* $\square$

Corollary 4.3 *If v is a step function and f a linear step function then $A_v(f) = \lambda t.\, v(t, f(t))$ is step function with $\mathcal{N}_{A_v(f)} \leq \mathcal{N}_v \mathcal{N}_f$.* $\square$

The following derivative updating algorithm follows directly from Proposition 4.1.

Algorithm 4.4 *We assume that f and v are given in terms of linear and standard step functions respectively. The algorithm finds the collection of single-step functions whose lub is the derivative update $\lambda t.\, v(t, f(t))$ in $O(\mathcal{N}_v \mathcal{N}_f)$.*
Input: $A = \{a_i \times b_i \searrow c_i \mid 1 \leq i \leq n\}$ *with* $v = \bigsqcup A$, *and* $B = \{d_j \searrow [g_j^-, g_j^+] \mid 1 \leq j \leq m\}$ *with* $f = \bigsqcup B$.
Output: $C = \{a_i \sqcup d_{ji} \searrow c_i \mid b_i \prec g_j \,\&\, a_i \Uparrow d_{ji}\}$ *with* $\lambda t.\, v(t, f(t)) = \bigsqcup C$.
\# INITIALISATION
$C := \emptyset$
`for` $i = 1 \ldots n$ and $j = 1 \ldots m$
  `if` $b_i \prec g_j$ `obtain` d_{ji}
  `if` $a_i \Uparrow d_{ji}$ `put` $C := C \cup \{a_i \sqcup d_{ji} \searrow c_i\}$.

Suppose $f \in D^0$ is the initial function, which gives the initial approximation to the function component of the solution $(f_s, g_s) \in D^1$ of the fix-point equation $\mathsf{Up} \circ \mathsf{Ap}_v(f_s, g_s) = (f_s, g_s)$ that we seek, i.e. $f \sqsubseteq f_s$. Then, $\lambda t.v(t, f(t))$ is the initial approximation to the derivative component g_s of the solution, i.e. $\lambda t.v(t, f(t)) \sqsubseteq g_s$. We thus require $(f, \lambda t.v(t, f(t))) \in \mathsf{Cons}$. Furthermore, we need to ensure that for all $n \geq 1$ the iterates $(\mathsf{Up} \circ \mathsf{Ap}_v)^n(f, \lambda t.v(t, f(t)))$ of $(f, \lambda t.v(t, f(t)))$, which by monotonicity are above $(f, \lambda t.v(t, f(t))$, are consistent. This leads us to the notion of strong consistency.

4.1. Strong Consistency

The pair $(f,g) \in D^0 \times D^0$ is *strongly consistent*, written $(f,g) \in \mathsf{SCons}$, if for all $h \sqsupseteq g$ we have $(f,h) \in \mathsf{Cons}$. It was shown in [7] that the lub of a directed set of strongly consistent pairs is strongly consistent, i.e. $\mathsf{SCons} \subset D^1$ is a sub-dcpo. Strong consistency of the initial pair $(f, \lambda t.v(t, f(t)))$ will ensure that its orbit under the domain-theoretic Picard operator remains consistent.

We will establish necessary and sufficient conditions for strong consistency in a general setting and will show that on basis elements strong consistency is decidable. Assume $(f,g) \in \mathsf{Cons}$. Let $Q : D^0 \times D^0 \to ([0,1]^2 \to \overline{\mathbb{R}}_\perp, \leq)$ with $Q(f,g)(x,y) = f^-(y) + K^{-+}(g)(x,y)$. Also, put $R : D^0 \times D^0 \to ([0,1]^2 \to \overline{\mathbb{R}}_\perp, \geq)$ with $R(f,g)(x,y) = f^+(y) + K^{+-}(g)(x,y)$. Note that we use the standard convention that $\infty - \infty = \perp$ in $\overline{\mathbb{R}}_\perp$. Compare Q with S and R with T. Then Q and R are Scott continuous. We finally put $q(f,g) = \lambda x. \sup_{y \in \mathrm{dom}(g)} Q(f,g)(x,y)$ and $r(f,g) = \lambda x. \inf_{y \in \mathrm{dom}(g)} R(f,g)(x,y)$.

Proposition 4.5 *Assume $f, g \in D^0$ with g^- and g^+ continuous almost everywhere and let O be any connected component of $dom(g)$ such that $O \cap dom(f) \neq \emptyset$. Then $(f,g) \in \mathsf{SCons}$ implies $f^- \leq q(f,g) \leq f^+$ and $f^- \leq r(f,g) \leq f^+$ on $O \cap dom(f)$.*

Proof The relations $f^- \leq q(f,g)$ and $r(f,g) \leq f^+$ hold by definition. Let $g_l, g_u \in D^0$ be given by $g_l = [g^-, \overline{\lim} g^-]$ and $g_u = [\underline{\lim} g^+, g^+]$. Then we have $g \sqsubseteq g_l$ and $g \sqsubseteq g_u$, which imply $(f, g_l) \in \mathsf{Cons}$ and $(f, g_u) \in \mathsf{Cons}$. Furthermore, since g^- and g^+ are continuous a.e., it follows that $g^+ = \underline{\lim} g^+$ a.e., and $g^- = \overline{\lim} g^-$ a.e. Thus for $y \leq x$ we have:

$$\begin{aligned} f^-(y) + \int_y^x g^+(z)\, dz &= f^-(y) + \int_y^x g_u^-(z)\, dz \\ &= S(f, g_u)(x,y) \leq f^+(x). \end{aligned}$$

Similarly, for $x \leq y$:

$$\begin{aligned} f^-(y) + \int_y^x g^-(z)\, dz &= f^-(y) + \int_y^x g_l^+(z)\, dz \\ &= S(f, g_l)(x,y) \leq f^+(x). \end{aligned}$$

Thus, $q(f,g) \leq f^+$. Similarly, $f^- \leq r(f,g)$. □

Conversely, we have the following:

Proposition 4.6 *Assume $(f,g) \in D^1$ with f^-, g^- and g^+ bounded. Suppose, for each connected component O of $dom(g)$ such that $O \cap dom(f) \neq \emptyset$, we have $f^- \leq q(f,g) \leq f^+$ on $O \cap dom(f)$. Then $(f,g) \in \mathsf{SCons}$.*

Proof Take $h \sqsupseteq g$ and let O be a connected component of $\mathrm{dom}(g)$ such that $O \cap \mathrm{dom}(f) \neq \emptyset$. Let $[x_0, x_1] := \overline{O \cap \mathrm{dom}(f)}$ and put $f^-(x_i) := \underline{\lim}_{x \to x_i^-} f(x)$ for $i = 0, 1$, which are real numbers since f^- is bounded. Let $k = \sup_{x \in [x_0, x_1]} f^-(x) - (f^-(x_0) + \int_{x_0}^{x} h^+(u)\,du)$. Since f^- is bounded we have $k < \infty$. Choose a sequence $y_n \in [x_0, x_1]$ such that $a_n := k - [f^-(y_n) - (f^-(x_0) + \int_{x_0}^{y_n} h^+(u)\,du)] < 1/n$. By going to a subsequence if necessary assume, without loss of generality, that $y_n \leq y$ and $\lim_{n\to\infty} y_n = y \in [x_0, x_1]$. Define $l : O \cap \mathrm{dom}(f) \to \mathbb{R}$ by

$$l(x) = k + F(y) + \begin{cases} \int_y^x h^+(u)\,du & y \leq x \\ \int_y^x h^-(u)\,du & x \leq y \end{cases},$$

where $F(y) = f^-(x_0) + \int_{x_0}^{y} h^+(u)\,du$. We have $l \in \int h$; see [7, Lemma 3.9]. For $y \leq x$, it follows immediately from the definition of k that $l(x) \geq f^-(x)$. Since, for $x < y$, we have $\int_y^x h^-(u)\,du \geq \int_y^x h^+(u)\,du$, it follows that $l(x) \geq f^-(x)$ for $x < y$ as well. We show that $l(x) \leq f^+(x)$ for $x \in \mathrm{dom}(f) \cap O$. For $x < y$, we have:

$$\begin{aligned}
&l(x)\\
&= k + F(y) + \textstyle\int_y^x h^-(u)\,du\\
&= k + F(y) + \textstyle\int_y^x h^-(u)\,du - (f^-(y_n) + \int_{y_n}^x h^+(u)\,du)\\
&\quad + (f^-(y_n) + \textstyle\int_{y_n}^x h^+(u)\,du)\\
&= k - [f^-(y_n) - F(y_n)] - \textstyle\int_y^x h^+(u)\,du + \int_y^x h^-(u)\,du\\
&\quad + f^-(y_n) + \textstyle\int_{y_n}^x h^+(u)\,du\\
&= a_n + \textstyle\int_{y_n}^y h^+(u)\,du + f^-(y_n) + \int_y^x h^-(u)\,du\\
&= a_n + b_n + f^-(y_n) + \textstyle\int_y^x h^-(u)\,du\\
&= a_n + b_n + \textstyle\int_y^{y_n} h^-(u)\,du + f^-(y_n) + \int_{y_n}^x h^-(u)\,du,
\end{aligned}$$

where $b_n = \int_{y_n}^y h^+(u)\,du$. Since for large n we have $x < y_n$, it follows, by putting $c_n = \int_y^{y_n} h^-(u)\,du$ and using $g^- \leq h^-$, that:

$$\begin{aligned}
l(x) &\leq a_n + b_n + c_n + f^-(y_n) + \textstyle\int_{y_n}^x g^-(u)\,du\\
&\leq a_n + b_n + c_n + f^+(x),
\end{aligned}$$

as $q(f,g) \leq f^+$. Taking the limit as $n \to \infty$, we have $a_n, b_n, c_n \to 0$ since h^- and h^+ are bounded. Thus, we get: $l(x) \leq f^+(x)$. For $x \geq y$, a similar calculation with $c_n' = \int_y^{y_n} h^+(u)\,du$ yields

$$\begin{aligned}
&l(x)\\
&= a_n + b_n + c_n' + f^-(y_n) + \textstyle\int_{y_n}^x h^+(u)\,du\\
&\leq a_n + b_n + c_n' + f^-(y_n) + \textstyle\int_{y_n}^x g^+(u)\,du\\
&\leq a_n + b_n + c_n' + f^+(x).
\end{aligned}$$

And it follows again that $l(x) \leq f^+(x)$. □

Corollary 4.7 *Assume that for $f, g \in D^0$, the functions f^-, f^+, g^-, g^+ are bounded and g^-, g^+ are continuous a.e. Then $(f,g) \in \mathsf{SCons}$ iff for each connected component O of $dom(g)$ such that $O \cap dom(f) \neq \emptyset$, we have $f^- \leq q(f,g) \leq f^+$ and $f^- \leq r(f,g) \leq f^+$ on $O \cap dom(f)$.* □

Corollary 4.8 *For a pair of basis elements $f, g \in D^0$, we have $(f,g) \in \mathsf{SCons}$ iff for each connected component O of $dom(g)$ such that $O \cap dom(f) \neq \emptyset$, we have $f^- \leq q(f,g) \leq f^+$ and $f^- \leq r(f,g) \leq f^+$ on $O \cap dom(f)$.* □

Corollary 4.9 *For a pair of basis elements $f, g \in D^0$, we can test whether or not $(f,g) \in \mathsf{SCons}$ with complexity $O(\mathcal{N}_f + \mathcal{N}_g)$.*

Proof We use the two stage function updating algorithm (Algorithm 3.3) with g^- and g^+ interchanged to compute $q(f,g)$ in $O(\mathcal{N}_f + \mathcal{N}_g)$. We then test if $q(f,g) \leq f^+$ again in $O(\mathcal{N}_f + \mathcal{N}_g)$. □

The following example will show that we cannot relax the assumption that g^- and g^+ be continuous a.e. in Proposition 4.5 and Corollary 4.7. It will also show that it is not always possible to approximate a strongly consistent pair of functions by strongly consistent pairs of basis elements.

Lemma 4.10 *A continuous function $[g^-, g^+] \in D^0$ is maximal iff $\underline{\lim} g^+ = g^-$ and $\overline{\lim} g^- = g^+$.*

Proof Since $\underline{\lim} g^+$ is the largest lower semi-continuous function below g^+, we have $g^- \leq \underline{\lim} g^+$. Similarly, $\overline{\lim} g^- \leq g^+$. Hence, $[\underline{\lim} g^+, \overline{\lim} g^-] \in D^0$ and $[g^-, g^+] \sqsubseteq [\underline{\lim} g^+, \overline{\lim} g^-]$. □

Example 4.11 We construct a fat Cantor set of positive Lebesgue measure on $[0,1]$ as in [8]. The unit interval $[0,1] = L \cup R \cup C$ is the disjoint union of the two open sets L and R and the Cantor set C with $\mu(L) = \mu(R) = (1-\epsilon)/2$ and $\mu(C) = \epsilon$, where $0 < \epsilon < 1$, such that $\overline{L} = L \cup C$ and $\overline{R} = R \cup C$. Define

$f^-, f^+ : [0,1] \to \mathbb{R}$ by $f^- = \lambda x.0$, $f^+ = \lambda x.(1-\epsilon)/2$, and $g^-, g^+ : [0,1] \to \mathbb{R}$ by

$$g^+(x) = \begin{cases} 1 & x \in R \cup C \\ 0 & x \in L \end{cases}$$

$$g^-(x) = \begin{cases} 1 & x \in R \\ 0 & x \in L \cup C \end{cases}$$

The Cantor set C is precisely the set of discontinuities of g^- and g^+, which has positive Lebesgue measure. Put $f = [f^-, f^+]$ and $g = [g^-, g^+] = ([0,1] \searrow [0,1]) \sqcup (R \searrow \{1\}) \sqcup (L \searrow \{0\})$. Note that $s(f,g) = \lambda x. \int_0^x g^-(u)\, du$ is monotonically increasing and $s(f,g)(1) = (1-\epsilon)/2$. It follows that $f^- \leq s(f,g) \leq f^+$ and thus $(f,g) \in \mathsf{Cons}$. Since $\underline{\lim}\, g^+ = g^-$ and $\overline{\lim} g^- = g^+$, it follows, by Lemma 4.10, that $g \in D^0$ is maximal and thus $(f,g) \in \mathsf{SCons}$. However, we have $q(f,g)(1) = \int_0^1 g^+(u)\, du = (1+\epsilon)/2 > (1-\epsilon)/2 = f^+(1)$ and thus the conclusion of Proposition 4.5 is not satisfied. □

Proposition 4.12 *The dcpo* $\mathsf{SCons} \subset D^1$ *is not continuous.*

Proof Let (f,g) be as in Example 4.11. We construct $g_0 \ll g$ such that for all $g_1 \in D^0$ with $g_0 \sqsubseteq g_1 \sqsubseteq g$, we have $(f, g_1) \notin \mathsf{SCons}$. Let $R' \subset R$ and $L' \subset L$ be finite unions of open intervals in R and L respectively with $R' \ll R$ and $L' \ll L$ in the lattice of open sets of $[0,1]$. Let $g_1 = ([0,1] \searrow [-c, 1+b]) \sqcup (R' \searrow [1-a, 1+b]) \sqcup (L' \searrow [-c, d])$ where a, b, c, d are positive. Then g_1 is a basis element, $g_1 \ll g$ and $\mu((g_1^+)^{-1}(1+b)) = 1 - \mu(L') > \frac{1+\epsilon}{2}$. Thus, $q(f, g_1)(x) = \int_0^x g_1^+$ which implies that $q(f,g_1)(1) > \frac{1+\epsilon}{2} > f^+(1)$. By Corollary 4.8, $(f, g_1) \notin \mathsf{SCons}$. From this it now follows that $(f_1, g_1) \notin \mathsf{SCons}$ for all $f_1 \ll f$ and for all $g_1 \in D^0$ with $g_0 \sqsubseteq g_1 \sqsubseteq g$ and the result follows. □

5. The initial value problem

We consider a Scott continuous time-dependent scalar field $v : [0,1] \times \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ and an initial function $f \in D^0$. We assume that $(f, A_v(f)) \in \mathsf{SCons}$ and define the sub-dcpo

$$D_{v,f} = \{(h,g) \in \mathsf{SCons} \mid (f, A_v(f)) \sqsubseteq (h,g)\},$$

The domain-theoretic Picard operator for the scalar field v and initial function f is now given by

$$P_{v,f} : D_{v,f} \to D_{v,f}$$

with $P_{v,f} = \mathsf{Up} \circ \mathsf{Ap}_v$. This has a least fix-point given by $(f_s, g_s) = \bigsqcup_{i \in \omega} P^i_{v,f}(f, A_v(f))$.

Lemma 5.1 *Suppose* $f = \bigsqcup_{i \in \omega} f_i$ *and* $v = \bigsqcup_{i \in \omega} v_i$, *where* $(f_i)_{i \in \omega}$ *and* $(v_i)_{i \in \omega}$ *are increasing chains, then for each* $i, j, n \geq 0$ *we have*

(i) $(\mathsf{Up} \circ \mathsf{Ap}_{v_i})^n (f_j, A_{v_i}(f_j)) \in \mathsf{Cons}$.

(ii) $(\mathsf{Up} \circ \mathsf{Ap}_{v_i})^n (f_j, A_{v_i}(f_j)) \sqsubseteq (\mathsf{Up} \circ \mathsf{Ap}_v)^n (f, A_v(f))$.

Proof We use induction on $n \geq 0$. For $n = 0$, by monotonicity of $v \mapsto A_v$, we have $(f_j, A_{v_i}(f_j)) \sqsubseteq (f, A_v(f))$ and hence $(f_j, A_{v_i}(f_j)) \in \mathsf{Cons}$ as $(f, A_v(f)) \in \mathsf{Cons}$. Now assume $(\mathsf{Up} \circ \mathsf{Ap}_{v_i})^n (f_j, A_{v_i}(f_j)) \in \mathsf{Cons}$ and $(\mathsf{Up} \circ \mathsf{Ap}_{v_i})^n (f_j, A_{v_i}(f_j)) \sqsubseteq (\mathsf{Up} \circ \mathsf{Ap}_v)^n (f, A_v(f))$. Let $(f_{ijn}, g_{ijn}) := (\mathsf{Up} \circ \mathsf{Ap}_{v_i})^n (f_j, A_{v_i}(f_j))$ and $(h_n, g_n) := (\mathsf{Up} \circ \mathsf{Ap}_v)^n (f, A_v(f))$. Then, $f_{ijn} \sqsubseteq h_n$ implies $A_{v_i}(f_{ijn}) \sqsubseteq A_v(h_n)$, and hence $(f_{ijn}, A_{v_i}(f_{ijn})) \sqsubseteq (h_n, A_v(h_n))$. Since $(f, A_v(f)) \sqsubseteq (h_n, g_n) \sqsubseteq (h_n, A_v(h_n))$, it follows that $(h_n, A_v(h_n)) \in \mathsf{SCons}$ and, thus, $(f_{ijn}, A_{v_i}(f_{ijn})) \in \mathsf{Cons}$. By monotonicity of Up we also obtain: $(\mathsf{Up} \circ \mathsf{Ap}_{v_i})^{n+1} (f_j, A_{v_i}(f_j)) = \mathsf{Up}(f_{ijn}, A_{v_i}(f_{ijn})) \sqsubseteq \mathsf{Up}(h_n, A_v(h_n)) = (\mathsf{Up} \circ \mathsf{Ap}_v)^{n+1} (f, A_v(f))$. This completes the inductive proof. □

Lemma 5.2 *Suppose* $f = \bigsqcup_{i \in \omega} f_i$ *and* $v = \bigsqcup_{i \in \omega} v_i$, *where* $(f_i)_{i \in \omega}$ *and* $(v_i)_{i \in \omega}$ *are increasing chains of standard basis elements. Then, for each* $i \geq 0$ *and each* $j \geq 0$, *the function and the derivative components of* $(\mathsf{Up} \circ \mathsf{Ap}_{v_i})^n (f_j, A_{v_i}(f_j))$ *are, respectively, a linear step function and a standard step function.*

Proof We use induction on $n \geq 0$. Since, by Corollary 4.3, $v_i(t, f_j(t))$ is a step function, the base case follows. As for the inductive step, we have

$$(\mathsf{Up} \circ \mathsf{Ap}_{v_i})^{n+1} (f_j, A_{v_i}(f_j))$$

$$= (\mathsf{Up} \circ \mathsf{Ap}_{v_i})((\mathsf{Up} \circ \mathsf{Ap}_{v_i})^n (f_j, A_{v_i}(f_j)))$$

$$= (\mathsf{Up} \circ \mathsf{Ap}_{v_i})(f_{ijn}, g_{ijn}),$$

where, by the inductive hypothesis, f_{ijn} is a linear step function and g_{ijn} a standard step function. But $\mathsf{Ap}_{v_i}(f_{ijn}, g_{ijn}) = (f_{ijn}, A_{v_i}(f_{ijn}))$ and by Corollary 4.3 $A_{v_i}(f_{ijn})$ is a standard step function. Hence, $\mathsf{Up}(\mathsf{Ap}_{v_i}(f_{ijn}, g_{ijn})) = \mathsf{Up}(f_{ijn}, A_{v_i}(f_{ijn})) = (\mathsf{Up}_1(f_{ijn}, A_{v_i}(f_{ijn})), A_{v_i}(f_{ijn}))$, and the inductive step follows since by Theorem 3.1 $\mathsf{Up}_1(f_{ijn}, A_{v_i}(f_{ijn}))$ is a linear step function. □

Theorem 5.3 *Suppose that* $f \in D^0$ *and* $v \in [0,1] \times \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ *are computable, and assume that* $(f, A_v(f)) \in \mathsf{SCons}$. *Then the least fixed point of* $P_{v,f}$ *is computable.*

Proof By assumption, there exist two effective chains $(f_i)_{i\in\omega}$ and $(v_i)_{i\in\omega}$ of standard basis elements with $f = \bigsqcup_{i\in\omega} f_i$ and $v = \bigsqcup_{i\in\omega} v_i$. We will construct the least fixed point of $P_{v,f}$ as the lub of an effective chain of basis elements with linear step functions in D^1.

Recall that if a triple sequence $(a_{ijn})_{i,j,n\in\omega}$ in a dcpo is monotone separately in each index for fixed values of the other indices, then $\bigsqcup_{i,j,n\in\omega} a_{ijn} = \bigsqcup_{m_1\in\omega}\bigsqcup_{m_2\in\omega}\bigsqcup_{m_3\in\omega} a_{ijn} = \bigsqcup_{n\in\omega} a_{nnn}$, where m_1, m_2, m_3 is any permutation of i, j, n. We now define the triple sequence $a_{ijn} := (\mathsf{Up}\circ\mathsf{Ap}_{v_i})^n(f_j, A_{v_i}(f_j))$ and note that by Lemma 5.1, $a_{ijn} \in D^1$. Thus, we have:

$$(f_s, g_s) = \bigsqcup_{n\in\omega} P^n_{v,f}(f, A_v(f))$$

$$= \bigsqcup_{n\in\omega} (\mathsf{Up}\circ\mathsf{Ap}_{\bigsqcup_i v_i})^n(\bigsqcup_{j\in\omega} f_j, A_{\bigsqcup_i v_i}(\bigsqcup_{j\in\omega} f_j))$$

$$= \bigsqcup_{n\in\omega} (\mathsf{Up}\circ\mathsf{Ap}_{v_n})^n(f_n, A_{v_n}(f_n)).$$

By Lemma 5.2 and Algorithm 3.3, for $n \geq 0$, $a_{nnn} = (\mathsf{Up}\circ\mathsf{Ap}_{v_n})^n(f_n, A_{v_n}(f_n))$ is a linear basis element of D^1. Since $(f_i)_{i\in\omega}$ and $(v_i)_{i\in\omega}$ are effective chains of basis elements, it follows that $(a_{nnn})_{n\in\omega}$ is an effective chain of basis elements and thus its lub is computable. □

6. The Classical Problem

We now return to the classical initial value problem as in Section 1, and assume, by a translation of the origin, that $(t_0, x_0) = (0, 0)$. Thus, $O \subset \mathbb{R}^2$ is a neighbourhood of the origin, $v : O \to \mathbb{R}$ is continuous function and we consider the initial value problem:

$$\dot{x} = v(t, x)\ , \qquad x(0) = 0.$$

By the Peano-Cauchy theorem [6], this equation has a solution which is in general not unique. It is also known that even if v is a computable function, the above differential equation may have no computable solution [15]. We will show that all the classical solutions are contained within the least fix-point of the domain-theoretic Picard operator. Moreover, if v is computable then this least fix-point is indeed computable.

Let $R \subset O$ be a compact rectangle, whose interior contains the origin. Then the continuous function v is bounded on R and therefore for some $M > 0$ we have $|v| \leq M$ on R. Let $(a_n)_{n\in\omega}$ be any positive strictly decreasing sequence with $\lim_{n\to\infty} a_n = 0$. The standard choice is $a_n = a_0/2^n$, for some rational or dyadic number $a_0 > 0$. For large enough n, say $n \geq m$, for some $m \geq 0$, we have $[-a_n, a_n] \times [-Ma_n, Ma_n] \subset O$. For $i \in \omega$, put $T_i = [-a_{i+m}, a_{i+m}]$ and $X_i = [-a_{i+m}M, a_{i+m}M]$. Let $f = \bigsqcup_{i\in\omega} f_i$, where $f_i = \bigsqcup_{j\leq i} T_j \searrow X_j$, and consider the canonical extension $v : T_0 \times \mathbf{I}X_0 \to \mathbf{I}\mathbb{R}$ with $v(t, X) = \{v(t, x) \mid x \in X\}$. We work in the domains $D^0(T_0)$ and $D^1(T_0)$. By [7, Proposition 8.11], $(f, T_0 \searrow [-M, M]) \in \mathsf{SCon}$ and thus $(f, A_v(f)) \in \mathsf{SCon}$ since $(T_0 \searrow [-M, M]) \sqsubseteq A_v(f)$. Therefore, $P_{v,f}$ has a least fix-point.

Theorem 6.1 *The least fix-point (f_s, g_s) of $P_{v,f}$ satisfies:*

$$\left[\frac{df_s^-}{dt}, \frac{df_s^+}{dt}\right] = v(t, [f_s^-(t), f_s^+(t)]),$$

$$g_s^-(t) = \frac{df_s^-}{dt}, \qquad g_s^+(t) = \frac{df_s^+}{dt}.$$

Proof It follows from Lemma 5.1 that

$$(f_s, g_s) = \bigsqcup_{n\in\omega} P^n_v(\bigsqcup_{j\in\omega} f_j, A_v(\bigsqcup_{j\in\omega} f_j))$$

$$= \bigsqcup_{j\in\omega}\bigsqcup_{n\in\omega} (\mathsf{Up}\circ\mathsf{Ap}_v)^n(f_j, A_v(f_j)).$$

Let $(f_{jn}, g_{jn}) := (\mathsf{Up}\circ\mathsf{Ap}_v)^n(f_j, A_v(f_j))$ for a fixed $j \geq 0$ and all $n \geq 0$. Then, $g_{j(n+1)} = A_v(f_{jn}) = \lambda t.\, v(t, f_{jn}(t))$. We show by induction on $n \geq 0$ that for $a_j \leq t \leq a_0$ we have:

$$f^{\pm}_{j(n+1)}(t) = b^{\pm}_{jn} + \int_{a_j}^{t} v^{\pm}(u, f_{jn}(u))\, du,$$

where $|b^{\pm}_{jn}| \leq Ma_j$. We have $f_{j1} = \mathsf{Up}_1(f_j, A_v(f_j))$. Note that maps of the form

$$x \mapsto f_j^+(y) - \int_x^y v^-(u, f_j(u))\, du,$$

for $a_j \leq y \leq a_0$ in Equation 2 for f^+_{j1} will all be greater than the linear map $x \mapsto Mx$ for $a_j \leq x \leq y$. Furthermore, maps of the form

$$x \mapsto f_j^+(y) + \int_y^x v^+(u, f_j(u))\, du,$$

for $a_j \leq y \leq a_0$ in Equation 2 for f^+_{j1} will all be greater than $x \mapsto Ma_j + \int_{a_j}^x v^+(u, f_j(u))\, du$. It follows that in determining f^+_{j1}, for $a_j \leq t \leq a_0$, from Equation 2 the infimum only needs to be taken over maps of the form

$$x \mapsto Ma_j + \int_y^x v^+(u, f_j(u))\, du,$$

for $y \in (-a_j, a_j)$. Thus, for $a_j \le t \le a_0$, we obtain

$$\begin{array}{rcl} f^+_{j1}(t) & = & \inf_{y\in(-a_j,a_j)} Ma_j + \int_y^t v^+(u, f_j(u))\, du \\ & = & \inf_{y\in(-a_j,a_j)} Ma_j + \int_y^{a_j} v^+(u, f_j(u))\, du \\ & & + \int_{a_j}^t v^+(u, f_j(u))\, du \\ & = & b^+_{j0} + \int_{a_j}^t v^+(u, f_j(u))\, du, \end{array}$$

where $b^+_{j0} = \inf_{y\in(-a_j,a_j)} Ma_j + \int_y^{a_j} v^+(u, f_j(u))\, du$ with $|b_{j0}| \le Ma_j$. Assume the inductive hypothesis:

$$f^+_{jn}(t) = b^+_{j(n-1)} + \int_{a_j}^{t} v^+(u, f_{j(n-1)}(u))\, du,$$

where $|b^+_{j(n-1)}| \le Ma_j$. Then put $b^+_{j(n+1)} := f^+_{j(n+1)}(a_j)$, which satisfies $|b^+_{j(n+1)}| \le Ma_j$ and by monotonicity $b^+_{j(n+1)} \le b^+_{jn}$. Maps of the form

$$x \mapsto f^+_{jn}(y) - \int_x^y v^-(u, f_{jn}(u))\, du,$$

are above f^+_{jn} for $a_j \le x \le y$. Thus, Equation 2 for $f^+_{j(n+1)}$ implies that there exists $y \in [-a_j, a_j]$ such that

$$\underline{\lim} f^+_{jn}(y) + \int_y^{a_j} v^+(u, f_{jn}(u)) = b^+_{j(n+1)}.$$

For any $y \in (-a_j, a_j)$, the relation

$$f^+_{jn}(y) + \int_y^{a_j} v^+(u, f_{jn}(u))\, du \ge b^+_{jn}$$

implies that for $a_j \le t \le a_0$ we have:

$$f^+_{j(n+1)}(t) = b^+_{jn} + \int_{a_j}^{t} v^+(u, f_{jn}(u))\, du.$$

This completes the inductive proof for $f^+_{j(n+1)}$. The proof for $f^-_{j(n+1)}$ is similar. Next, we take the limit as $n \to \infty$ for a fixed $j \ge 0$. The bounded sequences b^-_{jn} and b^+_{jn} are increasing and decreasing respectively; we put $\lim_{n\to\infty} b^-_{jn} = b^-_j$ and $\lim_{n\to\infty} b^+_{jn} = b^+_j$. Then, by the monotone convergence theorem we obtain: $f^-_{js}(t) = b^-_j + \int_{a_j}^t v^-(u, f_{js})\, du$ and $f^+_{js}(t) = b^+_j + \int_{a_j}^t v^+(u, f_{js}(u)\, du$, for $a_j \le t \le a_0$, where $(f_{js}, A_v(f_{js}))$ is the least fix-point of P_{v,f_j}. Finally taking the limit, we have $\lim_{j\to\infty} b^-_j = 0$ and $\lim_{j\to\infty} b^-_j = 0$, as well as $\lim_{j\to\infty} a_j = 0$. Note that the sequence $(f^-_{js})_{j\ge 0}$ is increasing with limit f^-_s and $(f^+_{js})_{j\ge0}$ is decreasing with limit f^+_s. Since v^+ and v^- are bounded in $[-a_j, a_j]$, the monotone convergence theorem gives us: $f^-_s(t) = \int_0^t v^-(u, f_s(u))\, du$ and $f^+_s(t) = \int_0^t v^+(u, f_s(u))\, du$. From these, it follows that f^-_s and f^+_s are continuous. Furthermore, v^- and v^+ are also continuous since v is continuous. Therefore, the maps $u \mapsto v^-(u, f_s(u))$ and $u \mapsto v^+(u, f_s(u))$ are both continuous as well. Thus, $g^-_s(t) = \frac{df^-_s}{dt}$ and $g^+_s(t) = \frac{df^+_s}{dt}$, for $0 \le t \le a_0$. A similar proof provides the result for $-a_0 \le t \le 0$. □

Corollary 6.2 *If $f^- = f^+$ or $g^- = g^+$ hold, then both equalities hold and the domain-theoretic solution $f^- = f^+$ gives the unique solution of the classical initial value problem.* □

If v is Lipschitz in its second component then we know from [7, Theorem 8.12] that $f^-_s = f^+_s$ is the unique solution of the classical problem. We can now use Theorem 5.3 to deduce a domain-theoretic proof of the following known result [15].

Corollary 6.3 *If v is computable and Lipschitz in its second component then the unique solution of the classical initial value problem, $\dot{h} = v(t, h(t))$ with $h(0) = 0$, is computable.* □

Algorithm 3.3 for function updating, Algorithm 4.4 for derivative updating, and Corollary 6.3 together provide a new technique based on domain theory to solve the classical initial value satisfying the Lipschitz condition. It is distinguished by the property that the solution can be obtained up to any desired accuracy. For a continuous piecewise linear map $f : [0, 1] \to \mathbb{R}$, let J_f be the partition of $[0, 1]$ such that f is linear in each subinterval of J_f. If $f, g : [0, 1] \to \mathbb{R}$ are continuous piecewise linear maps, then $|f - g| = \max(d(f, g), d(g, f))$ where $d(f, g) = \max_{x\in J_f} |f(x) - g(x)|$. Thus, $|f - g|$ can be obtained in $O(\mathsf{card}(J_f) + \mathsf{card}(J_g))$, where $\mathsf{card}(D)$ is the number of elements in the finite set D.

Algorithm 6.4 *We solve $\frac{dh}{dt} = v(t, h(t))$ with the initial condition $h(0) = 0$ up to a given precision $\epsilon > 0$. The Function Updating Algorithm 3.3 and the Derivative Updating Algorithm 4.4 will be used as subroutines.*
Input:

(i) *Positive rational numbers a_0, M, such that $v : [-a_0, a_0] \times [-Ma_0, Ma_0] \to \mathbb{R}$ is continuous, satisfies a Lipschitz condition in the second argument uniformly in the first, and $|v| \le M$.*

(ii) *An increasing chain $(v_n)_{n\in\omega}$ of step functions with $v_n = \bigsqcup_{i\in I_n} a_i \times b_i \searrow c_i \in ([-a_0, a_0] \times [-Ma_0, Ma_0] \to \mathbf{I}\mathbb{R})$ is given recursively for $n \in \omega$*

such that $v = \bigsqcup_{n\in\omega} v_n$. (Note that for each elementary function v, the step functions v_n can be obtained from available interval arithmetic libraries.)

(iii) *A rational number $\epsilon > 0$.*

Output: *Two continuous and piecewise linear maps $f_\epsilon^-, f_\epsilon^+ : [-a_0, a_0] \to \mathbb{R}$ which satisfy $f_\epsilon^- \le h \le f_\epsilon^+$ and $|f_\epsilon^+ - f_\epsilon^-| \le \epsilon$, where $h : [-a_0, a_0] \to \mathbb{R}$ is the unique solution of the initial value problem.*

for j=0,1,2...
  # INITIALISATION
  $f_j := \bigsqcup_{i\le j} [-\frac{a_0}{2^i}, \frac{a_0}{2^i}] \searrow [-\frac{a_0 M}{2^i}, \frac{a_0 M}{2^i}]$
  # use Algorithm 3.4 as subroutine
  $(f_{j0}, g_{j0}) := (f_j, \lambda t.\, v_j(t, f_j(t)))$
  for n=0,...,j
    # use Algorithms 3.4 and 2.3 as subroutines
    $(f_{jn}, g_{jn}) := (\mathsf{Up} \circ \mathsf{Ap}_{v_j})^n (f_{j0}, g_{j0})$.
    if $|f_{jn}^+ - f_{jn}^-| \le \epsilon$ then
      $f_\epsilon^- := f_{jn}^-$ and $f_\epsilon^+ := f_{jn}^+$
      return f_ϵ^- and f_ϵ^+

The algorithm is incremental: a better precision ϵ' with $0 < \epsilon' < \epsilon$ can be obtained by continuing with the work already achieved for ϵ.

The function and derivative updating algorithms 3.3 and 4.4 can be extended to the semi-rational polynomial basis, which enables us to solve the differential equation with $v = \bigsqcup_{n\in\omega} v_n$, where v_n^- and v_n^+ are piecewise semi-rational polynomials. However, this will in general involve solving for the algebraic roots of rational polynomials. Moreover, each function updating will in this case increase the degree of each polynomial by one, in contrast to Algorithm 3.3 which always produces a piecewise linear function update. We illustrate this in Figure 6 with two iterations for solving $\dot{x} = 2t + x + \frac{9}{2}$ with $x(t_0) = x_0$, where $v = \lambda t.\lambda x.2t + x + 9/2$ is itself a rational polynomial. The exact solution is $x(t) = 6.5e^t - 2t - 6.5$.

More generally, with the assumption that v is only continuous, all classical solutions are contained in the domain-theoretic solution as follows:

Theorem 6.5 *Any solution h of the classical initial value problem, with $\frac{dh}{dt} = v(t, h(t))$ and $h(0) = 0$, satisfies $f_s^- \le h \le f_s^+$ in a neighbourhood of the origin, where $(f_s, \lambda t.\, v(t, f_s(t)))$ is the domain-theoretic solution.*

Proof The classical solution h gives a domain-theoretic solution (r, g) with $r^- = r^+ = h$ and $g = \lambda t.\, v(t, r(t))$ which satisfies $f \sqsubseteq r$. Since (f_s, g_s) is the least solution with $f \sqsubseteq f_s$ we get $f_s \sqsubseteq r$. □

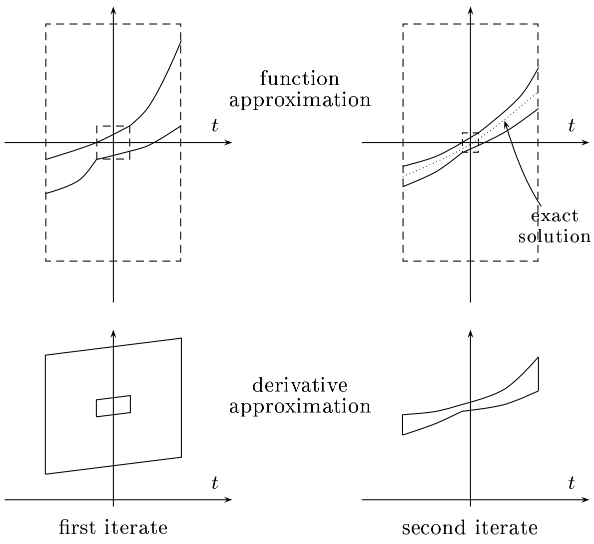

Figure 3. Two iterates of the updating operators for solving $\dot{x} = 2t + x + \frac{9}{2},\ x(0) = 0$

The classical initial-value problem may have no computable solutions even if v is computable [15]. However, from the above result, all the classical solutions will be contained in the domain-theoretic solution, which, by Theorem 5.3, is computable if v is computable.

7. Conclusion and Implementation

Algorithm 6.4 enables us to solve classical initial value problems up to any desired accuracy, overcoming the problems of the round-off error, the local error and the global truncation error in the current established methods such as the multi-step or the Runge-Kutta techniques. It also allows us to solve initial value problems for which the initial value or the scalar field is imprecise or partial. We can implement Algorithm 6.4 in rational arithmetic for differential equations given by elementary functions, by using available interval arithmetic packages to construct libraries for elementary functions expressed as lubs of step functions. Since rational arithmetic is in general expensive, we can obtain an implementation in floating point or fixed precision arithmetic respectively by carrying out a sound floating point or dyadic rounding scheme after each output of the updating operators. For polynomial scalar fields, an implementation with the semi-rational polynomial basis can provide a viable alternative.

As for future work, generalization to higher dimensions, systems of ordinary differential equations and the boundary value problem will be addressed. It is also a great challenge to extend the domain-theoretic framework for differential calculus to obtain domains for functions of several real variables as a platform to tackle partial differential equations.

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