Domain of Computation of a Random Field in Statistical Physics

(Extended Abstract)

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Abstract

We present a domain-theoretic analysis of the invariant measure of the one-dimensional Ising model in a random external magnetic field. The invariant measure is obtained as a fixed point of the Markov transition operator of an iterated function system with probabilities acting on the probabilistic power domain of the upper space of a closed real interval. This enables us to use the generalised Riemann integral in combination with Elton's ergodic theorem to obtain an algorithm to compute the free energy density of the system. We also develop the generalised double Riemann integral, which we use, together with a two-dimensional version of Elton's theorem, to deduce algorithms to compute the magnetisation per spin and the Edwards-Anderson parameter of the system.

1 Introduction

The *Ising model* was introduced by Ising as a model for ferromagnetism some seventy years ago; it also describes such systems as lattice gases, binary alloys and "melting" of DNA. The model deals with the configuration of a system of interacting objects, say two-valued spins, situated at the sites of a d-dimensional lattice (grid) in thermal equilibrium with a reservoir. The basic assumption in the Ising model is that objects interact only with their nearest neighbours as well as possibly with an external field. An increase in the temperature can produce a radical change in the configuration of the system, i.e. a phase transition. See, for example, [17] for an introduction to this subject. The Ising model can be solved exactly in one and two dimensions; in two dimensions it already exhibits the phenomenon of phase transition, where the global configuration of the interacting objects changes. It is one of the few exactly soluble models which exhibits phase transition. In higher dimensions, the model has continued to be the subject of intensive theoretical and experimental research with the use of standard *renormalisation techniques* [15] and has been one of the main areas of study in the modern science of *complex systems* in the past decade.

In recent years, the problem of a one-dimensional $\frac{1}{2}$ spin Ising model in a *random* external field has also been studied but with a different kind of renormalisation method [19, 10, 5, 2]. We will briefly review this work here. Consider a one-dimensional chain of N Ising spins $\langle s_n \rangle_{n=1}^{N}$ with $s_n = \pm 1$ for each $1 \leq n \leq N$ as in Figure one.

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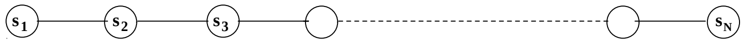


Figure 1. The one-dimensional $\frac{1}{2}$ spin Ising model.

Each state of the system is determined by a given set of values of the spins $s_n = \pm 1$. Assume now that there is a magnetic field h_n at each site $n \geq 1$. Then, we can write the Hamiltonian of the system in any state $\langle s_n \rangle_{n=1}^N$ as

$$\mathcal{H}_N = -\sum_{n=1}^{N-1} J s_n s_{n+1} - \sum_{n=1}^{N} h_n s_n$$

where $J > 0$ is the *coupling* constant. This equation says that the only interactions which contribute to the energy of the system are between neighbouring spins on the one hand and between each spin and the magnetic field on the other hand.

In statistical physics, all physical quantities of a system, such as free energy and entropy, can be obtained from the *canonical partition function*

$$Z = \sum_i e^{-\beta \epsilon_i}$$

of the system, where ϵ_i is the energy of the state i and the summation is over all the possible quantum states i of the system [13]. Here, $\beta = (k_B T)^{-1}$, where T is the temperature of the system and k_B is the universal Boltzmann constant. The probability of finding the system in quantum state i is given by $e^{-\beta \epsilon_i}/Z$.

For the one-dimensional Ising model of N spins, the canonical partition function is given by

$$Z_N = \sum_{s_1,\ldots,s_N=\pm 1} \exp \beta\Big(\sum_{n=1}^{N-1} J s_n s_{n+1} + \sum_{n=1}^{N} h_n s_n\Big). \tag{1}$$

Note that for each $n = 1, \ldots, N-1$ and real variable x, we have:

$$\begin{aligned}\sum_{s_n=\pm 1} \exp \beta(J s_n s_{n+1} + x s_n) &= 2\cosh \beta(J s_{n+1} + x) \\ &= \exp(A(x) s_{n+1} + B(x)),\end{aligned}$$

where the real functions $A, B : \mathbb{R} \to \mathbb{R}$ are given by

$$A(x) = (2\beta)^{-1} \log(\cosh \beta(x+J)/\cosh \beta(x-J)), \tag{2}$$

$$B(x) = (2\beta)^{-1} \log(4\cosh \beta(x+J)\cosh \beta(x-J)), \tag{3}$$

as one can easily check for the two possible values $s_{n+1} = \pm 1$. Using the above formula, we carry out the summation in Equation (1) over the first spin s_1 to obtain:

$$Z_N = \sum_{s_2,\ldots,s_N=\pm 1} \exp \beta\Big\{B(h_1) + \sum_{n=2}^{N-1} J s_n s_{n+1} + (h_2 + A(h_1)) s_2 + \sum_{n=3}^{N} h_n s_n\Big\}. \tag{4}$$

We now put $\xi_1 = h_1$ and

$$\xi_n = h_n + A(\xi_{n-1}) \tag{5}$$

for $2 \leq n \leq N$. Therefore, Equation (4) can be rewritten as

$$Z_N = \sum_{s_2,\ldots,s_N=\pm 1} \exp \beta\Big\{B(\xi_1) + \sum_{n=2}^{N-1} J s_n s_{n+1} + \xi_2 s_2 + \sum_{n=3}^{N} h_n s_n\Big\}.$$

If we continue to do the summation over the left spins, then after $N-1$ iterations we get [19]:

$$Z_N = \sum_{s_N=\pm 1} \exp \beta\Big(\xi_N s_N + \sum_{n=1}^{N-1} B(\xi_n)\Big). \tag{6}$$

In other words, the partition function is reduced to that of a *single* spin s_N.

If the magnetic field is constant, i.e. $h_n = h$ for $1 \le n \le N$, then it is easily shown that the smooth function

$$\begin{aligned} g: \mathbb{R} &\to \mathbb{R} \\ x &\mapsto h + A(x), \end{aligned}$$

satisfies $|g'(x)| = |A'(x)| < 1$ for all $x \in \mathbb{R}$ and has a unique fixed point x^*. This shows that x^* is an attractor and $\lim_{n\to\infty} g^n(x) = x^*$ for all $x \in \mathbb{R}$. Therefore, in this case, $\lim_{n\to\infty} \xi_n = x^*$, and one can compute the various physical quantities such as free energy and magnetisation per spin [10], as we will see in Section 5.

In [10, 5, 2], the problem of a one-dimensional Ising model in a random magnetic field has been studied as follows. Let $h > 0$ be a constant. Suppose h_n is a random field taking values $\pm h$ with equal probabilities, i.e. $\langle h_n \rangle_{n\ge1}$ is a sequence of independent and identically distributed random variables with $P(h_n = h) = P(h_n = -h) = 1/2$ for all $n \ge 1$. Then, Equation (5) is a stochastic equation for the Markov process $\xi_1, \xi_2, \xi_3, \ldots$ with transitional probabilities

$$P(\xi_n = h + A(\xi_{n-1})) = P(\xi_n = -h + A(\xi_{n-1})) = \frac{1}{2}. \tag{7}$$

The stochastic equation induces in the nth step a probability density $p_n(x)$ for the effective random field ξ_n satisfying the Perron-Frobenius (or Kolmogorov-Chapman) equation [20]:

$$p_n(y) = \int p_{n-1}(x)\{\frac{1}{2}\delta(y - h - A(x)) + \frac{1}{2}\delta(y + h - A(x))\}\, dx, \tag{8}$$

where $\delta(z)$ is the delta function centred at $z = 0$. Consider the two functions

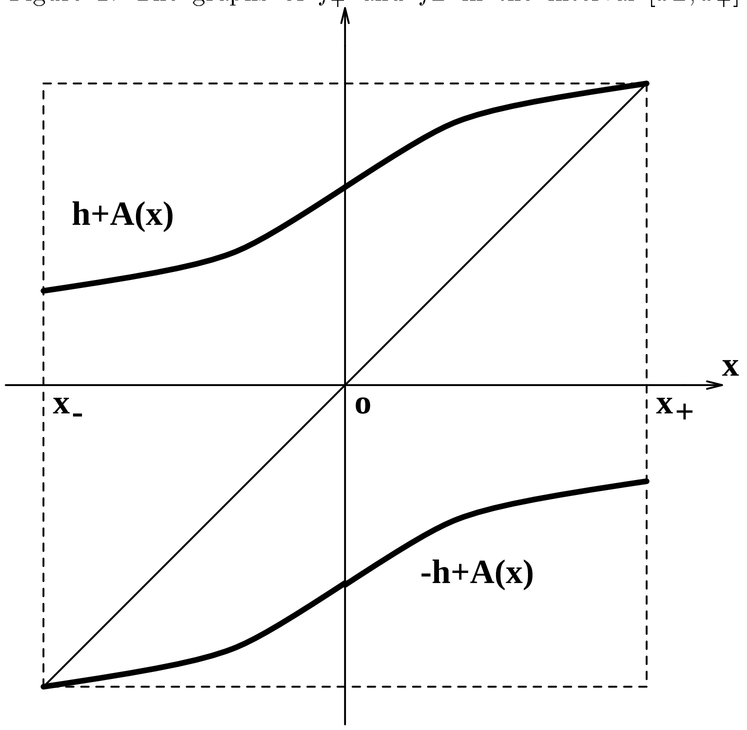
$$f_+, f_- : \mathbb{R} \to \mathbb{R} \tag{9}$$

defined by $f_+(x) = h + A(x)$ and $f_-(x) = -h + A(x)$. Each f_σ satisfies $|f'_\sigma(x)| < 1$ and has a unique fixed point x_σ, where $\sigma = \pm$, with

$$x_+ = -x_- = h/2 + (2\beta)^{-1}\text{arcsinh}(e^{2\beta J}\sinh \beta h) > 0. \tag{10}$$

Furthermore $f_\sigma[x_-, x_+] \subseteq [x_-, x_+]$ for $\sigma = \pm$.

Figure 2. The graphs of f_+ and f_- in the interval $[x_-, x_+]$



When the temperature $T > 0$, an analysis of iterates of the Perron-Frobenius Equation (8), including analysis by symbolic dynamics, has been presented in [1, 3] to show that the equation has a unique fixed point, i.e. an invariant probability distribution with support in $[x_-, x_+]$, which has a multifractal structure, and that starting with any initial probability distribution, the iterates tend to this invariant distribution. Various physical quantities such as free energy and magnetisation per spin can be expressed as integrals over this invariant distribution [10].

In this paper, we will present a domain-theoretic analysis of Equation (5), for $T > 0$, by interpreting it as the dynamics of an iterated function system (IFS) with probabilities, consisting of the maps $f_+, f_- : \mathbb{R} \to \mathbb{R}$ with equal probabilities $p_+ = p_- = 1/2$. Following [6, 7], we will examine the dynamics of the IFS on the normalised probabilistic power domain of the upper space of $[x_-, x_+]$, i.e. instead of working in the space of probability distributions over $[x_-, x_+]$, we work in the bigger space of probability measures, or normalised continuous valuations, on the upper space of $[x_-, x_+]$. The domain-theoretic formulation enables us to show some further properties of the invariant probability distribution of the Perron-Frobenius equation. In particular, using the generalised Riemann integral, we obtain finite algorithms to compute various physical quantities of the Ising model in the random field to any given threshold of accuracy.

In Section 2, we will review the basic domain-theoretic framework for measures on compact metric spaces. In Section 3, we present some general results about the invariant measure of an IFS with probabilities. In Section 4, we extend the theory of generalised Riemann integration to multiple integration. Finally, in Section 5, we apply these general results to the one-dimensional random field Ising model.

2 Domain-theoretic framework

Let X be a compact metric space, and let $(\mathbf{U}X, \supseteq)$ be its upper space, i.e. the non-empty compact subsets of X ordered by reverse inclusion, which is a bounded complete ω-continuous dcpo (directed complete partial order). The singleton map embeds X onto the set of maximal elements of $\mathbf{U}X$. Similarly, the set of probability measures on X can be embedded into the set of normalised measures on $\mathbf{U}X$. To make this precise, we need some basic definitions and results.

A *valuation* on a topological space Y is a map $\nu : \Omega(Y) \to [0, \infty)$ which satisfies:

(i) $\nu(a) + \nu(b) = \nu(a \cup b) + \nu(a \cap b)$,

(ii) $\nu(\emptyset) = 0$, and

(iii) $a \subseteq b \Rightarrow \nu(a) \leq \nu(b)$.

A *normalised* valuation ν further satisfies $\nu(Y) = 1$. A *continuous* valuation [14, 12, 11] is a valuation such that whenever $A \subseteq \Omega(Y)$ is a directed set (wrt $\subseteq$) of open sets of Y, then

$$\nu(\bigcup_{O \in A} O) = \sup_{O \in A} \nu(O).$$

For any $b \in Y$, the *point valuation* based at b is the valuation $\eta_b : \Omega(Y) \to [0, \infty)$ defined by

$$\eta_b(O) = \begin{cases} 1 & \text{if } b \in O \\ 0 & \text{otherwise.} \end{cases}$$

Any finite linear combination

$$\sum_{i=1}^{n} r_i \eta_{b_i}$$

of point valuations η_{b_i} with constant coefficients $r_i \in [0, \infty)$, $(1 \leq i \leq n)$ is a continuous valuation on Y; we call it a *simple valuation*.

The *normalised probabilistic power domain*, $\mathbf{P}Y$, of a topological space Y consists of the set of normalised continuous valuations ν on Y with $\nu(Y) = 1$ and is ordered as follows:

$$\mu \sqsubseteq \nu \text{ iff for all open sets } O \text{ of } Y,\ \mu(O) \leq \nu(O).$$

The partial order $(\mathbf{P}Y, \sqsubseteq)$ is a dcpo in which the lub of a directed set $\langle \mu_i \rangle_{i \in I}$ is given by $\bigsqcup_i \mu_i = \nu$ where for any open subset $O \in \Omega(Y)$ we have

$$\nu(O) = \sup_{i \in I} \mu_i(O).$$

If Y is an ω-continuous dcpo with bottom $\perp$, then $\mathbf{P}Y$ is also an ω-continuous dcpo with bottom $\eta_\perp$ and has a basis consisting of simple valuations [7]. Furthermore, any $\mu \in \mathbf{P}Y$ extends uniquely to a Borel measure on Y [16, Theorem 3.9]. For convenience, we denote this unique extension by μ as well.

For two simple valuations

$$\mu_1 = \sum_{b \in B} r_b \eta_b \qquad\qquad \mu_2 = \sum_{c \in C} s_c \eta_c$$

in $\mathbf{P}^1 Y$, where B, C are finite subsets of Y, we have by the *splitting lemma* [12, 7]: $\mu_1 \sqsubseteq \mu_2$ iff, for all $b \in B$ and all $c \in C$, there exists a nonnegative number $t_{b,c}$ such that

$$\sum_{c \in C} t_{b,c} = r_b \qquad\qquad \sum_{b \in B} t_{b,c} = s_c \tag{11}$$

and $t_{b,c} \neq 0$ implies $b \sqsubseteq c$. We can consider any $b \in B$ as a source with mass r_b, any $c \in C$ as a sink with mass s_c, and the number t_{bc} as the flow of mass from b to c. Then, the above property can be regarded as conservation of total mass.

Since $(\mathbf{U}X, \supseteq)$ is an ω-continuous dcpo with bottom X, $\mathbf{P}^1\mathbf{U}X$ is an ω-continuous dcpo with bottom η_X. The singleton map

$$\begin{array}{rccc} s: & X & \rightarrow & \mathbf{U}X \\ & x & \mapsto & \{x\} \end{array}$$

embeds X onto the set $s(X)$ of maximal elements of $\mathbf{U}X$. Furthermore, $s(X) \subseteq \mathbf{U}X$ is a Borel subset; in fact any Borel subset $B \subseteq X$ induces a Borel subset $s(B) \subseteq \mathbf{U}X$ [6, Corollary 5.10]. A valuation $\mu \in \mathbf{P}^1\mathbf{U}X$ is said to be *supported* in $s(X)$ if its unique extension to a Borel measure on $\mathbf{U}X$ satisfies $\mu(s(X)) = 1$. It can be shown that $\mu \in \mathbf{P}^1\mathbf{U}X$ iff

$$\mu(s(a)) = \mu(\Box a) \tag{12}$$

for all open subsets $a \subseteq X$. If μ is supported in $s(X)$, then the *support* of μ is the set of points $y \in s(X)$ such that $\mu(O) > 0$ for any Scott-neighbourhood $O \subseteq \mathbf{U}X$ of y. Any element of $\mathbf{P}^1\mathbf{U}X$ which is supported in $s(X)$ is a maximal element of $\mathbf{P}^1\mathbf{U}X$ [6, Proposition 5.18]; we denote the set of all valuations which are supported in $s(X)$ by $\mathbf{S}^1 X$. We can identify $\mathbf{S}^1 X$ with the set $\mathbf{M}^1 X$ of normalised Borel measures on X as follows. Let

$$\begin{array}{rccc} e: & \mathbf{M}^1 X & \rightarrow & \mathbf{S}^1 X \\ & \mu & \mapsto & \mu \circ s^{-1} \end{array}$$

and

$$\begin{array}{rccc} j: & \mathbf{S}^1 X & \rightarrow & \mathbf{M}^1 X \\ & \nu & \mapsto & \nu \circ s. \end{array}$$

Theorem 2.1 *[6, Theorem 5.21] The maps e and j are well-defined and induce an isomorphism between $\mathbf{S}^1 X$ and $\mathbf{M}^1 X$.* $\Box$

For $\mu \in \mathbf{M}^1 X$ and an open subset $a \subseteq X$, Equation (12) implies

$$\mu(a) = e(\mu)(s(a)) = e(\mu)(\Box a). \tag{13}$$

Furthermore, it follows from Theorem 2.1 that there exists an increasing chain of simple valuations $\langle \nu_i \rangle_{i \geq 0}$ with

$$\mu \circ s^{-1} = \bigsqcup_{i \geq 0} \nu_i. \tag{14}$$

In other words, any probability measure on X is the least upperbound of a chain of simple valuations on $\mathbf{U}X$. This fundamental property is used in [7] to develop a theory of generalised Riemann integration, which we will be the basis of work in Section 4.

3 Invariant measure of an IFS

Suppose $\mathcal{A} \subseteq \mathbf{P}^1\mathbf{U}X$ is a directed set of simple valuations. We would like to know when the lub $\bigsqcup \mathcal{A}$ of $\mathcal{A}$ determines a Borel measure on X; in other words, under what conditions we have $\bigsqcup \mathcal{A} \in \mathbf{S}^1 X$. The following proposition gives the necessary and sufficient condition for this. We will denote the diameter of any compact subset $c \subseteq X$ by $|c|$.

Proposition 3.1 *The following conditions are equivalent.*

(i) $\bigsqcup \mathcal{A} \in \mathbf{S}^1 X$.

(ii) *For all* $\epsilon > 0$ *and all* $\delta > 0$, *there exists* $\sum_{a \in A} r_a \eta_a \in \mathcal{A}$ *with*

$$\sum_{|a| \geq \delta} r_a < \epsilon. \qquad \Box$$

Suppose now we have an increasing chain $\langle \nu_i \rangle_{i \geq 0}$ of simple valuations ν_i in $\mathbf{P}^1\mathbf{U}X$. Using Equation (11), it is easy to see that we can obtain a finitary branching tree whose nodes on the ith level are precisely the elements of B_i and a child of any node is a subset of that node. We call this a tree *associated* with the chain $\langle \nu_i \rangle_{i \geq 0}$. Note that a chain can have more than one associated tree. Clearly, the lub in $\mathbf{U}X$ of the nodes of any branch of an associated tree is an element of $\mathbf{U}X$, which is to say that the intersection of a shrinking sequence of non-empty compact subsets of X is a non-empty compact subset. Using Proposition 3.1 and König's Lemma, we can prove the following useful result.

Proposition 3.2 *Suppose* $\langle \nu_i \rangle_{i \geq 0} \in \mathbf{P}^1\mathbf{U}X$ *is an increasing chain of simple valuations. If, for any associated tree of the chain, the lub of the nodes of any branch is a singleton subset of* X, *then the lub of the chain* $\langle \nu_i \rangle_{i \geq 0}$ *is in* $\mathbf{S}^1 X$. $\Box$

Let $(X; f_1, \ldots, f_N; p_1, \ldots, p_N)$ be an IFS with probabilities on the compact metric space X, i.e. for each $i = 1, \ldots, N$, the map $f_i : X \to X$ is continuous and we have $0 < p_i < 1$ with $\sum_{i=1}^N p_i = 1$. Define the *transition* operator

$$\begin{array}{rccc} H: & \mathbf{P}^1\mathbf{U}X & \to & \mathbf{P}^1\mathbf{U}X \\ & \mu & \mapsto & H(\mu) \end{array}$$

by $H(\mu)(O) = \sum_{i=1}^N p_i \mu(f_i^{-1}(O))$. We know from [6, Proposition 6.1] that H is Scott continuous and, therefore, has a least fixed point $\nu^* = \bigsqcup_{m \geq 0} H^m \eta_X$. The mth iterate $H^m \eta_X$ is the simple valuation given by

$$H^m \eta_X = \sum_{i_1, i_2, \ldots, i_m = 1}^{N} p_{i_1} p_{i_2} \ldots p_{i_m} \eta_{f_{i_1} f_{i_2} \cdots f_{i_m} X}. \tag{15}$$

Moreover, the chain $\langle H^m \eta_X \rangle_{m \geq 0}$ has a unique associated tree which we can label with the transitional probabilities as in [8]. See Figure 3. We call this the *IFS tree with transitional probabilities*.

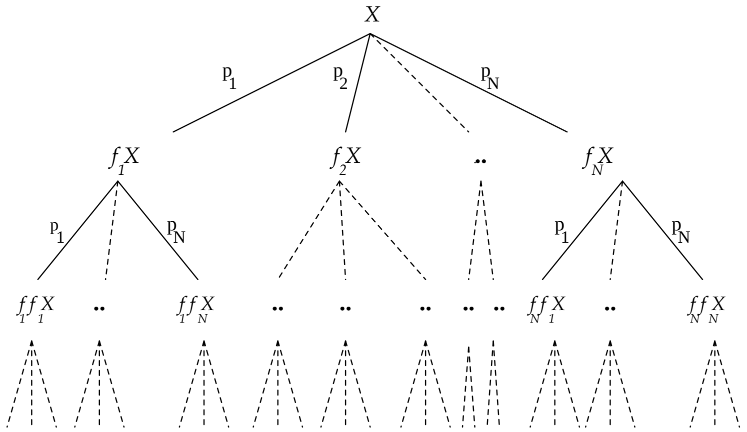


Figure 3. The IFS tree with transitional probabilities.

Assume now that the least fixed point ν^* of H is in $\mathbf{S}^1X$. Then, by the remark following Equation (12), ν^* is a maximal element of $\mathbf{P}^1\mathbf{U}X$ and, therefore, the unique fixed point of H. If $\nu \in \mathbf{P}^1\mathbf{U}X$, then one can show using the monotonicity of H that ν^* is the maximal topological limit of $H^m(\nu)$ as $m \to \infty$. Writing the maximal topological limit as $\lim_{m\to\infty} H^m(\nu)$, we have:

Proposition 3.3 *If $\nu^* \in \mathbf{S}^1X$, then ν^* is the unique fixed point of H, and, for any $\nu \in \mathbf{P}^1\mathbf{U}X$, we have $\lim_{m\to\infty} H^m(\nu) = \nu^*$.* $\square$

A particular instance of the above result occurs if the IFS is *hyperbolic*, i.e. if $f_i : X \to X$ is contracting with contractivity factor $s_i < 1$. Then it follows from Proposition 3.2 that $\nu^* \in \mathbf{S}^1X$, since a typical node $f_{i_1}f_{i_2}\dots f_{i_m}X$ of the above tree on level m has diameter at most $s^m|X|$ which tends to 0 as $m \to \infty$, where $s = \max_i s_i$ and $|X|$ is the diameter of X.

Corollary 3.4 *For a hyperbolic IFS, H has a unique fixed point ν^* and $\lim_{m\to\infty} H^m(\nu) = \nu^*$ for any $\nu \in \mathbf{P}^1\mathbf{U}X$.* $\square$

4 Multiple R-integration

In this section, we extend the generalised Riemann integration, R-integration, introduced in [7] to multiple integration. We will only consider the double integral since the generalisation to any multiple integral is straightforward. The double R-integral is used in the next section to compute the magnetisation per spin and the Edwards-Anderson parameter in the Ising model.

4.1 The double R-integral

Recall that given Borel measures $\mu \in \mathbf{M}X$ and $\nu \in \mathbf{M}Y$ on topological spaces X and Y, the product measure $\mu \times \nu \in \mathbf{M}(X \times Y)$ satisfies $(\mu \times \nu)(B_1 \times B_2) = \mu(B_1)\nu(B_2)$ for all Borel subsets B_1 and B_2 of X and Y respectively [18]. Similarly, given dcpo's D and E, and valuations $\mu \in \mathbf{P}D$ and $\nu \in \mathbf{P}E$, we have the product valuation $\mu \times \nu \in \mathbf{P}(D \times E)$ which for any "rectangle" $O_1 \times O_2$, where O_1 and O_2 are open subsets of D and E respectively, satisfies $(\mu \times \nu)(O_1 \times O_2) = \mu(O_1)\nu(O_2)$ [11]. Moreover, the product operation is continuous. In other words, we have the following property.

Proposition 4.1 *Suppose $\mu = \bigsqcup_{i\in I} \mu_i \in \mathbf{P}D$ and $\nu = \bigsqcup_{i\in I} \nu_i \in \mathbf{P}E$, where $\langle \mu_i \rangle_{i\in I}$ and $\langle \nu_i \rangle_{i\in I}$ are directed sets in $\mathbf{P}D$ and $\mathbf{P}E$ respectively. Then,*

$$\mu \times \nu = \bigsqcup_{i\in I} \mu_i \times \nu_i. \qquad \square$$

Furthermore, the product of two simple valuations is another simple valuation.

Proposition 4.2 *Let*

$$\alpha = \sum_{a \in A} r_a \eta_a \qquad \textit{and} \qquad \beta = \sum_{b \in B} s_b \eta_b$$

be simple valuations on dcpo's D and E . Then, the product valuation $\alpha \times \beta$ is a simple valuation on $D \times E$ and is given by

$$\alpha \times \beta = \sum_{a \in A, b \in B} r_a s_b \eta_{(a,b)}. \qquad \square$$

Let X and Y be compact metric spaces, $\mu \in \mathbf{M}^1 X$, $\nu \in \mathbf{M}^1 Y$. We have the product measure $\mu \times \nu \in \mathbf{M}^1(X \times Y)$. On the other hand, by the remark following Theorem 2.1, there exist increasing chains $\langle \mu_i \rangle_{i \geq 0}$ and $\langle \nu_i \rangle_{i \geq 0}$ of simple valuations $\mu_i \in \mathbf{P}^1\mathbf{U}X$ and $\nu_i \in \mathbf{P}^1\mathbf{U}Y$ for $i \geq 0$ such that

$$\mu \circ s_X^{-1} = \bigsqcup_{i \geq 0} \mu_i \qquad \text{and} \qquad \nu \circ s_Y^{-1} = \bigsqcup_{i \geq 0} \nu_i$$

where $s_X : X \to \mathbf{U}X$ and $s_Y : Y \to \mathbf{U}Y$ are the singleton maps. For each $i \geq 0$, we have the product valuation $\mu_i \times \nu_i \in \mathbf{P}^1(\mathbf{U}X \times \mathbf{U}Y)$ which is a simple valuation by Proposition 4.2. Furthermore, by Proposition 4.1, we have

$$\mu \circ s_X^{-1} \times \nu \circ s_Y^{-1} = \bigsqcup_{i \geq 0} \mu_i \times \nu_i. \tag{16}$$

Note that $\mathbf{U}X \times \mathbf{U}Y$ is a subspace of $\mathbf{U}(X \times Y)$. Let $i : \mathbf{U}X \times \mathbf{U}Y \to \mathbf{U}(X \times Y)$ be the inclusion map. Then, i is Scott continuous and we have the continuous embedding

$$\begin{array}{rccc} \hat{\imath}: & \mathbf{P}^1(\mathbf{U}X \times \mathbf{U}Y) & \to & \mathbf{P}^1\mathbf{U}(X \times Y) \\ & \lambda & \mapsto & \lambda \circ i^{-1}. \end{array}$$

It is easy to check that

$$\hat{\imath}(\mu \circ s_X^{-1} \times \nu \circ s_Y^{-1}) = (\mu \times \nu) \circ s_{X \times Y}^{-1}$$

where $s_{X \times Y} : X \times Y \to \mathbf{U}(X \times Y)$ is the singleton map. Furthermore, the product of the two simple valuations

$$\alpha = \sum_{a \in A} r_a \eta_a \in \mathbf{P}^1\mathbf{U}X \qquad \beta = \sum_{b \in B} s_b \eta_b \in \mathbf{P}^1\mathbf{U}Y \tag{17}$$

satisfies

$$\begin{aligned} \hat{\imath}(\alpha \times \beta) &= (\alpha \times \beta) \circ i^{-1} \\ &= \Big(\sum_{a \in A, b \in B} r_a s_b \eta_{(a,b)}\Big) \circ i^{-1} \\ &= \sum_{a \in A, b \in B} r_a s_b \eta_{i(a,b)} \\ &= \sum_{a \in A, b \in B} r_a s_b \eta_{(a,b)} \\ &= \alpha \times \beta \end{aligned}$$

By Equation (16) and the continuity of $\hat{\imath}$, we finally deduce:

Proposition 4.3

$$(\mu \times \nu) \circ s_{X \times Y}^{-1} = \bigsqcup_{i \geq 0} \mu_i \times \nu_i. \qquad \square$$

Let $f : X \times Y \to \mathbb{R}$ be a bounded function. For the simple valuations α and β given by Equation (17), we have the lower sum of f with respect to $\alpha \times \beta$ defined by

$$S^{\ell}(f, \alpha \times \beta) = \sum_{a \in A, b \in B} r_a s_b \inf f(a, b),$$

where $f(a, b)$ is the image of $(a, b) \subseteq X \times Y$ under f. Similarly, the upper sum of f with respect to $\alpha \times \beta$ is defined by

$$S^{u}(f, \alpha \times \beta) = \sum_{a \in A, b \in B} r_a s_b \sup f(a, b). \quad \square$$

Furthermore, for a given choice of $x_a \in a$ and $y_b \in b$ for each $a \in A$ and $b \in B$, the generalised Riemann sum of f with respect to $\alpha \times \beta$ is defined by

$$S(f, \alpha \times \beta) = \sum_{a \in A, b \in B} r_a s_b f(x_a, y_b).$$

See [7] for details. The main result of this section is the following.

Theorem 4.4 *If $f : X \times Y \to \mathbb{R}$ is continuous almost everywhere with respect to the product measure $\mu \times \nu$, then*

$$S^{\ell}(f, \mu_i \times \nu_i) \nearrow \int_{X \times Y} f \, d\mu \times \nu, \qquad S^{u}(f, \mu_i \times \nu_i) \searrow \int_{X \times Y} f \, d\mu \times \nu,$$

$$S(f, \mu_i \times \nu_i) \to \int_{X \times Y} f \, d\mu \times \nu$$

as $i \to \infty$. $\square$

4.2 Integration over the invariant measure of an IFS

Suppose now we have an IFS with probabilities $(X; f_1, \ldots, f_N; p_1, \ldots, p_N)$ such that $\nu^* = \bigsqcup_{m \geq 0} H^m \eta_X$, where $H^m \eta_X$ is given by Equation (15), is a maximal element of $\mathbf{P}^1\mathbf{U}X$. If $g : X \to \mathbb{R}$ is continuous almost everywhere with respect to $\mu^* = \nu^* \circ s$, then for any point $x \in X$ we have the generalised Riemann sum for $H^m \eta_X$ given by

$$S(g, H^m \eta_X) = \sum_{i_1, \ldots, i_m = 1}^{N} p_{i_1} \ldots p_{i_m} g(f_{i_1} \ldots f_{i_m} x)$$

and it follows that

$$S(g, H^m \eta_X) \to \int g \, d\mu^*$$

as $m \to \infty$. If the maps f_i are contacting and g satisfies a Lipschitz condition, then we can obtain a finite algorithm to calculate the integral to any given accuracy as follows [8]. Suppose there exist $k > 0$ and $c > 0$ such that g satisfies

$$|g(x) - g(y)| \leq c(d(x, y))^k$$

for all $x, y \in X$. (When $k = 1$, we call c a *Lipschitz constant* for g.) Let $\epsilon > 0$ be given. Then

$$|S(g, H^m \eta_X) - \int g d\mu^*| \leq \epsilon$$

for $m = \lceil \log((\epsilon/c)^{1/k}/|X|)/\log s \rceil$, where s is the contractivity of the IFS and $\lceil a \rceil$ is the least integer greater than or equal to a.

Next, assume that $h : X\times X \to \mathbb{R}$ is continuous almost everywhere with respect to $\mu^*\times\mu^*$. Then, for any $x\in X$, we have the following generalised Riemann sum for $H^m\eta_X \times H^m\eta_X$:

$$S(h, H^m\eta_X \times H^m\eta_X) = \sum_{i_1,\ldots,i_m=1}^{N} \sum_{j_1,\ldots,j_m=1}^{N} p_{i_1}\ldots p_{i_m} p_{j_1}\ldots p_{j_m} h(f_{i_1}\ldots f_{i_m}x, f_{j_1}\ldots f_{j_m}x)$$

and it follows that

$$S(h, H^m\eta_X \times H^m\eta_X) \to \int\int h(x,y)\, d\mu^*(x) d\mu^*(y) \tag{18}$$

as $m\to\infty$. If h satisfies a Lipschitz condition then again we can obtain a finite algorithm to estimate $\int h\, d\mu^*\times\mu^*$ to within any given threshold.

We also have ergodic theorems for IFSs with probabilities. Let $(X; f_1,\ldots,f_N; p_1,\ldots,p_N)$ be a hyperbolic IFS with probabilities with invariant measure μ^*. Let $g : X\to\mathbb{R}$ be a continuous function. Suppose $i_1, i_2, \ldots$ is is a sequence of independent identically distributed random variables on $\{1,2,\ldots,N\}$ with probabilities $P(i_n = k) = p_k$ $(1\le k\le N)$ for all $n\ge 1$. Let $\xi_{i_1}\in X$, and put $\xi_{i_{n+1}} = f_{i_n}(\xi_{i_n})$ for all $n\ge 1$. Then Elton's ergodic theorem states:

Theorem 4.5 *[9] For almost all sequences $i_1, i_2, \ldots$, and all $\xi_1\in X$,*

$$\int\int g(x)\, d\mu^*(x) = \lim_{k\to\infty} \frac{1}{k}\sum_{n=1}^{k} g(\xi_{i_n}). \qquad \square$$

It is also possible to deduce a two-dimensional version of Elton's Theorem. Let $h : X\times X\to\mathbb{R}$ be a continuous function. Let $i_1, i_2, \ldots$ and $j_1, j_2, \ldots$ be two sequences of independent identically distributed random variables on $\{1,2,\ldots,N\}$ with probabilities $P(i_n = k) = P(j_n = k) = p_k$ $(1\le k\le N)$ for all $n\ge 1$. Let $\xi_{i_1}, \zeta_{j_1}\in X\times X$, and put $\xi_{i_{n+1}} = f_{i_n}(\xi_{i_n})$ and $\zeta_{j_{n+1}} = f_{j_n}(\zeta_{j_n})$ for all $n\ge 1$. Then, we have:

Theorem 4.6 *For almost all sequences $i_1, i_2, \ldots$ and $j_1, j_2, \ldots$, and all $\xi_1, \zeta_1\in X$,*

$$\int\int h(x,y)\, d\mu^*(x) d\mu^*(y) = \lim_{k\to\infty} \frac{1}{k}\sum_{n=1}^{k} h(\xi_{i_n}, \zeta_{j_n}). \qquad \square$$

5 Domain-theoretic analysis of the Ising model

In this section, we apply the results of the previous two sections to the one-dimensional random field Ising model (1d RFIM) that we described in the introduction. We consider a more general case and allow the external magnetic field be asymmetric, i.e. $h_n = h_\pm$, for real numbers h_+ and h_- with $h_+\ge -h_- > 0$. We assume, however, for convenience that the two value of the field are still equally probable, i.e. $P(h_n = h_+) = P(h_n = h_-) = 1/2$ for all $n\ge 1$.

5.1 Invariant measure of the 1d RFIM

Consider the IFS $([x_-, x_+]; f_-, f_+; 1/2, 1/2)$ where the two maps $f_+, f_- : \mathbb{R}\to\mathbb{R}$ are defined by $f_\pm(x) = h_\pm + A(x)$, with $A(x) = (2\beta)^{-1}\log(\cosh\beta(x+J)/\cosh\beta(x-J))$, as given by Equations (9) and (2). Their unique fixed points x_+, x_- are given by $x_\pm = h_\pm/2 + (2\beta)^{-1}\operatorname{arcsinh}(e^{2\beta J}\sinh\beta h_\pm)$, with $x_+\ge -x_- > 0$, cf. equation (10). A simple calculation shows that the first and the second derivatives of the two maps are given by

$$A'(x) = \frac{1}{2}(\tanh\beta(x+J) - \tanh\beta(x-J))$$
$$A''(x) = \frac{\beta}{2}\left(\frac{1}{\cosh^2\beta(x+J)} - \frac{1}{\cosh^2\beta(x-J)}\right).$$

It follows that $1 > A'(x) > 0$ for all $x\in[x_-, x_+]$ (in fact for all $x\in\mathbb{R}$). Furthermore, since A'' is negative in $(0, x_+]$ and $A(x) = -A(-x)$, we easily see that A', and hence f'_+ and f'_-, have maximum value

$$s = A'(0) = \tanh \beta J \tag{19}$$

in the interval $[x_-, x_+]$ with $0 < s < 1$. By the intermediate value theorem, it follows that A, f_+, f_- are contracting with contractivity factor s. Therefore, by Corollary 3.4, the IFS has a unique invariant measure ν^* and starting with any initial valuation $\nu \in \mathbf{P}^1\mathbf{U}X$ we have $\lim_{m\to\infty} H^m(\nu) = \nu^*$. Moreover, there is a finite algorithm to generate ν^* on any digitised screen [8].

An analysis of the multifractal nature of the invariant measure $\mu^* = \nu^* \circ s^{-1}$ is given in [1, 3]. When $f_+[x_-, x_+] \cap f_-[x_-, x_+] = \emptyset$ the support of μ^* is totally disconnected and homoeomorphic to the Cantor set. On the other hand, when $f_+[x_-, x_+] \cap f_-[x_-, x_+] \neq \emptyset$, the support of ν^* is the whole interval $[x_-, x_+]$ and is a fat fractal as in Figure 4, which is taken from [2].

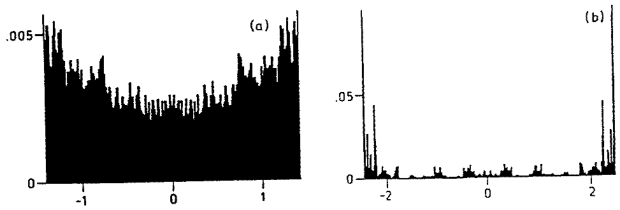


Figure 4. Distribution of the local magnetisation ($kT = 1$, $J = 1$). The multifractal can be fat (a) for $h_+ = -h_- = .6$ or thin (b) for $h_+ = -h_- = 1.5$.

5.2 Expected value of physical quantities

We will now use the results of Section 4, to obtain finite algorithms to compute the expected values of various physical quantities to any given threshold.

5.2.1 Free energy density

The free energy F plays the same fundamental role in thermal physics that ordinary energy plays in mechanics. It is given in terms of the partition function Z by $F(\beta) = -\frac{1}{\beta}\log Z$. See for example [13]. Therefore, for the Ising model of N spins, Equation (6) gives

$$\begin{aligned} F_N(\beta) &= -\frac{1}{\beta}\log Z_N \\ &= -\frac{1}{\beta}\log(\exp\beta\{\frac{1}{\beta}\log(2\cosh\beta\xi_N) + \sum_{n=1}^{N-1} B(\xi_n)\}) \\ &= -\frac{1}{\beta}\log(2\cosh\beta\xi_N) - \sum_{n=1}^{N-1} B(\xi_n), \end{aligned}$$

where the function B is given by Equation (3). Therefore, the free energy density $f(\beta)$ is given in the limit $N \to \infty$ by

$$\begin{aligned} f(\beta) &= \lim_{N\to\infty}\frac{F_N(\beta)}{N} = -\frac{1}{\beta}\lim_{N\to\infty}\frac{\log Z_N}{N} \\ &= -\lim_{N\to\infty}\frac{1}{N}\log(2\cosh\beta\xi_N) - \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1} B(\xi_n) \\ &= -\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1} B(\xi_n) \end{aligned}$$

If the field h_n is constant then $\lim_{n\to\infty}\xi_n = x^*$ and by the continuity of B we have $f(\beta) = -B(x^*)$ as in [10].

For the random field Ising model, Elton's ergodic Theorem 4.5 implies

$$f(\beta) = -\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1} B(\xi_n) = -\int B\,d\mu^*,$$

since the IFS is hyperbolic and B is a continuous function. A simple calculation, similar to the case of A, shows that $B : [x_-, x_+] \to [x_-, x_+]$ is contracting with contractivity factor

$$c = B'(x_+) = \frac{1}{2}(\tanh\beta(x_+ + J) + \tanh\beta(x_+ - J)). \tag{20}$$

Therefore, for the choice of $0 \in [x_-, x_+]$, we consider the generalised Riemann sums

$$S(B, H^n\eta_{[x_-,x_+]}) = \frac{1}{2^n}\sum_{i_1,\ldots,i_n=\pm} B(f_{i_1}\ldots f_{i_n}(0)).$$

Given $\epsilon > 0$, we have

$$|S(B, H^n\eta_{[x_-,x_+]}) - \int B\,d\mu^*| < \epsilon,$$

with $n = \lceil\log((\epsilon/c)/(x_+ - x_-))/\log s\rceil$, where s is the contractivity of the IFS given by Equation (19), c is given by Equation (20) and $\lceil a\rceil$ is, as before, the least integer greater than or equal to a.

5.2.2 Magnetisation per spin

The magnetisation per spin is given by

$$m(\beta) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\langle s_k\rangle_{\mathcal{H}_N},$$

where $\langle s_k\rangle_{\mathcal{H}_N}$ is the average value of s_k with respect to the Hamiltonian $\mathcal{H}_N$ of the N spins $\langle s_n\rangle_{n=1}^N$. In order to find $\langle s_k\rangle_{\mathcal{H}_N}$, one reduces the partition function Z_N to that of a typical spin s_k with $1 \le k \le N$. This is done by applying the technique of summation, described in the introduction, over spins from left *and* right as in [4]. Let ξ_n be as before, and let $\zeta_N = h_N$ and

$$\zeta_{n-1} = h_{n-1} + A(\zeta_n)$$

for $1 \le n \le N-1$. Then, it is easy to deduce that

$$\begin{aligned} Z_N &= \sum_{s_k=\pm 1}\exp\beta((\xi_k + A(\zeta_k))s_k + \sum_{n=1}^{k-1} B(\xi_n) + \sum_{n=N}^{k-1} B(\zeta_n)) \\ &= 2\cosh\beta(\xi_k + A(\zeta_k)) \times \exp\beta(\sum_{n=1}^{k-1} B(\xi_n) + \sum_{n=N}^{k-1} B(\zeta_n)). \end{aligned}$$

Therefore,

$$\begin{aligned} \langle s_k\rangle_{\mathcal{H}_N} &= \frac{1}{Z_N}\{\sum_{s_k=\pm 1} s_k\exp(\beta(\xi_k + A(\zeta_k)))\} \times \exp\beta(\sum_{n=1}^{k-1} B(\xi_n) + \sum_{n=N}^{k+1} B(\zeta_n)) \\ &= \frac{1}{Z_N}\{2\sinh(\xi_k + A(\zeta_k))\} \times \exp\beta(\sum_{n=1}^{k-1} B(\xi_n) + \sum_{n=N}^{k+1} B(\zeta_n)) \\ &= \tanh\beta(\xi_k + A(\zeta_k)). \end{aligned}$$

If the field $h_n = h$ is constant, then $\xi_k, \zeta_k \to x^*$, where x^* is the unique solution of $A(x) = h + x$, and hence $m(\beta) = \tanh\beta(2x^* + h)$ as in [10].

In order to compute the magnetisation per spin for the case of the random field, we use the two-dimensional version of Elton's Ergodic Theorem 4.6 as follows.

$$\begin{aligned} m(\beta) &= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\langle s_k\rangle_{\mathcal{H}_N} \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\tanh\beta(\xi_k+A(\zeta_k)) \\ &= \int\int \tanh\beta(x+A(y))\,d\mu^*(x)d\mu^*(y), \end{aligned}$$

where the double integral can be computed by Equation (18). We can deduce a finite algorithm to compute $m(\beta)$ to within an accuracy of any given $\epsilon>0$. The function

$$\tanh\beta(-):\mathbb{R}\to\mathbb{R}$$

has Lipschitz constant β. Hence for $x_1,x_2,y_1,y_2\in[x_-,x_+]$ we have

$$\begin{aligned} |\tanh\beta(x_1+A(y_1))-\tanh\beta(x_2+A(y_2))| &\le \beta(|x_1-x_2|+|A(y_1)-A(y_2)|) \\ &\le \beta(|x_1-x_2|+s|y_1-y_2|) \end{aligned}$$

where $s=\tanh\beta T$ is the contractivity of $A:[x_-,x_+]\to[x_-,x_+]$. For convenience, we put $h(x,y)=\tanh\beta(x+A(y))$ and $\nu_n=H^n\eta_{[x_+,x_-]}$ where H is the transition operator. We choose $0\in[x_-,x_+]$ for the generalised Riemann sum. It follows that if $s^n(x_+-x_-)\le\epsilon/\beta(1+s)$, i.e. if

$$n=\lceil\frac{\log(\epsilon/\beta(1+s)(x_+-x_-))}{\log s}\rceil,$$

then

$$|S(h,\nu_n\times\nu_n)-m(\beta)|=|S(h,\nu_n\times\nu_n)-\int\int\tanh\beta(x+A(y))\,d\mu^*(x)d\mu^*(y)|\le\epsilon,$$

where

$$S(h,\nu_n\times\nu_n)=\frac{1}{2^{2n}}\sum_{i_1,\ldots,i_n=\pm}\ \sum_{j_1,\ldots,j_n=\pm}\tanh\beta(f_{i_1}\ldots f_{i_n}(0)+A(f_{j_1}\ldots f_{j_n}(0))).$$

5.2.3 Edwards-Anderson parameter

Similarly, we can obtain a finite algorithm for computing the Edwards-Anderson parameter q_{EA} given by

$$q_{EA}=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\langle s_k\rangle^2_{\mathcal{H}_N}=\int\int(\tanh\beta(x+A(y)))^2\,d\mu^*(x)d\mu^*(y).$$

The function

$$\tanh^2\beta(-):\mathbb{R}\to\mathbb{R}$$

has Lipschitz constant $4\beta\sqrt{3}/9$. It follows that if

$$n=\lceil\frac{\log(9\epsilon/4\sqrt{3}\beta(1+s)(x_+-x_-))}{\log s}\rceil,$$

then

$$|S(h^2,\nu_n\times\nu_n)-q_{EA}|=|S(h^2,\nu_n\times\nu_n)-\int\int\tanh^2\beta(x+A(y))\,d\mu^*(x)d\mu^*(y)|\le\epsilon,$$

where

$$S(h^2,\nu_n\times\nu_n)=\frac{1}{2^{2n}}\sum_{i_1,\ldots,i_n=\pm}\ \sum_{j_1,\ldots,j_n=\pm}\tanh^2\beta(f_{i_1}\ldots f_{i_n}(0)+A(f_{j_1}\ldots f_{j_n}(0))).$$

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