Uniform Differentiation and Domains of Computation for Real and Complex Lipschitz Vector Functions

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Abstract

We introduce the notion of a partial extension of an arbitrary, not necessarily continuous, map from a subset of a topological space into a continuous Scott domain and deduce a universal property for such a domain with respect to partial maps. It is then proved that the Clarke gradient of a Lipschitz map between two finite dimensional real Euclidean spaces is the partial extension of the classical derivative of the Lipschitz map, which exists on a dense subset. A uniform domain-theoretic formalism is developed for deriving, stage by stage, the so-called L-derivative of a Lipschitz map from a Hilbert space into a finite dimensional Hilbert space over the field of real or complex numbers respectively. In the real case, it is shown that the L-derivative of a map between finite dimensional Euclidean spaces always coincides with the Clarke gradient. In the complex case, we derive a generalisation of Cauchy Riemann equations for Lipschitz maps between finite dimensional complex vector spaces by introducing the notion of multi-dimensional angular derivative of a complex map. We then introduce the notion of distance to the maximal elements for Scott continuous maps that are, like the L-derivative, compact valued, and prove that it has a Scott continuity property. Finally, we use Arzelà-Ascoli's theorem to construct for the first time continuous Scott domains of computation for Lipschitz maps between finite dimensional real or complex vector spaces.

1 Introduction

The aim of this paper is to show that a simple domain-theoretic method can be developed and uniformly used to define the notion of the so-called L-derivative for Lipschitz vector maps on real or complex finite or infinite dimensional Hilbert spaces. The domain-theoretic L-derivative of a Lipschitz map at a point is obtained simply as the intersection of a family of non-empty convex and compact subsets. In previous work on the subject [10, 7, 11], only scalar maps had been treated and it was shown that for real-valued Lipschitz maps on finite dimensional real Euclidean spaces the L-derivative coincides with the Clarke gradient [4], which had been introduced using several higher order concepts in non-smooth optimisation and control in 1970's. We note, in passing, that another application of domain theory in differential calculus, not directly related to this work, has been established in [3].

We have developed new techniques and results in two different areas in order to extend the theory further to vector functions on infinite dimensional vector spaces. These extensions are required to develop domain-theoretic models of real or complex Lipschitz manifolds on the one hand and for applications in quantum computation which involves maps between finite dimensional complex vector spaces.

First, in the theory of Hilbert spaces, we show that a certain logical predicate, with quantifiers of depth one, on a non-empty, convex weak* compact subset of a finite product of the dual of the Hilbert space is equivalent to the containment of the zero functional. This property actually allows us to show that the L-derivative of any vector Lipschitz map on a finite or infinite dimensional Hilbert space is a Scott continuous map extending the classical counterparts in differentiable calculus and complex analysis: the L-derivative of a C^1 vector map and that of an analytic vector map coincide with their classical derivatives. To make the paper more accessible, we will first introduce the result for the logical predicate in the Basic Lemma 3.4 for finite dimensional real or complex Euclidean spaces before presenting the result for infinite dimensional Hilbert spaces.

Second, in domain theory, the notion of extension of a domain map is generalised to that of a partial extension. We define a partial extension of an arbitrary (not necessarily continuous) map from a subset of a topological space into a continuous Scott domain, by stipulating that the extended map is below the original map wherever the latter is defined. This can be interpreted by saying that a partial extension is sound but not necessarily complete. We then show that continuous Scott domains have the universal property that every map from a dense subset of a topological space into a continuous Scott domain has a maximal continuous partial extension to the whole space. This should be compared with the characterisation of continuous Scott domains as densely injective spaces, which were introduced by Dana Scott [22]. Continuous Scott domains are characterised by the universal property that any continuous map from a dense subset of a topological space into a continuous Scott domain has a continuous extension to the whole space. Our result strengthens this universal property by showing that the original map need not be continuous and yet we obtain a continuous partial extension. In addition, if the original map is continuous at a given point then the partial extension coincides with the original map at that point. We then show that the generalised Jacobians of a Lipschitz map between finite dimensional Euclidean spaces are simply the partial extension of the classical derivative which exists on a dense set for the Lipschitz map by Rademacher's theorem [5, p. 148]. This allows us to prove the equivalence of the Clarke gradient and the L-derivative for maps between finite dimensional Euclidean spaces.

The uniform and simple method of deriving the L-derivative of a complex Lipschitz map between finite dimensional complex vector spaces allows us to find a connection between this L-derivative and the L-derivatives of the real and imaginary parts of the function, which gives a multi-dimensional generalisation of the Cauchy-Riemann equations, which generalises the one dimensional result in [8]. This is achieved by introducing a multi-dimensional notion of complex angular derivative which extends the one-dimensional angular derivative, which is attributed to Riemann [15, p. 14].

We also formulate a measure of proximity of the L-derivative to a C^1 map or to an analytic map as the distance of the map to the set of maximal elements of the function space which correspond to the classical derivatives of maps. We use König's lemma to show that this distance is Scott continuous if we restrict the maps to a compact set.

Two main results in this paper use the L-derivative for Lipschitz vector maps to construct, for the first time, continuous domains for Lipschitz maps between finite dimensional real or complex Euclidean spaces. We employ two fundamental theorems in mathematical analysis for this construction: (i) Kirszbraun's theorem [5, p. 8] that a Lipschitz map can be extended from any subset of a Hilbert space to the whole space with the same Lipschitz constant and (ii) the celebrated Arzelà-Ascoli theorem for a uniformly bounded and equi-continuous family of maps.

Finally, we show that for the case of real Lipschitz vector maps, we can obtain an effective structure for our continuous Scott domain if we approximate the L-derivative of the vector map by the product of the best axis aligned rectangular approximations, as is done in interval analysis [19]. We also construct a bigger continuous Scott domain for Lipschitz complex maps which we propose would be suitable for developing an effective structure.

1.1 Notation and terminology

Let $\mathbb{F}$ be the field of real numbers $\mathbb{R}$ or that of complex numbers $\mathbb{C}$ and $\mathcal{H}$ a Hilbert space over $\mathbb{F}$. For $u \in \mathbb{F}$, we denote by $|u|$ its absolute value if $u \in \mathbb{R}$ and its modulus $|u| = \sqrt{u_1^2 + u_2^2}$ (also called its absolute value) if $u = u_1 + iu_2 \in \mathbb{C}$. For any positive integer m, we equip $\mathbb{F}^m$, with the Euclidean norm $\|v\| = \sqrt{\sum_{i=1}^m |v_i|^2}$. If $u, v \in \mathbb{F}$ then uw denotes their real product if $u, w \in \mathbb{R}$ and their complex product $uw = (u_1w_1 - u_2w_2) + i(u_1w_2 + u_2w_1)$ if $u, w \in \mathbb{C}$ with $u = u_1 + iu_2$ and $w = w_1 + iw_2$. From these usual conventions, it follows that for a vector $v \in \mathbb{F}^n$ and a linear map of type $\mathbb{F}^n \to \mathbb{F}^m$, represented in the standard coordinate system by the matrix $A \in \mathbb{F}^{m\times n}$, the value of the linear map A at v is written as usual by $Av \in \mathbb{F}^m$ with $(Av)_i = \sum_{j=1}^n A_{ij}v_j$ where $A_{ij}v_j$ is the product in $\mathbb{R}$ or $\mathbb{C}$.

We equip the vector space $\mathbb{F}^{m\times n}$ of $m \times n$ matrices over $\mathbb{F}$ with the Frobenius norm $\|A\|$ given by

$$\|A\|^2 = \sum_{1\leq i\leq m} \sum_{1\leq j\leq n} |A_{ij}|^2$$

for $A \in \mathbb{F}^{m\times n}$. Note that the Frobenius norm is subordinate to the Euclidean norm of vectors in $\mathbb{F}^n$, i.e., $\|Av\| \leq \|A\|\|v\|$. Consider the domain $\mathbf{C}(\mathbb{F}^{m\times n})$ of non-empty, convex and compact subsets of $\mathbb{F}^{m\times n}$ ordered by reverse inclusion and augmented with $\mathbf{C}(\mathbb{F}^{m\times n})$. By [14, Proposition II-4.20(iv)], it is a continuous Scott domain with the way-below relation given by $b \ll c$ iff $c \subseteq b^\circ$, where b° is the interior of b. For $n = 1$ we identify $\mathbf{C}(\mathbb{F}^{m\times 1})$ with the domain $\mathbf{C}(\mathbb{F}^m)$ of non-empty compact and convex subsets of $\mathbb{F}^m$. We extend the notion of Frobenius norm to $\mathbf{C}(\mathbb{F}^{m\times n})$, by putting $\|b\| = \{\|A\| : A \in b\}$ for $b \in \mathbf{C}(\mathbb{F}^{m\times n})$. We denote by $\mathbf{I}\mathbb{R}$ the continuous Scott domain of the non-empty compact intervals of $\mathbb{R}$ ordered with reverse inclusion and augmented with $\mathbb{R}$ as the bottom element. As usual, we identify any singleton element $\{x\}$ of $\mathbf{I}\mathbb{R}$ or $\mathbf{C}(\mathbb{F}^{m\times n})$ with $x \in \mathbb{R}$ or $x \in \mathbb{F}^{m\times n}$, respectively. The characteristic function of a set A is denoted by χ_A.

For an open subset $U \subseteq \mathbb{F}^n$, consider locally Lipschitz maps of type $U \to \mathbb{F}^n$, i.e., there exists $k \geq 0$ such that $\|f(v) - f(w)\| \leq k\|v - w\|$. The *total derivative* of a map $f : U \to Y$ at $x \in U$, when it exists, is a bounded linear map $L : \mathbb{F}^n \to \mathbb{F}^m$ with

$$\lim_{\|h\|\to 0} \frac{\|f(x + h) - f(x) - L(h)\|}{\|h\|} = 0.$$

When $\mathbb{F} = \mathbb{R}$, this is simply the *Fréchet* derivative [24]. In this case, if the total derivative f' exists in an open set $O \subset U$ and the map $x \mapsto f'(x)$ is continuous, then we say f is C^1 in O. When $\mathbb{F} = \mathbb{C}$, if the total derivative exists then all partial derivatives $\frac{\partial f_j}{\partial z_i}$ exists for $1 \leq j \leq m$ and

$1 \leq i \leq n$. Thus, if the total derivative exists in a an open set, then $f_j : (z_1, \ldots, z_n) \mapsto f_j(z_1, \ldots, z_n)$ is analytic separately in each complex variable z_i, which by definition implies that f_i, and thus f, is analytic [20] as a function of several complex variables.

1.2 Partial Extension of Domain Maps

Recall that a T_0 topological space Z is densely injective if every continuous map $f : X \to Z$ extends continuously to any space Y containing X as a dense subset; and furthermore, a T_0 space is densely injective iff it is a continuous Scott domain equipped with its Scott topology [14, p. 181-182]. This provides a topological characterisation of continuous Scott domains, also referred to as bounded complete domains. We will now observe, with a simple proof, that continuous Scott domains have in fact a stronger property, regarding extension of functions that are not necessarily continuous. This gives a domain-theoretic generalisation of the notion of the lower envelop of a real valued function in mathematical analysis.

Definition 1.1. Assume that Z is a continuous Scott domain and $f : X \to Z$ is *any* map where X is a subset of a topological space Y. We say a map $f^* : Y \to Z$ is a *partial extension* of f if $f^*(x) \sqsubseteq f(x)$ for $x \in X$. We say a continuous partial extension f^* is *maximal* if for any other continuous partial extension g of f, we have $g(y) \sqsubseteq f^*(y)$ for $y \in Y$.

Proposition 1.2. *If X is a dense subset of Y then, any map $f : X \to Z$ has a maximal continuous partial extension $f^* : Y \to Z$ given by*

$$f^*(y) := \sup\{\inf f(O \cap X) : O \textit{ is open, } y \in O\}.$$

Moreover, if f is continuous at $x \in X$ then $f^*(x) = f(x)$.

Proof. Clearly $f^*(x) \sqsubseteq f(x)$ for $x \in X$, and thus f^* is a partial extension. We now note that the proof of continuity of f^* given in [14, p. 181], for the case when f is itself continuous, does not actually use the continuity of f. Moreover, in the above proof, in showing that $f^*(x) = f(x)$ for any $x \in X$, only the continuity of f at x is used. Thus, by using these two parts of the proof in [14], it follows that in our case too f^* is continuous and if f is continuous at $x \in X$ then $f(x) = f^*(x)$. It remains to show that f^* is the maximal extension. Let $y \in Y$ and $q \ll g(y)$, where g is another continuous partial extension of f. Then, there exists an open set O containing y such that $q \ll u$ for all $u \in g(O)$. Thus,

$$q \sqsubseteq \inf g(O) \sqsubseteq \inf g(O \cap X) \sqsubseteq f(O \cap X) \sqsubseteq f^*(y).$$

Since this holds for all $q \ll g(y)$, we obtain $g(y) \sqsubseteq f^*(y)$. □

There are many simple examples of partial extension of maps in analysis. For example, take the step function $S : \mathbb{R} \to \mathbb{R} \subseteq \mathbf{I}\mathbb{R}$ defined by $S(x) = 0$ if $x < 0$, $S(x) = 1$ if $x > 0$ and $S(0) = a$ for some $a \in [0, 1]$. Then $S^*(x) = 0$ if $x < 0$, $S^*(x) = 1$ if $x > 0$ and $S^*(0) = [0, 1]$. We give two other examples.

Example 1.3. Consider the sawtooth wave $S : \mathbb{R} \to \mathbb{R} \subseteq \mathbf{I}\mathbb{R}$ as a periodic function defined by $S(x) = x - \lfloor x \rfloor$. Clearly, S has a discontinuity at each $n \in \mathbb{Z}$. Its partial extension $S^* : \mathbb{R} \to \mathbf{I}[0, 1]$ is given by $S^*(x) = x - \lfloor x \rfloor$ if $x \notin \mathbb{Z}$ and $S^*(x) = [0, 1]$ for $x \in \mathbb{Z}$.

Example 1.4. We now give an example, presented in [9, Example 6.6], of a map $f : [-1, 1] \to [0, 1] \subseteq \mathbf{I}[0, 1]$ whose set of discontinuities is uncountable with positive Lebesgue measure λ. Take any positive real number $a < 1$. The construction in [9] gives two disjoint open subsets $B, C \subseteq [-1, 1]$ with $\lambda(B) = \lambda(C) = a$ and $B = (C^c)^\circ$, i.e., the interior of the complement C^c of C. Define $f : [-1, 1] \to \mathbf{I}[0, 1]$ with $f(x) = 0$ if $x \in \mathrm{cl}(B)$ and $f(x) = 1$ if $x \in C$, where $\mathrm{cl}(B)$ is the closure of B. Then $D := [-1, 1] \setminus (B \cup C)$ is the set of discontinuities of f and has Lebesgue measure $\lambda(D) = 2 - 2a$. The partial extension is given by $f^* : [-1, 1] \to \mathbf{I}[0, 1]$ with $f^*(x) = 0$ if $x \in B$, $f(x) = 1$ if $x \in C$ and $f(x) = [0, 1]$ if $x \in D$.

We will see a prominent example of partial extension of domain maps in the next section.

2 Generalised Jacobians

The notion of partial extension can be applied to nonsmooth analysis. We recall the notion of generalised Jacobians of real Lipschitz vector functions as introduced by Clarke and presented in [4, section 2.6]. First, note that by Rademacher's theorem [5, page 148], a locally Lipschitz map $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable almost everywhere with respect to the Lebesgue measure. Let Ω_f be the null set where f fails to be differentiable. The generalised (Clarke) Jacobian ∂f is defined to be:

$$\partial f(x) = \mathrm{conv}\{\lim_{j \to \infty} Jf(x_j) : x_j \to x, \quad x_j \notin \Omega_f\}, \quad (1)$$

where $Jf(x)$ denotes the Jacobian of f at $x \in U$. The above formula is to be interpreted as follows. There are many sequences (x_j) on $U \setminus \Omega_f$, which converge to x such that $Jh(x_j)$ also converges to a limit; the generalised Jacobian $\partial f(x)$ is the convex hull of all such limits. Let the the vector space, $\mathbb{R}^{m \times n}$, of $m \times n$ matrices over real numbers be equipped with the Frobenius norm. By [4, Proposition 2.6.2], $\partial f_j(x)$ is a non-empty convex compact subset of $\mathbb{R}^{m \times n}$, and the map $\partial f : U \to \mathbf{C}(\mathbb{R}^{m \times n})$ is upper semi-continuous (equivalently Scott continuous). We have: $\partial f(x) \subseteq \partial f_1(x) \times \ldots \times \partial f_m(x)$, where the latter denotes the set of $m \times n$ matrices whose jth row belong to $f_j(x)$. Note that from the definition, we may in principle obtain a different set $\partial_A f(x)$ if instead of Ω_f we use a null set $A \supset \Omega_f$ instead of Ω_f. The problem whether $\partial f(x)$ has an intrinsic value independent of any such null set A remained an open problem until a non-elementary proof was given in [23] to show that $\partial f(x)$ has indeed an intrinsic value.

2.1 Representation by Partial Extension

We will now show that for any Lipschitz map $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, the generalised Jacobian ∂f is simply the partial extension of the derivative map $f' : U_0 \to \mathbb{R}^{m\times n}$, where $U_0 \subseteq U$ is the dense subset where f is differentiable. Note that $\mathbb{R}^{m\times n}$ can be considered as a subset of the maximal elements of the bounded complete domain $\mathbf{C}(\mathbb{R}^{m\times n})$, i.e., the set of non-empty compact and convex subsets of the space of $m \times n$ real matrices $\mathbb{R}^{m\times n}$ ordered by reverse subset inclusion and augmented with a bottom element that can be regarded as the whole space $\mathbb{R}^{m\times n}$. Thus, we can consider f' as a map of type $U_0 \to \mathbf{C}(\mathbb{R}^{m\times n})$. We need the following theorem of Carathéodory on convex hulls:

Theorem 2.1. *[6] Any point of the convex hull of a subset $S \subseteq R^p$ lies in the convex hull of at most $p + 1$ points in S.*

By allowing some of the points in Carathéodory's theorem to be the same points if necessary, we can assume that any point of the convex hull of a subset $S \subseteq R^p$ lies in the convex hull of $p + 1$ points in S.

Recall that the Hausdorff metric d_H on the set of non-empty compact subsets of $\mathbb{R}^p$ is defined by $d_H(A, B) = \inf\{r > 0 : A \subseteq B_r \text{ and } B \subseteq A_r\}$ where for any non-empty compact set A and $r > 0$, the r-neighbourhood A_r of A is defined by $A_r = \{x \in \mathbb{R}^p : \exists y \in A.\ |x - y| \leq r\}$.

Theorem 2.2. *For any Lipschitz map $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, the generalised Jacobian is the partial extension of the derivative map:*

$$\partial f = (f')^*$$

Proof. Let $x \in U$. We will show that $\partial f(x) = (f')^*(x)$. Suppose $y \in \partial f(x)$. Then by Clarke's definition of generalised Jacobian, there exists a sequence $x_n \in U_0$, $n \geq 0$, such that the limit $\lim_{n\to\infty} f'(x_n)$ exists and $y = \lim_{n\to\infty} f'(x_n)$. Let $O \subseteq U$ be an open set with $x \in O$. Then there exists $N \geq 0$ such that $n \geq N$ implies $x_n \in O$, i.e., $f'(x_n) \in f'(O \cap U_0)$ for all $n \geq N$. If $k \geq 0$ is a Lipschitz constant for f then $\|f'(x)\| \leq k$ for any $x \in U_0$ and thus $f'(O \cap U_0) \subseteq B_k(0)$, where $B_k(0)$ is the compact unit ball of radius k in $\mathbb{R}^{m\times n}$. Hence, $\inf f'(O \cap U_0) \in \mathbf{C}(\mathbb{R}^{m\times n})$ is the convex hull of the closure of $f'(O \cap U_0)$, which implies $y \in f'(O \cap U_0)$, i.e., $f'(O \cap U_0) \sqsubseteq y$. Since this holds for any open set O containing x, it follows that

$$y \in \sup\{\inf f'(O \cap U_0) : x \in O,\ O \text{ open}\}$$

i.e., $y \in (f')^*(x)$.

Next suppose $y \in (f')^*(x)$. Let $\mathrm{cl}(A)$ denote the closure of A. For any $k \in \mathbb{N}$, let $O_k := O_x(1/2^k)$ be the open ball of radius $1/2^k$ centred at x. Since

$$y \in \inf f'(O_k \cap U_0) = \mathrm{Conv}(\mathrm{cl}(f'(O_k \cap U_0))),$$

the convex hull of the closure of $f'(O_k \cap U_0)$, it follows from Carathéodory's theorem, applied to the point y of the mn dimensional Euclidean space $\mathbb{R}^{m\times n}$, that there exist $mn+1$ points $y_{ik} \in \mathrm{cl}(f'(O_k \cap U_0))$, for $1 \leq i \leq mn+1$, such that

$$y \in \mathrm{Conv}\{y_{ik} : 1 \leq i \leq mn + 1\}.$$

By the definition of the closure of a set, let $x_{ik} \in f'(O_k) \cap U_0$ be such that $\|f'(x_{ik}) - y_{ik}\| < 1/2^k$ for each $k \in \mathbb{N}$. Since the subset $\inf f'(O_k \cap U_0)$ is compact for each $k \in \mathbb{N}$, there is a subsequence y_{ik_ℓ} such that the limits $y_i := \lim_{\ell\to\infty} y_{ik_\ell}$ exist for $1 \leq i \leq mn + 1$. By continuity, we have

$$y \in \mathrm{Conv}\{y_i : 1 \leq i \leq mn + 1\}.$$

In fact the polyhedron $P_\ell := \mathrm{Conv}\{y_{ik_\ell} : 1 \leq i \leq mn+1\}$ converges to $P := \mathrm{Conv}\{y_i : 1 \leq i \leq mn + 1\}$ in the Hausdorff metric d_H on $\mathbf{C}(\mathbb{R}^{m\times n})$ which implies $d(P, y) = \lim_{\ell\to\infty} d(P_\ell, y) = 0$ since $d(P_\ell, y) = 0$ for each $\ell \in \mathbb{N}$, where $d(A, y)$ is the minimum distance from the point y to a compact set A. By construction, we have $\lim_{\ell\to\infty} x_{ik_\ell} = x$ and $\lim_{\ell\to\infty} f'(x_{ik_\ell}) = y_i$. By the definition of the generalised Jacobians, we conclude that $y \in \partial f(x)$.

□

We note that we cannot directly use the standard extension of continuous maps to obtain Theorem 2.2. In fact, there are Lipschitz functions which are not continuously differentiable at any point. For example, in [17] a Lipschitz map $f : [0, 1] \to \mathbb{R}$ has been constructed with $\partial f(x) = [0, 1]$ for all $x \in [0, 1]$. It follows that f is not continuously differentiable at any point $x \in [0, 1]$ since at such a point we would have $\partial f(x) = f'(x) \neq [0, 1]$.

A number of properties of the generalised Jacobians that are proved in [4] now simply follow as a corollary of Theorem 2.2.

Corollary 2.3. *For any Lipschitz map $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $x \in U$ we have:*

(i) The set $\partial f(x)$ is non-empty, convex and compact.

(ii) The map ∂f is the maximal Scott continuous partial extension of $f' : U_0 \to \mathbf{C}(\mathbb{R}^{m\times n})$.

(iii) If f is continuously differentiable at x then $\partial f(x) = f'(x)$.

Proof. (i) This follows immediately from Theorem 2.2.

(ii) Since $(f')^*$ is a partial extension of f', we have $\partial f(x) = (f')^*(x) \sqsubseteq f'(x)$.

(iii) By Proposition 1.2, $\partial f(x) = (f')^*(x) = f'(x)$. $\square$

Theorem 2.2 also shows by an elementary proof that the generalised Jacobian has an intrinsic value independent of any null set used in defining it.

3 L-derivative for vector functions

The definition of generalised Jacobians in Equation 1, uses several notions of higher order type. We will show that the simple domain-theoretic definition used in [10, 7, 11] can be extended to provide a uniform method to define an L-derivative for both real and complex Lipschitz vector functions which also works in the case of vector functions defined on infinite dimensional spaces.

The local differential properties of a function are formalised in the domain-theoretic framework by the notion of a set-valued Lipschitz constant. Assume $U \subseteq \mathbb{F}^n$ is an open subset. A map $f : U \subset \mathbb{F}^n \to \mathbb{F}^m$ is a vector function with components $f_i : U \to \mathbb{F}$ for $i = 1, \ldots, m$. Given a matrix $v \in \mathbb{F}^{m\times n}$, we let $v_i \in \mathbb{F}^n$ be the ith row of v. We define the *vector-wise action* of f on v by the column vector

$$f[v] = (f_1(v_1), \ldots, f_m(v_m))^T \in \mathbb{F}^m$$

We call $f[v]$ the *vector-wise action* of f to distinguish it from $f(u)$ for $u \in \mathbb{F}^n$, which is the usual action of f. Similarly, a matrix $A \in \mathbb{F}^{m\times n}$ is considered as representation of m linear maps by the m rows A_i of A. Thus, for $A \in \mathbb{F}^{m\times n}$ and $v \in \mathbb{F}^{m\times n}$ we define the vector-wise evaluation

$$\mathrm{ev}(A, v) = A[v] = (A_1 v_1^T, \ldots, A_m v_m^T)^T \in \mathbb{F}^m$$

where $A_i v_i^T \in \mathbb{F}$ is the usual action of the row vector A_i on the column vector v_i^T (i.e., the scalar product of A_i and v_i). We also use below the notation $Av := \sum_{i=1}^m \sum_{j=1}^n A_{ij} v_{ij}$, (in contrast to $A[v]$), which is the usual action of A considered as a single row vector $A = (A_1, A_2, \ldots, A_m)$ on the single column vector $v = (v_1, v_2, \ldots, v_m)^T$. This means that we consider $v \in \mathbb{F}^{m\times n}$, depending on the context, both as an $m \times n$ matrix of $\mathbb{F}$ elements and also as a single column vector in the mn dimensional vector space $\mathbb{F}^{mn}$. The vector-wise evaluation map is extended pointwise to

$$\mathrm{ev} : \mathbf{C}(\mathbb{F}^{m\times n}) \times \mathbb{F}^{m\times n} \to \mathbf{C}(\mathbb{F}^m)$$

by $\mathrm{ev}(b, v) = b[v] = \{A[v] : A \in b\}$.

In the following, we use the notation: $f[x + v] := (f_1(x + v_1), \ldots, f_m(x + v_m))^T$, i.e., a copy of x is used for each input to f_i. For $v = 0$ this convention means that $f[x] = (f_1(x), \ldots, f_m(x))^T = f(x)$.

Definition 3.1. The continuous function $f : U \to \mathbb{F}^m$ has *a non-empty, convex and compact set-valued Lipschitz constant* $b \in \mathbf{C}(\mathbb{F}^{m\times n})$ in an open subset $a \subseteq U$ if for all $u, w \in a^m$ we have: $b[u - w] \sqsubseteq f[u] - f[w]$. The *single-step tie* $\delta(a, b)$ of a with b is the collection of all partial functions f on U with $a \subseteq \mathrm{dom}(f) \subseteq U$ in $C^0(U)$ which have b as a non-empty convex compact set-valued Lipschitz constant in a.

For a single-step tie $\delta(a, b)$, one can think of b as the non-empty compact-set Lipschitz constant for the family of functions in $\delta(a, b)$. The classical Lipschitz constant for f would simply be $k = \|b\| \geq 0$. By generalising the concept of a Lipschitz constant in this way, one is able to obtain essential information about the differential properties of the function.

In particular, if $f \in \delta(a, b)$ for $a \neq \emptyset$ and $b \neq \bot$, then the induced function $f : a \to \mathbb{F}^m$ is Lipschitz: for all $x, y \in a$ we have $\|f(x) - f(y)\| \leq \|b\|\|x - y\|$. For the rest of this section, we assume we are in dimension $n \geq 2$ and for convenience we write C^0 for $C^0(U)$.

Definition 3.2. A *step tie* of C^0 is any finite intersection $\bigcap_{i\in I} \delta(a_i, b_i) \subseteq C^0$, where I is a finite indexing set. A *tie* of C^0 is any intersection $\Delta = \bigcap_{i\in I} \delta(a_i, b_i) \subseteq C^0$, for an arbitrary indexing set I. The *domain* of a non-empty tie Δ is defined as $dom(\Delta) = \bigcup_{i\in I}\{a_i \mid b_i \neq \bot\}$.

A non-empty step tie with rational intervals gives us a family of functions with a *finite* set of consistent differential properties, and a non-empty general tie gives a family of functions with a consistent set of differential properties. Recall that a function $f : U \to \mathbb{R}$ defined on the open set $U \subseteq \mathbb{R}^n$ is locally Lipschitz if it is Lipschitz in a neighbourhood of any point in U.

The following lemma plays a crucial role in deducing our results. Let the unit sphere in $\mathbb{F}^{m\times n}$ by be denoted S and let 0 denote the zero vector in $\mathbb{F}^m$. For a matrix $A \in \mathbb{F}^{m\times n}$, and $1 \leq i \leq m$, let A_i denote the ith row of A, i.e., $A_i = (A_{i1}, A_{i2}, \ldots, A_{ij}, \ldots, A_{in})$.

Definition 3.3. Given $b \in \mathbf{C}(\mathbb{F}^{m\times n})$, we say b satisfies the *zero containment predicate* if the following holds:

$$\mathsf{Z}(b, \mathbb{F}^{m\times n}) \equiv \forall \epsilon > 0.\, \forall v \in S.\, \exists A \in b.\, \|A[v]\| \leq \epsilon$$

As the name implies, we have the following Basic Lemma.

Lemma 3.4. *If $b \in \mathbf{C}(\mathbb{F}^{m\times n})$ then $\mathsf{Z}(b, \mathbb{F}^{m\times n})$ iff $0 \in b$.*

Proof. The right to left implication is trivial. For Left to Right implication: Suppose $0 \notin b$. Let $r = \inf\{\|A\| : A \in b\} > 0$. By compactness of b we have $r > 0$ and there exists $C \in b$ with $\|C\| = r$. (Note that the Frobenius norm of the matrix $C \in \mathbb{F}^{m\times n}$ is the same as the Euclidean norm of C considered as a vector in $\mathbb{F}^{mn}$). Now we put $v = \overline{C}/\|C\| \in S$, where $\overline{C}$ is the complex conjugate of C, i.e., $(\overline{C})_{ij} = \overline{C_{ij}}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. For any $A \in b$ we have $A = k\frac{C}{\|C\|} + B$ where $k \in \mathbb{F}$ with $|k| \geq r$ and B is in the orthogonal complement of C, i.e., $Bv = 0$. Thus, $Av = k$ and hence $|Av| \geq r$. Recall that for any vector $x \in \mathbb{F}^n$ we have $\|x\|_1^2 \leq n\|x\|_2^2$ where $\|x\|_1$ and $\|x\|_2$ are respectively the ℓ_1 norm and the ℓ_2 (i.e., Euclidean) norm of x. Therefore for any $A \in b$ and each $i = 1, \ldots, m$ we have: $|A_i v_i^T|^2 \geq \frac{1}{n}\sum_{j=1}^n |A_{ij}v_{ij}|^2$ and thus:

$$\|A[v]\|^2 =$$
$$\sum_{i=1}^m |A_i v_i^T|^2 \geq \frac{1}{n}\sum_{i=1}^m\sum_{j=1}^n |A_{ij}v_{ij}|^2 = \frac{1}{n}|Av| \geq r^2/n$$

We therefore obtain a contradiction by taking $\epsilon = r/(2\sqrt{n})$ with $v = \overline{C}/\|C\| \in S$. □

We now collect some simple properties of step ties, which we will use later.

Proposition 3.5. *Suppose the total derivative $f'(x)$ of $f : U \to \mathbb{F}^m$ exists for all $x \in a$. If $f \in \delta(a, b)$ then $f'(x) \in b$ for all $x \in a$.*

Proof. Suppose $f \in \delta(a, b)$ and $x \in a$. We show that $\mathsf{Z}(b - f'(x), \mathbb{F}^{m\times n})$. Let $\epsilon > 0$ and $v \in S$ be given, where S is the unit sphere in $\mathbb{F}^{m\times n}$. Consider the rows v_i of v for $i = 1, \ldots, m$. For each $i = 1, \ldots, m$, for sufficiently small $k > 0$ we have by the existence of the total derivative $f'(x) \in \mathbb{F}^{m\times n}$ of f at x: $\|f(x+kv_i) - f(x) - f'(x)kv_i\| \leq \epsilon k/\sqrt{m}$ and thus

$$\|f_i(x + kv_i) - f_i(x) - k(f'(x))_i v_i\| \leq \epsilon k/\sqrt{m} \qquad (2)$$

for each i, where, we recall, $(f'(x))_i$ is the ith row of the matrix $f'(x) \in \mathbb{F}^{m\times n}$. Therefore, from Equation 2, using our notation explained before Definition 3.1, we obtain:

$$\|f[x + kv] - f[x] - kf'(x)[v]\| \leq \epsilon k \qquad (3)$$

Take $k > 0$ to be small enough so that $x + kv_i \in a$, for all $i = 1, \ldots, m$. Then from our assumption that $f \in \delta(a, b)$ we have: $f[x + kv] - f[x] \in kb[v]$. Thus, there exists $B \in b$ with $f[x+kv]-f[x] = kB[v]$, which by Equation 3, implies $\|Bkv - kf'(x)[v]\| \leq \epsilon k$ or $\|(B - f'(x))[v]\| \leq \epsilon$. Thus, $\mathsf{Z}(b - f'(x), \mathbb{F}^{m\times n})$ and we conclude that $f'(x) \in b$. □

Proposition 3.6. *For any indexing set I, the family of step functions $(b_i\chi_{a_i})_{i\in I}$ is consistent if $\bigcap_{i\in I}\delta(a_i, b_i) \neq \emptyset$.*

Proof. Suppose $f \in \bigcap_{i\in I}\delta(a_i, b_i)$. We will show that every finite subfamily $(b_i\chi_{a_i})_{i\in J}$, where $J \subseteq I$ is a finite subset, is consistent, from which the result follows as $\mathbf{C}(\mathbb{F}^{m\times n})$ is bounded complete. It suffices to prove that for any finite subset $J \subseteq I$ we have $\bigcap_{j\in} b_j \neq \emptyset$ if $\bigcap_{j\in J} a_j \neq \emptyset$. This we will show by induction on the cardinality $|J|$ of J. For $|J| = 1$, this vacuously holds. Suppose now $|J| > 1$ and $a := \bigcap_{j\in J} a_j \neq \emptyset$. Let $k \in J$. Then by the inductive hypothesis $\bigcap_{j\in J\setminus\{k\}} b_j \neq \emptyset$. If $\bigcap_{j\in J} b_j = \emptyset$, then put $b := \bigcap_{j\in J\setminus\{k\}} b_j$. assume, for a contradiction, that $b \cap b_k = \emptyset$. Since b and b_k are non-empty, compact and convex sets, so is $b - b_k = \{A - B : A \in b, B \in b_k\}$, which is the Minkowski sum of two compact and convex sets b and $-b_k$. We will show that $0 \in b - b_k$ which implies that $b \cap b_k \neq \emptyset$, a contradiction. Fix $x \in a$. Let $\epsilon > 0$ and $v \in S$, where S is the unit sphere in $\mathbb{F}^{m\times n}$. Let $c \in \mathbb{F}$ with $|c| > 0$ sufficiently small such that $x + cv_i \in a$ for $i = 1, \ldots, m$. Then, from $f \in \bigcap_{i\in I}\delta(a_i, b_i)$ we obtain $f[x + cv] - f[x] \in cb[v]$ and $f[x + cv] - f[x] \in cb_k[v]$, which imply that there exists $A \in b$ and $B \in b_k$ with $cA[v] = cB[v]$ or $(A - B)[v] = 0$, where $A - B \in b - b_k$. Since, $\|(A - B)[v]\| = 0$ we have $\mathsf{Z}(b - b_k, \mathbb{F}^{m\times n})$ and Lemma 3.4 gives $0 \in b - b_k$, which completes the proof. □

One important corollary of this is that consistency of a family of step functions can be determined from the associated ties in a finitary manner: The family $(b_i\chi_{a_i})_{i\in I}$ is consistent if for any finite subfamily $J \subseteq I$ we have $\bigcap_{i\in J}\delta(a_i, b_i) \neq \emptyset$.

Let $(T^1(U), \supseteq)$ be the dcpo of ties of C^0 ordered by reverse inclusion.

Definition 3.7. The *primitive map* $\int : (U \to \mathbf{C}\mathbb{F}^{m\times n}) \to T^1(U)$ is defined by $\int(g) = \bigcap_{i\in I}\delta(a_i, b_i)$, where $g = \sup_{i\in I} b_i\chi_{a_i}$. We usually write $\int(f)$ as $\int f$ and call it the set of *primitives* of f.

The primitive map is well-defined, onto and continuous, which is a generalisation of [7, Proposition 3.17].

Given a continuous function $f : U \to \mathbb{F}^m$, the relation $f \in \delta(a, b)$ provides, as we have seen, finitary information about the local differential properties of f. By collecting all such local information, we obtain the complete differential properties of f, namely its derivative.

Definition 3.8. The *derivative* of a continuous function $f : U \to \mathbf{C}(\mathbb{F}^{m\times n})$ is the map

$$\mathcal{L}f = \bigsqcup_{f\in\delta(a,b)} b\chi_a : U \to \mathbf{C}(\mathbb{F}^{m\times n}).$$

Theorem 3.9. (i) *$\mathcal{L}f$ is Scott continuous.*

(ii) *If $f'(x)$ exists, then $f'(x) \in \mathcal{L}f(x)$.*

(iii) *If $f \in C^1(U)$ then $\mathcal{L}f = f'$.*

(iv) *If $\mathcal{L}f(x)$ is a singleton, then $f'(x)$ exists and $f'(x) = \mathcal{L}f(x)$.*

Proof. (i) This follows from Proposition 3.6.

(ii) This follows from Proposition 3.5 and the definition of $\mathcal{L}f(x)$.

(iii) and (iv) See the Appendix.

□

We now show that when $\mathbb{F} = \mathbb{R}$, the L-derivative $\mathcal{L}f$ of a locally Lipschitz map $f : U \to \mathbb{R}^m$ coincides with Clarke's generalised Jacobian. We will now prove that the L-derivative and the generalised Jacobian of a locally Lipschitz map coincide which gives an elementary proof that the generalised Jacobian has an intrinsic value.

Theorem 3.10. *For any locally Lipschitz map $f : U \to \mathbb{R}^m$, we have $\partial f = \mathcal{L}f$.*

Proof. By Corollary 2.3(ii), ∂f is the maximal Scott continuous partial extension of the map $f' : U_0 \to \mathbb{R}^{m\times n} \subseteq \mathbf{C}(\mathbb{R}^{m\times n})$. By Theorem 3.9(i) and (ii), $\mathcal{L}f$ is a continuous partial extension of $f' : U_0 \to \mathbb{R}^{m\times n} \subseteq \mathbf{C}(\mathbb{R}^{m\times n})$. Thus, by Proposition 1.2, all we need to check is that for all $x \in U$, we have $\mathcal{L}f(x) \subseteq \partial f(x)$.

Let $b \ll \partial f(x)$. By the Scott continuity of ∂f at x (Corollary 2.3(ii)), there is an open set $a \subseteq U$ with $x \in a$ such that $b \ll \partial f(y)$ for all $y \in a$. Let $u, v \in a^m$ with $v \neq u$. Fix i with $1 \leq i \leq m$. Consider the $n-1$ dimensional open discs $D_{u_i}(\delta)$ and $D_{v_i}(\delta)$, orthogonal to the vector $v_i - u_i$, with centres at u_i and v_i respectively and each of radius δ. Take small enough $\delta > 0$ such that $D_{u_i}(\delta)$ and $D_{v_i}(\delta)$ are both contained in a. Since f_i has a total derivative for almost all points $y \in a$, by Fubini's theorem the set of points $w \in D_{u_i}$ such that f_i has a total derivative for almost all points on the line segment $\{w + t(v_i - u_i) : t \in [0,1]\}$, with respect to the 1-dimensional Lebesgue measure on the line segment, has full $n-1$ dimensional Lebesgue measure on D_{u_i}. Let $w \in D_{u_i}$ be such a point and consider the path $p : [0,1] \to a$ given by $p(t) = w + t(v_i - u_i)$ with $p'(t) = v_i - u_i$. Since f_i has for almost all $t \in [0,1]$ total derivative at $p(t)$, and since f equals to the integral of its derivative on a path that the derivative exists almost everywhere, we have:

$$f_i(w + (v_i - u_i)) - f_i(w) = \int_0^1 f_i'(p(t))p'(t)\,dt$$

$$= \int_0^1 f_i'(w + t(v_i - u_i))(v_i - u_i)\,dt.$$

From $b \ll \partial f(y)$ for all $y \in a$ and by Rademacher's theorem, we deduce that $f'(y) \in b$ for almost all $y \in a$. Thus for almost all $t \in [0,1]$, $f_i'(w + t(v_i - u_i))(v_i - u_i) \in (b(v_i - u_i))_i = \{A_i(v_i - u_i) : A \in b\}$. Since $\{A_i(v_i - u_i) : A \in b\}$ is compact and convex, it follows from [8, Lemma 4.5], that $f_i(w + (v - u)) - f_i(w) \in \{A_i(v_i - u_i) : A \in b\}$. By continuity and compactness this holds in the limit as $w \to u_i$ and thus $f_i(v_i) - f_i(u_i) \in \{A_i(v_i - u_i) : A \in b\}$. Since this holds for all $i = 1, \ldots, m$, we obtain: $f[v] - f[u] \in b[v - u]$ and we conclude that $f \in \delta(a, b)$ or equivalently $b \sqsubseteq \mathcal{L}f(x)$. Since $b \ll \partial f(x)$ is arbitrary, we get $\partial f(x) \sqsubseteq \mathcal{L}f(x)$.

□

4 Infinite Dimensions

We will now show that the L-derivative extends to Lipschitz vector maps on any Hilbert space $\mathcal{H}$ over the field $\mathbb{F}$, of real or complex numbers, with inner product $\langle ., . \rangle$. Recall that the dual $\mathcal{H}^*$ of $\mathcal{H}$ is itself a Hilbert space as it is equipped with its operator norm which is complete and induces an inner product. By Riesz's representation theorem, $\mathcal{H}^*$ is anti-isomorphic to $\mathcal{H}$ if $\mathbb{F} = \mathbf{C}$ and is isomorphic to $\mathcal{H}$ if $\mathbb{F} = \mathbb{R}$: For any $u \in \mathcal{H}$, there is a unique $\phi_u \in \mathcal{H}^*$ such that $\phi_u(v) = \langle v, u \rangle$ for all $v \in \mathcal{H}$. And for each $\phi \in \mathcal{H}^*$ there exists a unique $u_\phi \in \mathcal{H}$ such that $\langle x, u_\phi \rangle = \phi(x)$ for all $x \in \mathcal{H}$. Moreover, we have $\langle \phi, \psi \rangle = \langle u_\psi, u_\phi \rangle$.

In both case of real and complex Hilbert spaces, the unit ball of $\mathcal{H}$ is, by Alaoglu's theorem [21, 3.15], compact with respect to the weak* topology.

Let $S = \{u \in \mathcal{H} : \|u\| = 1\}$ be the unit sphere in $\mathcal{H}$. Consider bounded linear maps of type $\mathcal{H} \to \mathbb{F}^m$ equipped with the product weak* topology $(\mathcal{H}^*)^m$, i.e., the weakest topology on the function space of bounded linear functionals $f : \mathcal{H} \to \mathbb{F}^m$ such that for all $v \in \mathcal{H}$ the maps $v \mapsto f_i(v) : \mathcal{H} \to \mathbb{F}$ are continuous for all $i = 1, \ldots, m$. By Tychonoff's theorem the unit ball in $(\mathcal{H}^*)^m$ is compact with respect to the weak* product topology. Let $\mathbf{C}((\mathcal{H}^*)^m)$ be the bounded complete directed complete partial order of the non-empty and convex subsets of $(\mathcal{H}^*)^m$ that are compact with respect to the product weak* topology ordered by reverse inclusion and augmented with $(\mathcal{H}^*)^m$ as a bottom element. Note that in the infinite dimensional case, $\mathbf{C}((\mathcal{H}^*)^m)$ is no longer a continuous Scott domain but most of the results in the finite dimensional case do extend, perhaps surprisingly, to infinite dimensional real and complex Hilbert spaces.

The definition of a single-step ties is syntactically the same as before using the notion of vector-wise evaluation. Let $0 \in (\mathcal{H}^*)^m$ denote the trivial linear function with value zero. It is easy to extend the Basic Lemma 3.4 to infinite dimensional Hilbert spaces. Using this lemma, it is straightforward to check that Proposition 3.5 and Proposition 3.6

extend to the infinite dimensional case. It then follows, in particular, that we can define the L-derivative of a continuous map $f : X \to \mathbb{F}^m$ to be the Scott continuous map:

$$\mathcal{L}f = \bigsqcup_{f \in \delta(a,b)} b\chi_a : U \to \mathbf{C}((\mathcal{H}^*)^m),$$

which has the property that $f'(x) \in \mathcal{L}f(x)$ whenever $f'(x)$ exists.

In the full version of the paper, we will show that the L-derivative can also be defined for a continuous function $f : B \to \mathbb{R}^m$ where B is any real Banach space and that for $m = 1$ the L-derivative coincides with Clarke's gradient on Banach spaces.

5 Generalised Cauchy Riemann equations

In this section we establish a relation between the L-derivative of a complex map $f : U \to \mathbb{C}^m$, where $U \subseteq \mathbb{C}^n$ is an open subset, and the L-derivative of its real and imaginary part. This will provide a new interpretation for the total derivative of a complex function and presents a generalisation of Cauchy Riemann equations [8] to Lipschitz maps in higher dimensions.

We now fix some notations for dealing with a complex map which can also be interpreted as a pair of real maps. For any $z = x + iy \in \mathbb{C}$, the conjugate complex number is denoted by $\overline{z} = x - iy$; moreover, the same notation is used to denote the pointwise extension of conjugation to sets of complex numbers, i.e., for $C \subseteq \mathbb{C}$, we write $\overline{C} = \{\overline{z} : z \in C\}$. Furthermore, all basic arithmetic operations on complex numbers are extended pointwise to subsets of complex numbers, e.g., for a subset $C \subseteq \mathbb{C}$, we write $iC := \{iz : z \in C\}$. In order to keep the presentation as simple as possible, we use the isomorphism $\mathcal{H} : x + iy \mapsto (x, y)$ to identify the complex plane $\mathbb{C}$, regarded as a group under addition of complex numbers, and the real two dimensional Euclidean plane $\mathbb{R}^2$, regarded as a group under the addition of vectors: we move from one plane to the other plane while suppressing any explicit reference to $\mathcal{H}$ or its pointwise extension to subsets of $\mathbb{C}$ wherever convenient. In particular, an expression containing complex conjugation or multiplication by i can in fact be interpreted, after carrying out the complex number operations, in the real plane, for example, $\overline{iA}$, for a subset $A \subseteq \mathbb{R}^2$, denotes the set $\mathcal{H}[\overline{i[\mathcal{H}^{-1}[A]]}] \subseteq \mathbb{R}^2$, i.e., the subset A is rotated by $\pi/2$ around the origin and is then reflected through the x-axis. i.e., an overall reflection through the line $y = -x$.

We now introduce the generalised multi-dimensional directional derivative of f which extends the one dimensional notion [15, p. 27].

Definition 5.1. A map $f : U \subseteq \mathbb{C}^n \to \mathbb{C}^m$ has *generalised directional derivative* for $\theta \in [0, 2\pi)^n$ at $z \in \mathbb{C}^n$ if there exists a linear map $L_\theta : U \subseteq \mathbb{C}^n \to \mathbb{C}^m$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(z + h) - f(z) - L_\theta h\|}{\|h\|} = 0 \tag{4}$$

where $h \in \mathbb{C}^n$ with $h_j = |h_j|e^{i\theta_j}$ for $1 \leq j \leq n$.

In order to determine a relation between the generalised directional derivative of $f = V + iW$ and the directional derivatives of V and W we write $z = (z_1, \ldots, z_n) \in \mathbb{C}^n \simeq (\mathbb{R}^2)^n$ as $z_j = x_j + iy_j$ with $(x_j, y_j) \in \mathbb{R}^2$ for $1 \leq j \leq n$. To any $\phi \in [0, 2\pi)$, we associate the unit vector $N_\phi = (\cos\phi, \sin\phi)^T \in \mathbb{R}^2$.

Proposition 5.2. *If $f = V + iW : U \subseteq \mathbb{C}^n \to \mathbb{C}^m$ is Lipschitz then for almost all $z \in \mathbf{C}^n$ it has a generalised directional derivative at $z \in \mathbf{C}^n$ for all $\theta = (\theta_1, \ldots, \theta_j, \ldots, \theta_n) \in [0, 2\pi)^n$, which is given by*

$$(L_\theta)_{kj} = e^{-i\theta_j}(V'_{kj} \cdot N_{\theta_j} + iW'_{kj} \cdot N_{\theta_j}),$$

where V' and W' are the total derivatives of V and W at $(x_j, y_j)_{1 \leq j \leq n} \in \mathbb{R}^{2^n}$ with $z_j = (x_j, y_j)$ for $1 \leq j \leq n$. [Note that in our notation $V'_{kj} = (\frac{\partial V_k}{\partial x_j}, \frac{\partial V_k}{\partial y_j})$ and similarly for W'_{kj}.]

Proof. Let $f = V + iW$ where $V, W : \mathbb{R}^{2^n} \to \mathbb{R}^m$ are the real and imaginary parts of f. Then V and W are Lipschitz maps and are therefore differentiable for almost all $(x_j, y_j)_{1 \leq j \leq n} \in \mathbb{R}^{2^n}$ with respect the Lebesgue measure on $\mathbb{R}^{2n}$. Let $(x_j, y_j)_{1 \leq j \leq n} \in \mathbb{R}^{2^n}$ be such a point. We will compute the kth component of the term in Equation 4 at $z \in \mathbb{C}^n$ with $z_j = x_j + iy_j$ with L_θ given above with $N_{\theta_j} = (\cos\theta_j, \sin\theta_j)^T$. First, we note that since V and W have total derivative at $(x_j, y_j)_{1 \leq j \leq n} \in \mathbb{R}^{2^n}$, it follows that

$$\lim_{\|h\| \to 0} \frac{\Delta V_k - \sum_{j=1}^n |h_j| V'_{kj} \cdot N_{\theta_j}}{\|h\|} = 0 \tag{5}$$

where

$$\Delta V_k \\ = V_k((x_j + |h_j|\cos\theta_j, y_j + |h_j|\sin\theta_j)_j) - V_k((x_j, y_j)_j).$$

Similarly,

$$\lim_{\|h\| \to 0} \frac{\Delta W_k - \sum_{j=1}^n |h_j| W'_{kj} \cdot N_{\theta_j}}{\|h\|} = 0 \tag{6}$$

where

$$\Delta W_k \\ = W_k((x_j + |h_j|\cos\theta_j, y_j + |h_j|\sin\theta_j)_j) - W_k((x_j, y_j)_j).$$

But

$$f_k(z + h) - f_k(z) - (L_\theta h)_k \\ = \Delta V_k + i(\Delta W_k)$$

$$-\sum_{j=1}^{n}|h_j|V'_{kj}\cdot N_{\theta_j}+i\left(\sum_{j=1}^{n}|h_j|W'_{kj}\cdot N_{\theta_j}\right)$$

and thus from Equations (5) and (6), we obtain:

$$\lim_{\|h\|\to 0}\frac{\|f_k(z+h)-f_k(z)-(L_\theta h)_k\|}{\|h\|}=0$$

□

We will show that the L-derivative of a complex Lipschitz map coincides with the convex hull of all its generalised directional derivatives, which will give a generalisation of Cauchy Riemann equations. We note that the kj entry of the matrix L_θ namely

$$(L_\theta)_{kj}=e^{-i\theta_j}(V'_{kj}\cdot N_{\theta_j}+iW'_{kj}\cdot N_{\theta_j})$$

with $\theta_j\in[0,2\pi)$ moves along a circle in $\mathbb{C}$ named after Kasner, though it appeared in some other notation in Riemann's PhD thesis [15, p. 27]. The two points $\overline{V'_{kj}}$ and $i\overline{W'_{kj}}$ are at the end of a diameter of this circle and we denote the disk in $\mathbb{C}$ whose boundary is the Kasner circle by $\mathsf{D}(\overline{V'_{kj}},i\overline{W'_{kj}})$ [8]. We now aim to express the L-derivative of f in terms of the L-derivative of the real-valued function $\hat{f}:(\mathbb{R}^2)^n\to(\mathbb{R}^2)^m$ given by $(\hat{f}((x,y))_k=(V_k((x,y)),W_k((x,y)))$.

Definition 5.3. The *total directional derivative* of a Lipschitz map $f=V+iW:U\subseteq\mathbb{C}^n\to\mathbb{C}^m$, where $V,W:U\subseteq(\mathbb{R}^2)^n\to\mathbb{R}^m$ is defined, via the dense embedding property, as the maximal continuous partial extension $\partial_d:=(\partial_d^0)^*$ of the map $\partial_d^0:U_0\subseteq U\subseteq\mathbb{C}^n\to(\mathbf{C}(\mathbb{C}))^{m\times n}$ with $(\partial_d^0f(z))_{kj}=\mathsf{D}(\overline{V'_{kj}}(x,y),i\overline{W'_{kj}(x,y)})$ where $\hat{f}:U_0\subseteq U\subseteq(\mathbb{R}^2)^n\to(\mathbb{R}^2)^m$ is assumed to be differentiable at $(x,y)\in U_0$ with $z_k=x_k+iy_k$, for $1\le k\le n$. With our notation, $\hat{f}_k:U\to\mathbb{R}^2$ is the kth $\mathbb{R}^2$ block in $(\mathbb{R}^2)^m$ which has m such blocks.

From the definition it follows that for $z=x+iy$:

Proposition 5.4.

$$(\partial_d f(z))_{kj}=Conv\bigcup\{\mathsf{D}(\overline{v},i\overline{w}):\begin{pmatrix}v\\w\end{pmatrix}\in(\mathcal{L}\hat{f}_k(x,y))_j\}$$

The proof of the following multi-dimensional generalisation of the Cauchy-Riemann equations, extending the 1 dimensional case in [8], is given in the Appendix.

Theorem 5.5. *If $f:U\subseteq\mathbb{C}^n\to\mathbb{C}^m$ is locally Lipschitz then $\mathcal{L}f=\partial_d f$.*

To see that we indeed have a multi-dimensional generalisation of the Cauchy-Riemann equations, suppose f is indeed analytic as a vector function of several complex variables. Then each component f_k ($1\le k\le m$) is separately analytic [20], i.e.

$$\frac{\partial f_i}{\partial\overline{z_j}}=0 \tag{7}$$

In this case, the two points $\overline{V'_{kj}}(x,y)$ and $i\overline{W'_{kj}(x,y)}$ coincide for all k and j and Theorem 5.5 gives $\frac{\partial f_k}{\partial z_j}=\overline{V'_{kj}}(x,y)=i\overline{W'_{kj}(x,y)}$ which is equivalent to Equation 7.

Example 5.6. Assume $f:U\subseteq\mathbb{C}^2\to\mathbb{C}$ with $f=g(\overline{z_1},\overline{z_2})+h(z_1,z_2)$, where g and h are both analytic. Then f is the sum of an anti-analytic (also called anti-holomorphic) and an analytic map. In this case, writing $z=(z_1,z_2)$, a simple calculation yields:

$$(\mathcal{L}f)_1(z)=\frac{\partial h}{\partial z_1}+\mathsf{D}\left(\frac{\partial g}{\partial z_1}(\overline{z}),-\frac{\partial g}{\partial z_1}(\overline{z})\right)$$

$$(\mathcal{L}f)_2(z)=\frac{\partial h}{\partial z_2}+\mathsf{D}\left(\frac{\partial g}{\partial z_2}(\overline{z}),-\frac{\partial g}{\partial z_2}(\overline{z})\right)$$

6 Distance to Maximal Elements

We have seen in Theorem 3.9(iii) and (iv) that a real or complex Lipschitz map $f:U\to\mathbb{F}^m$ is C^1 or analytic, respectively, in an open set if and only if its L-derivative is a singleton at each point in the open set. Then, the map $\mathcal{L}f:U\to\mathbf{C}(\mathbb{F}^{m\times n})$ is a maximal element of $U\to\mathbf{C}(\mathbb{F}^{m\times n})$ and moreover it sends every element of U to a maximal element of $\mathbf{C}(\mathbb{F}^{m\times n})$, namely an element of $\mathbb{F}^{m\times n}$. In domain-theoretic terminology, $\mathcal{L}f$ can be regarded as a total element of the function space $U\to\mathbf{C}(\mathbb{F})$ [12, 2, 13]. This motivates a quantitative measure for how close a vector Lipschitz map is to the maximal elements as follows. Consider the upper-extended real line $\mathbb{R}^+\cup\{\infty\}$ with its upper topology which coincides with the Scott topology on $\mathbb{R}^+\cup\{\infty\}$ if the latter is ordered by the reverse order of real numbers, with 0 as the top element.

Definition 6.1. The *t-distance* function

$$\mathsf{d}:(U\to\mathbf{C}(\mathbb{F}^{m\times n}))\to\mathbb{R}^+\cup\{\infty\},$$

is defined by $\mathsf{d}(g)=\sup\{\mathrm{diam}(g(z)):z\in U\}$, where $\mathrm{diam}(A)$ is the diameter of $A\in\mathbf{C}(\mathbb{F}^{m\times n})$ with $\mathrm{diam}(\perp)=\infty$.

Clearly, d is monotone; we will show in the Appendix that it satisfies a continuity property as well.

Proposition 6.2. *The t-distance is Scott continuous on Scott continuous functions restricted to a compact subset of U.*

One can easily compute the t-distance of the L-derivative of complex Lipschitz map f provided in Example 5.6, in which the two components of the L-derivative contain a disk of radius $|\frac{\partial g}{\partial z_2}(\overline{z})|$ and a disk of radius $|\frac{\partial g}{\partial z_2}(\overline{z})|$:

$$\mathsf{d}(f) = \max\left\{\sup_{z\in U}\left|\frac{\partial g}{\partial z_1}(\overline{z})\right|, \sup_{z\in U}\left|\frac{\partial g}{\partial z_2}(\overline{z})\right|\right\} \qquad (8)$$

Domain-theoretically, the t-distance relates to the more abstract notion of the distance of an element of a domain to the set of maximal elements of that domain as in the notion of measurement [18].

7 Domain for real Lipschitz vector functions

A domain for locally Lipschitz functions and for $C^1(U)$ is constructed as follows. First suppose that $\mathbb{F} = \mathbb{R}$. We will assume that U is a relatively compact open set, i.e, its closure is compact. The canonical example is $U = (0,1)^n$. The idea is to use $U \to \mathbf{I}\mathbb{R}^m$ to represent the function and $U \to \mathbf{C}\mathbb{R}^{m\times n}$ to represent the differential properties (partial derivatives) of the function. Thus, we aim to define a domain for Lipschitz vector functions as a sub-domain of $(U \to \mathbb{R}^m) \times (U \to \mathbf{C}(\mathbb{R}^{m\times n}))$, where $(U \to \mathbb{R}^m)$ is the set of continuous functions from U to $\mathbb{R}^m$ with the sup norm. Consider the *consistency* relation

$$\mathsf{Cons} \subseteq (U \to \mathbb{R}^m) \times (U \to \mathbf{C}(\mathbb{R}^{m\times n})),$$

defined by $(f,g) \in \mathsf{Cons}$ if $\uparrow\! f \cap \int g \neq \emptyset$, i.e., $\exists h : U \to \mathbb{R}^m$ with $g \sqsubseteq h$ and $f \sqsubseteq \mathcal{L}h$. For a consistent (f,g), we think of f as the *function part* or the *function approximation* and g as the *derivative part* or the *derivative approximation*. We will show that the consistency relation is Scott closed. It follows immediately from the definition that consistency is downward closed. Recall that by Kirszbraun's theorem [5, p. 8], a Lipschitz map can be extended from any subset of a Hilbert space to the whole space with the same Lipschitz constant.

Proposition 7.1. *The relation* Cons *is Scott closed.*

Proof. Let $(f_i, g_i)_{i\geq 0}$ be an increasing sequence of pairs of step functions with $(f_i, g_i) \in \mathsf{Cons}$ with $f = \sup_{i\geq 0} f_i$ and $g = \sup_{i\geq 0} g_i$, where $\mathrm{dom}(f) = \bigcup_{i\geq 0}\mathrm{dom} f_i$ and $\mathrm{dom}(g) = \bigcup_{i\geq 0}\mathrm{dom} g_i$. Without loss of generality assume $g_0(x) \neq \perp$ for some $x \in U$. Let $h_i : U \to \mathbb{R}^m$ be a partial map that is the Lipschitz witness of consistency such that $f_i^- \leq h_i \leq f_i^+$ and $g_i \sqsubseteq \mathcal{L}h_i$. By using Kirszbraun's theorem we extend the domain of definition of each h_i to $\mathrm{cl}(U)$ so that the extended map $\hat{h}_i$ is still Lipschitz with a Lipschitz constant given by $c_i := \max_{x\in\mathrm{dom}(g_i)} \|g_i(x)\|$, which is the maximum Lipschitz constant possible for consistency with g_i. Since $f_i$and g_i are increasing chains in $U \to \mathbb{R}^m$ and $U \to \mathbf{C}(\mathbb{R}^{m\times n})$ respectively, $(f_i^-)_{i\geq 0}$ and $(f_i^+)_{i\geq 0}$ are increasing and decreasing functions respectively and c_i is a decreasing sequence for $i \geq 0$. It follows for all $i \geq 0$ the map $\hat{h}_i$ has lower and upper bounds given by: $M_0^- - c_0 d \leq \hat{h}_i \leq M_0^+ + c_0 d$, where M_0^- and M_0^+ are, respective, the minimum of f_0^- and the maximum values of f_0^+, and d is the diameter of U. Thus, the family $(\hat{h}_i)_{i\geq 0}$ is uniformly bounded. In addition, $\|h_i(x) - h_i(y)\| \leq c_0\|x-y\|$ and thus the family $(\hat{h}_i)_{i\geq 0}$ is equi-continuous. It follows by Arzelà-Ascoli theorem [21, p.369] that the family $(\hat{h}_i)_{i\geq 0}$ has a convergent subsequence, say $(\hat{h}_{i_k})_{k\geq 0}$. Suppose $\lim_{k\to\infty} h_{i_k} = h$. Since $f_{i_k}^- \leq h_{i_k} \leq f_{i_k}^+$ for all $k \geq 0$, we have $f^- \leq h \leq f^+$. Let $A \subseteq \mathrm{dom}(g)$ be a connected component of g. Assume $A \cap \mathrm{dom}(f) \neq \emptyset$. We have $\mathcal{L}h_{i_k} \sqsupseteq g_{i_k}$ for all $k \geq 0$. Thus, $h_{\restriction A} \sqsupseteq g_{i_k \restriction A}$ for all $k \geq 0$, which implies $h_{\restriction A} \sqsupseteq g_{\restriction A}$. Thus, h is a witness for consistency of (f,g). $\square$

Define

$$\mathcal{D}_0(U) := \{(f,g) \in (U \to \mathbb{R}^m) \times (U \to \mathbf{C}(\mathbb{R}^{m\times n})) : \mathsf{Cons}(f,g)\}.$$

Corollary 7.2. *The domain $\mathcal{D}_0(U)$ is a bounded complete continuous dcpo, i.e., a continuous Scott domain.*

8 Domain for complex Lipschitz maps

Now let $\mathbb{F} = \mathbb{C}$ and let $U \subseteq \mathbb{C}^n$ be a relatively compact open set. In order to construct a domain for complex Lipschitz maps of type $(U \to \mathbb{C}^m)$, we use the continuous Scott domain $(U \to \mathbb{R}^2)^m$ to represent the function part of the map as $\mathbb{C}^m \simeq (\mathbb{R}^2)^m$, and the continuous Scott domain $(U \to \mathbb{C}^{m\times n}) \simeq (U \to (\mathbb{R}^2)^{m\times n})$ to represent the L-derivative, together with a consistency condition between the function part and the derivative part. We actually will construct two such domains, one that carries the minimal information required and a larger domain that carries extra information we can use to develop an effectively given domain.

The consistency relation is defined as

$$\mathsf{Cons}_0 \subseteq (U \to (\mathbb{R}^2)^m) \times (U \to \mathbf{C}(\mathbb{C}^{m\times n})),$$

defined by $(f,g) \in \mathsf{Cons}_0$ if $\uparrow\! f \cap \int g \neq \emptyset$. The condition is equivalent to the existence of $h = V + iW : \mathbb{C}^n \to \mathbb{C}^m$, where $V, W : (\mathbb{R}^2)^n \to \mathbb{R}^m$ with $f_j \sqsubseteq (V_j, W_j)$, for $1 \leq j \leq m$, and $g \sqsubseteq \mathcal{L}h$. In the Appendix, as in the case of real maps, we use the two theorems by Kirszbraun and Arzelà-Ascoli to show:

Proposition 8.1. *The predicate* Cons_0 *is Scott closed.*

We thus define our domain for complex maps as:

$\mathcal{C}_0(U) := \{(f,g) \in (U \to (\mathbf{I}\mathbb{R}^2)^m) \times (U \to \mathbf{C}(\mathbb{C}^{m\times n})) : (f,g) \in \mathsf{Cons}_0\}$

From Proposition (8.1), we obtain:

Corollary 8.2. *$\mathcal{C}_0(U)$ is a continuous Scott domain.*

9 Effective structure

In the domains we have constructed the L-derivative will be generated by a basis consisting of rational convex polytopes. Since the decidability of the consistency predicate Cons in this setting remains an open problem even in the case of $\mathbb{F} = \mathbb{R}$, $n = 2$ and $m = 1$ [11], we will construct two other domains, one for real and one complex vector maps, in which the L-derivative of the underlying real maps are approximated, as in interval analysis [19], by the smallest axis aligned hyper-rectangles. For the ease of presentation, for real maps we confine ourselves with $m = n = 2$, i.e., maps of type $U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, while for complex maps we assume $m = n = 1$, i.e., maps of type $U \subseteq \mathbb{C} \to \mathbb{C}$.

Let $\mathbf{I}\mathbb{R}^{2\times 2}$ be the set of all 2×2 matrices whose entries are non-empty compact real intervals ordered by component-wise reverse inclusion and augmented with a least element $\perp$ represented as the 2×2 matrix with $\mathbb{R}$ for all its entries. Let $\rho_i : \mathbb{R}^{2\times 2} \to \mathbb{R}^2$, for $i = 1, 2$, project any 2×2 matrix to the ith row and let $\pi_i : \mathbb{R}^2 \to \mathbb{R}$ for $i = 1, 2$ be the standard projection to the first and second coordinate axis. These maps extend pointwise to maps $\mathbf{C}(\mathbb{R}^{2\times 2}) \to \mathbf{C}(\mathbb{R}^2)$ and $\mathbf{C}(\mathbb{R}^2) \to \mathbf{I}\mathbb{R}$, which for convenience we still denote by ρ_i and π_j respectively. Note that $\rho_1(\mathcal{L}\hat{h}(x,y)) = \mathcal{L}V(x,y)$ and $\rho_2(\mathcal{L}\hat{h}(x,y)) = \mathcal{L}W(x,y)$ with $\mathcal{L}\hat{h}(x,y)) \subseteq \mathcal{L}V(x,y) \times \mathcal{L}W(x,y)$. Furthermore $\pi_1\mathcal{L}V(x,y) \times \pi_2\mathcal{L}V(x,y) \supset \mathcal{L}V(x,y)$ and $\pi_1\mathcal{L}W(x,y) \times \pi_2\mathcal{L}W(x,y) \supset \mathcal{L}W(x,y)$. Thus, we have the following two conservative approximations:

- A(i) $\mathcal{L}V(x,y) \times \mathcal{L}W(x,y)$ approximates $\mathcal{L}\hat{h}(x,y)$.
- A(ii) $\pi_1\mathcal{L}V(x,y) \times \pi_2\mathcal{L}V(x,y)$ and $\pi_1\mathcal{L}W(x,y) \times \pi_2\mathcal{L}W(x,y)$ approximate $\mathcal{L}V(x,y)$ and $\mathcal{L}W(x,y)$.

The combined result of A(i) and A(ii) is that we obtain the smallest axis aligned hyper-rectangle in $\mathbb{R}^{2\times 2}$ which contains $\mathcal{L}\hat{h}(x,y)$. More formally, for any Lipschitz map $f : \mathbb{R}^2 \to \mathbb{R}^2$, with components f_1 and f_2, we define $\mathcal{L}^{\mathsf{r}} f(x,y) \in \mathbf{I}\mathbb{R}^{2\times 2}$, with the ij entry given by

$$(\mathcal{L}^{\mathsf{r}} f(x,y))_{ij} = \pi_j(\rho_i(\mathcal{L}f(x,y))). \qquad (9)$$

Since the projections ρ_i and π are continuous, their extensions to non-empty compact sets are also continuous with respect to the Hausdorff metric. It follows that the map $\mathcal{L}^{\mathsf{r}} f : U \to \mathbf{I}\mathbb{R}^{2\times 2}$ is Scott continuous. We say $\mathcal{L}^{\mathsf{r}} f$ is the *rectangular* approximation to $\mathcal{L}f$ and define the domain,

$\mathcal{D}_1(U) := \{(f,g) \in (U \to \mathbf{I}\mathbb{R})^2 \times (U \to \mathbf{I}\mathbb{R}^2)^2 : \mathsf{Cons}(f_i, g_i), i = 1, 2\}$

For step functions $(f_i, g_i) \in (U \to \mathbf{I}\mathbb{R}) \times (U \to \mathbf{I}\mathbb{R}^2)$, with rational rectangles as values, the predicate $\mathsf{Cons}(f_i, g_i)$ is decidable, for $i = 1, 2$, by [11, Corollary 3.9]. This means that $\mathcal{D}_1(U)$ can be given an effective structure.

9.1 Another Domain for Complex Maps

We now construct a bigger domain for complex Lipschitz vector maps, restricting ourselves to $m = n = 1$. Consider tuples $((r,s),(v,w),g)$ with $r, s : U \subseteq \mathbb{R}^2 \to \mathbf{I}\mathbb{R}$, $v, w : U \to \mathbf{I}\mathbb{R}^2$ and $g : U \to \mathbf{C}(\mathbb{C})$. The pair r and s represent approximations, respectively, to the real and imaginary parts of a complex map $f = V + iW : U \to \mathbb{C}$, and g represents an approximation to $\mathcal{L}f$, while (v,w) represent rectangular approximations, respectively, to the L-derivatives $\mathcal{L}V$ and $\mathcal{L}W$ of the real functions $V, W : U \to \mathbb{R}$, In other words, we approximate $\mathcal{L}\hat{f}$ by the hyper-rectangle

$$((\pi_1\rho_1\mathcal{L}V) \times (\pi_2\rho_1\mathcal{L}V)) \times ((\pi_1\rho_2\mathcal{L}W \times (\pi_2\rho_2\mathcal{L}W)))$$

in $\mathbb{R}^{2\times 2}$. We therefore define

$$\mathsf{Cons}_1 \subseteq (U \to (\mathbf{I}\mathbb{R}^2 \times (\mathbf{I}\mathbb{R}^2)^2 \times \mathbf{C}(\mathbb{C})))$$

with $((r,s),(v,w),g) \in \mathsf{Cons}_1$ if there exists a complex map $f = V + iW : U \to \mathbb{C}$ such that $r \sqsubseteq V$, $s \sqsubseteq W$, $v \sqsubseteq \mathcal{L}V$, $w \sqsubseteq \mathcal{L}W$ and $g \sqsubseteq \mathcal{L}f$. We denote the sub-domain of consistent elements with respect to Cons_1 by $\mathcal{C}_1(U)$. From Proposition (7.1) and Proposition (8.1), we conclude:

Corollary 9.1. *$\mathcal{C}_1(U)$ is a continuous Scott domain.*

In fact, $\mathcal{C}_0(U)$ is a retraction of $\mathcal{C}_1(U)$, i.e., there exist Scott continuous maps $E : \mathcal{C}_0(U) \to \mathcal{C}_1(U)$, called a section, and $R : \mathcal{C}_1(U) \to \mathcal{C}_0(U)$, called a retraction, such that $R \circ E = \mathrm{Id}$ where Id is the identity map on $\mathcal{C}_0(U)$ [1]. To show this, we need the follow lemma, which follows from Proposition 5.4.

Lemma 9.2. *If $f = V + iW : U \to \mathbb{C}^m$, then $\mathcal{L}V \subseteq \overline{\mathcal{L}f}$ and $\mathcal{L}W \subseteq i\,\overline{\mathcal{L}f}$.*

Let $T : \mathbf{C}(\mathbb{C}) \to \mathbf{I}\mathbb{R}^2$ be the Scott continuous map that sends every convex compact polygon to the smallest axis aligned rectangle containing it. Put $T^+ : (U \to \mathbf{C}(\mathbb{C})) \to (U \to \mathbf{I}\mathbb{R}^2)$ by $(T^+(g))(z) = T(g(z))$ for all $z \in \mathbb{C}$. Then T^+ is Scott continuous and we define:

$$E((f,g)) = (f, T^+(\overline{g}), T^+(i\,\overline{g}), g)$$

and $F((f_1, f_2), (v, w), g) = (f, g)$ where $f = (f_1, f_2)$. Using the lemma it is easy to check that the E and F are well-defined and provide a section-retraction pair.

We propose that $\mathsf{C}_1(U)$ which contains a substructure equivalent to D_1 can be made into an effective continuous Scott domain.

10 Further work

As pointed out in the last section, one area for further work is to investigate if the consistency predicate Cons_1 used in defining the continuous Scott domain $\mathsf{C}_1(U)$ is decidable on rational step functions, i.e., those basis elements whose values are compact rational axis aligned rectangles or rational compact polygons. We propose that Tarski's theorem on the elimination of quantifiers in real closed fields can be invoked to prove this decidability result.

An area for future work is to use the continuous Scott domains constructed in this paper for Lipschitz maps between finite dimensional real and complex Euclidean spaces to develop domains of computation for real and complex Lipschitz manifolds, in which the transition maps in charts are Lipschitz.

Another area is to show that the results in the paper can be extended to real vector Lipschitz maps on real Banach spaces, using a separation property in the dual of a real Banach space, which has been proved in [7]. Since we do not know if the latter separation property holds for complex Banach spaces, the question remains if the same thing can be done for maps on complex Banach spaces.

References

[1] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3. Clarendon, 1994.

[2] U. Berger. Computability and totality in domains. *MSCS*, 12(3):281 – 294, 2002.

[3] R. Cazacu and J. D. Lawson. Quasicontinuous functions, domains, extended calculus, and viscosity solutions. *Applied General Topology*, 8:1–33, 2007.

[4] F. H. Clarke. *Optimization and Nonsmooth Analysis*. Wiley, 1983.

[5] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski. *Nonsmooth Analysis and Control Theory*. Springer, 1998.

[6] J. Eckhoff. *Handbook of Convex Geometry*, volume B, chapter Helly, Radon, and Carathodory type theorems, pages 389–448. North-Holland, 1993.

[7] A. Edalat. A continuous derivative for real-valued functions. In S. B. Cooper, B. Löwe, and A. Sorbi, editors, *New Computational Paradigms, Changing Conceptions of What is Computable*, pages 493–519. Springer, 2008.

[8] A. Edalat. A derivative for complex lipschitz maps with generalised cauchyriemann equations. *Theoretical Computer Science*, 564:89106, 2014.

[9] A. Edalat and A. Lieutier. Foundation of a computable solid modelling. *Theoretical Computer Science*, 284(2):319–345, 2002.

[10] A. Edalat and A. Lieutier. Domain theory and differential calculus (Functions of one variable). *Mathematical Structures in Computer Science*, 14(6):771–802, December 2004.

[11] A. Edalat, A. Lieutier, and D. Pattinson. A computational model for multi-variable differential calculus. *Information and Computation*, 224:22–45, 2013.

[12] Y. L. Ersov. Model C of partial continuous functionals. In R. O. Gandy and J. M. E Hyland, editors, *Logic Colloquium 76,*, pages 455–467. North Holland, 1977.

[13] M. Escardó and W. K. Ho. Operational domain theory and topology of sequential programming languages. *Information and Computation*, 207:411–437, 2009.

[14] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous Lattices and Domains*. Cambridge University Press, 2003.

[15] M. Gonzales. *Classical Complex Analysis*. CRC Press, 1991.

[16] S. C. Kleene. *Mathematical Logic*. Dover, 2002.

[17] G. Lebourg. Generic differentiability of Lipschitzian functions. *Transaction of AMS*, 256:125–144, 1979.

[18] K. Martin. The measurement process in domain theory. In *Lecture Notes in Computer Science*, volume 1853, pages 116–126. Springer, 2000.

[19] R. E. Moore. *Interval Analysis*. Prentice-Hall, Englewood Cliffs, 1966.

[20] R. M. Range. Complex analysis: A brief tour into higher dimensions. *The American Mathematical Monthly*, 110(2):89–108, 2003.

[21] W. Rudin. *Functional Analysis*. Mc-Graw-Hill, 1973.

[22] D. S. Scott. Continuous lattices. In F. W. Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, pages 97–136. Springer-Verlag, Berlin, 1972. LNM 274.

[23] J. Warga. Fat homeomorphisms and unbounded derivate containers. *J. Math. Anal. Appl.*, 81:545–560, 1981.

[24] S. Yamamuro. *Differential calculus in Topological Linear Spaces*, volume 374 of *Lecture Notes in Mathematics*. Springer-Verlag, 1970.

Appendix

Theorem 3.9

(i) $\mathcal{L}f$ is Scott continuous.

(ii) If $f'(x)$ exists, then $f'(x) \in \mathcal{L}f(x)$.

(iii) If $f \in C^1(U)$ then $\mathcal{L}f = f'$.

(iv) If $\mathcal{L}f(x)$ is a singleton, then $f'(x)$ exists and $f'(x) = \mathcal{L}f(x)$.

Proof. (i) and (ii) have already been proved.

(iii) By Proposition 3.5, we know that if $f'(x)$ exists then $f'(x) \in b$ whenever $f \in \delta(a,b)$ and $x \in a$. Thus $f'(x) \sqsupseteq \mathcal{L}f(x)$. To show equality, let $x \in U$ and put $L := f'(x)$. By the continuity of the total derivative $f' : U \to \mathbb{F}^{m\times n}$ at x, for each integer $n > 0$, there exists an open ball $a \subseteq U$ with $x \in a$ such that $f'(y) \in B_{1/n}(L)$ for $y \in a$, where $B_r(L) \subseteq \mathbb{F}^{m\times n}$ is the open ball of radius r and centre L. We have $f' \sqsupseteq \mathcal{L}f \sqsupseteq \mathrm{cl}(B_{1/n}(L))\,\chi_a$, where $\mathrm{cl}(B_r(L))$ is the closed ball centred at L with radius r. Since $\bigcap_{n\geq 0} \mathrm{cl}(B_{1/n}(L)) = f'(x)$ we conclude that $f'(x) = \mathcal{L}f(x)$.

(iv) Assume $\mathcal{L}f(x) = L \in \mathbb{F}^{m\times n} \subseteq \mathbf{C}(\mathbb{F}^{m\times n})$ is a singleton. Let $\epsilon > 0$ be given. Consider the open ball $O_\epsilon(L) \subseteq \mathbb{F}^{m\times n}$ with centre L and radius $\epsilon/2$ with respect to the Frobenius norm. By the definition of $\mathcal{L}f(x)$ and the intersection property of compact sets in a Hausdorff space, there exists $\delta(a,b)$, with $f \in \delta(a,b)$, such that $x \in a$ and $b \subseteq O_\epsilon(L)$. Thus, for $h \in \mathbb{F}^n$ with $\|h\|$ small enough so that $x + h \in a$ we have $f[x+h] - f[x] \in b[h]$ which is identical to:

$$f(x+h) - f(x) \in bh$$

or

$$f(x+h) - f(x) - Lh \in (b-L)h$$

But $\|b-L\| < \epsilon$ and thus $\|(b-L)h\| < \|h\|\epsilon$, since the Frobenius norm is subordinate to the Euclidean norm. Thus, we obtain:

$$\frac{\|f(x+h) - f(x) - Lh\|}{\|h\|} < \epsilon$$

which shows that $f'(x) = L$ and completes the proof.

□

Theorem 5.5 If $f : U \subseteq \mathbb{C}^n \to \mathbb{C}^m$ is locally Lipschitz then $\mathcal{L}f = \partial_d f$.

Proof. Let $z_0 \in U$. Let $b \in (\mathbf{C}(\mathbb{C}))^{m\times n}$ with $b \ll \mathcal{L}f(z_0)$. Then for some open set $a \subseteq U$ we have $b \sqsubseteq \mathcal{L}f(z)$ for all $z \in a$, which implies $f \in \delta(a,b)$. Let $h \in \mathbb{C}^n$ with $h_j \neq 0$ and $h_l = 0$ for $l \neq j$. Then we have $f_k(z+h) - f_k(z) \in b_{kj}h_j$. Assume $\hat{f} : (\mathbb{R}^2)^n \to (\mathbb{R}^2)^n$ is differentiable at (x,y). If $h_j = |h_j|e^{i\theta_j}$ then $\lim_{\|h\|\to 0}(f_k(z+h) - f_k(z))/\|h\|$ converges to a point on the boundary of $\mathsf{D}(\overline{V'_{kj}(x,y)}, i\overline{W'_{kj}(x,y)})$ and

$$\left\{ \lim_{\|h\|\to 0} (f_k(z+h) - f_k(z))/\|h\| : \theta_j \in [0, 2\pi) \right\}$$

is the whole boundary of $\mathsf{D}(\overline{V'_{kj}(x,y)}, i\overline{W'_{kj}(x,y)})$. Since b is convex, it follows that $\mathsf{D}(\overline{V'_{kj}(x,y)}, i\overline{W'_{kj}(x,y)}) \subseteq b_{kj}$. Since this holds for a dense set of points $z \in a$ where $\hat{f}$ is differentiable at the induced point (x,y), it follows that $\partial_d(z_0) \subseteq b$. Since $b \ll \mathcal{L}f$ is arbitrary, it follows that $\partial_d(z_0) \subseteq \mathcal{L}f(z_0)$.

Next assume $b \ll \partial_d(z_0)$. Taking the kj component of b the result follows as in the one dimensional generalisation of the Cauchy-Riemann equations [8, Theorem 4.7] and Theorem 3.10.

□

To prove the continuity property of t-distance, we first need a lemma. Recall that a crescent is the intersection of an open and a closed set. Each step function $g \in (X \to D)$, where X is a topological space and D is a domain, is given by a finite number of disjoint crescents in each of which the value of the step function is constant. From the Scott continuity of step functions, we can immediately deduce the following.

Lemma 10.1. *Suppose x belongs to the domain of definition of a step function $g \in (X \to D)$. If x is a point on the boundary of a crescent K of g, then $g(x) \sqsubseteq g(y)$ for $y \in K$.*

Moreover, any step function can be extended by Scott continuity to the closure of its domain of definition.

Proposition 6.2 The t-distance is Scott continuous on Scott continuous functions restricted to a compact subset of U.

Proof. Let $X \subseteq U$ be a fixed compact subset endowed with the relative subset topology induced from the open set $U \subseteq \mathbb{F}^n$. Let $g \in (U \to \mathbf{C}(\mathbb{C}))$ and suppose for some $c > 0$ we have $\mathsf{d}(g_{\restriction X}) < c$. Let $g = \sup_{i\geq 1} g_i$ where $g_i \ll g$ for all $i \geq 1$ is an increasing sequence of step functions. It is sufficient to find $i \geq 1$ such that $\mathsf{d}(g_{i\restriction X}) < c$. Assume, for a contradiction, that for each $i \geq 1$ we have $\mathsf{d}(g_{i\restriction X}) \geq c$. For each $i \geq 1$, consider the crescents, $K_{i1}, K_{i2}, \ldots, K_{in_i}$ of X, such that

(i) in each of which the value of $g_{i\restriction X}$ is constant and has diameter greater or equal to c and

(ii) the crescents $K_{i1}, K_{i2}, \ldots, K_{in_i}$ refine the crescents $K_{(i-1)1}, K_{(i-1)2}, \ldots, K_{(i-1)n_{i-1}}$, for $i > 1$.

Note that our second requirement (ii) implies that the value of g_i for $i > 1$ can be the same on some crescents among $K_{i1}, K_{i2}, \ldots, K_{in_i}$. By Lemma (10.1), it follows that in fact the value of $g_{i\restriction X}$ on any point of the closure $\mathrm{cl}(K_{ij})$ in X will have diameter greater or equal to c. Consider now the tree T whose root node is $\mathbb{C}$ and whose nodes on level $i \geq 1$ consist of the compact sets $\mathrm{cl}(K_{i1}), \mathrm{cl}(K_{i2}), \ldots, \mathrm{cl}(K_{in_i})$, and a node $\mathrm{cl}(K_{ij})$ for $1 \leq j \leq n_i$ is a parent of a node $\mathrm{cl}(K_{i+1,k})$ for some k with $1 \leq k \leq n_{i+1}$ if and only if $K_{ij} \supset K_{i+1,k}$. Since the crescents of $g_{i\restriction X}$ are disjoint, it follows that any node on level $i+1$ will have a unique parent node on level $i \geq 0$. By our assumption, the tree is infinite since for each $i \geq 1$, there is at least one crescent for which the constant value of g_i has diameter greater or equal to c. By König's lemma [16], T will have an infinite branch which gives a shrinking sequence of non-empty compact convex subsets of X with non-empty intersection. Let $x_0 \in X$ belong to this intersection. Then $\mathrm{diam}(g_{i\restriction X}(x_0)) \geq c$ for all $i \geq 1$. Since $g_{\restriction X}(x) = \bigcap_{i\geq 0} g_{i\restriction X}(x)$, it follows that $\mathrm{diam}(g_{\restriction X}(x) = \inf_{i\geq 0} \mathrm{diam}(g_{i\restriction X}(x) \geq c$, which gives a contradiction. □

Note that the t-distance map may fail to be continuous on non-compact sets. For example, let $\mathbb{F} = \mathbb{C}$ with $U = \{z \in \mathbb{C} : |z| < 1\}$ be the interior of the unit disk centred at the origin, and for $n \geq 1$ define $f_n : U \to \mathbf{C}(\mathbb{C})$ by $f_n(z) = z$ if $|z| < 1 - \frac{1}{n}$ and $f_n(z) = \bot$ otherwise. Then, clearly $f = \sup f_n$ is the identity map on U with $\mathsf{d}(f) = 0$ but $\mathsf{d}(f_n) = \infty$ for all $n \geq 1$.

Proposition 8.1 The predicate Cons_0 is Scott closed.

Proof. Let $(f_i, g_i)_{i \geq 0}$ be an increasing sequence of pairs of step functions with $(f_i, g_i) \in \mathsf{Cons}$, with $f_i \in (U \to \mathbb{R}^2)^m$ and $g_i \in (\mathbf{C}(\mathbb{C}))^{m \times n}$. Let $f = \sup_{i \geq 0} f_i$ and $g = \sup_{i \geq 0} g_i$, where $\mathrm{dom}(f) = \bigcup_{i \geq 0} \mathrm{dom} f_i$ and $\mathrm{dom}(g) = \bigcup_{i \geq 0} \mathrm{dom} g_i$. Without loss of generality assume $g_0(x) \neq \bot$ for some $x \in U$. Let $h_i : U \to \mathbb{C}^m$ be a partial map that is the Lipschitz witness of consistency such that $f_i^- \leq h_i \leq f_i^+$ and $g_i \sqsubseteq \mathcal{L} h_i$. Similar to the case of real vector maps, we can show that h_i is uniformly bounded and equi-continuous, from which we obtain a uniformly convergent subsequence $(h_{i_k})_{k \geq 0} d$, whose limit $h : U \to \mathbb{C}^m$, say, will satisfy $f \sqsubseteq h$ and $g \sqsubseteq \mathcal{L} h$, i.e., it will be a witness for the consistency of (f, g).

□