A. Visual Hull from Imprecise Polyhedral Scene - Supporting Material

A.1. Quadratic surface from 3 skewed lines

In the classical visual hull we encounter quadratic surfaces generated by three skewed lines.

Proposition A.1 *Given three mutually skewed lines l_0, l_1 and l_2, for every point q on any l_i, there is a unique line that intersects all three lines. With q going through every point on l_i (i = 0, 1 or 2), the line sweeps out a quadratic surface $Q(l_0, l_1, l_2)$ such that*

- *if l_0, l_1 and l_2 are parallel to a plane, the surface swept out is a hyperbolic paraboloid,*
- *In other cases, the surface generated is a hyperboloid of one piece.*

In [1], Blinn presents a scheme in generating a quadratic surface from 3 skewed lines as follows. We have used this representation to generate the positive and negative regions (half-spaces) induced by quadratic surfaces.

Given 2 points in homogeneous coordinates $S = [x_s, y_s, z_s, w_s]$ and $T = [x_t, y_t, z_t, w_t]$, we can represent the line L connecting S and T by the antisymmetric matrix:

$$L = \begin{bmatrix} 0 & p & -q & r \\ -p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{bmatrix}$$

where

$$p = \begin{vmatrix} z_s & w_s \\ z_t & w_t \end{vmatrix}, \quad q = \begin{vmatrix} y_s & w_s \\ y_t & w_t \end{vmatrix}, \quad r = \begin{vmatrix} y_s & z_s \\ y_t & z_t \end{vmatrix},$$
$$s = \begin{vmatrix} x_s & w_s \\ x_t & w_t \end{vmatrix}, \quad t = \begin{vmatrix} x_s & z_s \\ x_t & z_t \end{vmatrix}, \quad u = \begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix}.$$

Let us consider L as the intersection of 2 planes in their homogeneous representation: $G = [a_g, b_g, c_g, d_g]^T$ and $H = [a_h, b_h, c_h, d_h]^T$, we can represent L in another antisymmetric form, namely:

$$\tilde{L} = \begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ -g & j & -k & 0 \end{bmatrix}$$

where

$$e = \begin{vmatrix} c_g & c_h \\ d_g & d_h \end{vmatrix}, \quad f = \begin{vmatrix} b_g & b_h \\ d_g & d_h \end{vmatrix}, \quad g = \begin{vmatrix} b_g & b_h \\ c_g & c_h \end{vmatrix},$$
$$h = \begin{vmatrix} a_g & a_h \\ d_g & d_h \end{vmatrix}, \quad j = \begin{vmatrix} a_g & a_h \\ c_g & c_h \end{vmatrix}, \quad k = \begin{vmatrix} a_g & a_h \\ b_g & b_h \end{vmatrix}.$$

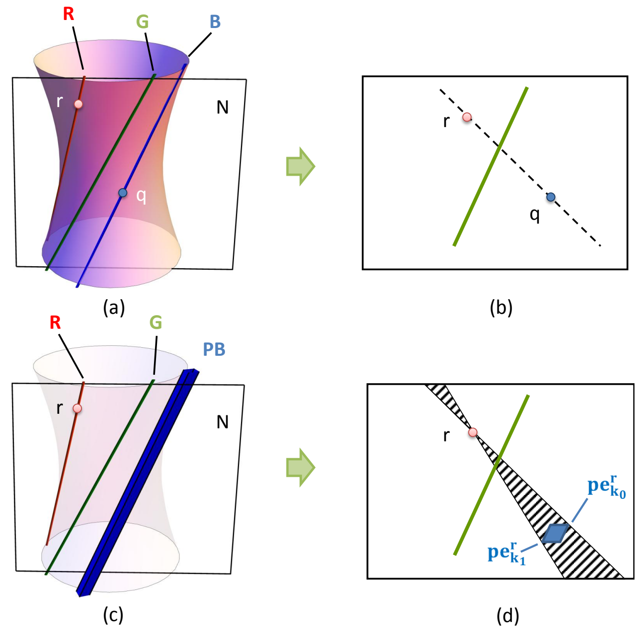

Figure 1. **(a)** A quadratic surface is determined by 3 skewed lines: red $[R]$, green $[G]$ and blue $[B]$. A point r on R and the line G determine a plane N which intersects B at the point q. **(b)** The view on the intersection plane N: the line G (the green line) is embedded in N and the dashed line connecting r and q is the line intersecting R, G and B. **(c)** Similar to (a) except that B has been extended into a partial line PB. **(d)** The intersection of N and the partial line PB is a polygon. The shaded area contains all the straight lines intersecting R, G and a cross-section of PB. Two edges $pe^r_{k_0}$ and $pe^r_{k_1}$ on PB satisfy Eq. (4)

There is a simple way to derive $\tilde{L}$ and L from each other.

$$e = u, \ f = -t, \ g = s, \ h = r, \ j = -q, \ k = p.$$

The quadratic surface $Q(K, L, M)$ generated by the 3 skewed lines K, L and M is given by [1].

$$Q(K, L, M) =$$
$$K\tilde{M}L - L\tilde{M}K = L\tilde{K}M - M\tilde{K}L = M\tilde{L}K - K\tilde{L}M.$$

The matrix $Q(K, L, M)$ is 4×4 and for any point $P = [x_p, y_p, z_p, w_p]$ on the surface of the quadratic surface we have:

$$PQP^T = 0.$$

A.2. Proofs of the two Theorems

Theorem 4.5 *Given 3 partial lines PL_0, PL_1 and PL_2, we have:*

$$\widehat{PQ}(PL_0, PL_1, PL_2) = \tag{1}$$

$$\bigcap \{\hat{Q}([pe_0], [pe_1], [pe_2]) | pe_i \text{ an edge of } PL_i, \ i = 0, 1, 2\}$$

$$\widetilde{PQ}(PL_0, PL_1, PL_2) = \tag{2}$$

$$\bigcap\{\check{Q}([pe_0], [pe_1], [pe_2]) | pe_i \text{ an edge of } PL_i,\ i = 0, 1, 2\}$$

Proof Recall that

$$\widehat{PQ}(PL_0, PL_1, PL_2) = \bigcap \hat{Q}(pl_0, pl_1, pl_2);$$
$$\widetilde{PQ}(PL_0, PL_1, PL_2) = \bigcap \check{Q}(pl_0, pl_1, pl_2),$$

where $pl_i \in SEL(PL_i)$, $i = 0, 1, 2$. We will only prove the positive region case Eq. (1) as the negative region case is similar.

The subset relation

$$\bigcap \hat{Q}(pl_0, pl_1, pl_2) \subseteq \bigcap \hat{Q}([pe_0], [pe_1], [pe_2])$$

is clear since each $Q([pe_0], [pe_1], [pe_2])$, is an instance of $Q(pl_0, pl_1, pl_2)$.

For the subset relation in the other direction, we first consider a simplified situation. Assume we have the partial quadratic surface formed by two classical lines l_0, l_1 and a partial line PL_2 that is formed by a set of edges $E = \{pe_0, pe_1, ..., pe_{n-1}\}$. In the case that partial points are hyper-rectangles, as we assume in this paper, we have $4 \leqslant n \leqslant 6$. For any point r on l_0, there is a classical plane $pln(r, l_1)$ defined by r and l_1. For any line selection $pl_2 \in SEL(PL_2)$, the plane $pln(r, l_1)$ has a unique intersection point q with pl_2. We have:

$$Q(l_0, l_1, pl_2) \cap pln(r, l_1) = ln(r, q) \cup l_1,$$

where $ln(r, q)$ is the straight line connecting r and q.

As r moves along l_0, the line $ln(r, q)$ sweeps out the quadratic surface $Q(l_0, l_1, pl_2)$[3, 1]. Therefore,

$$Q(l_0, l_1, pl_2) = \tag{3}$$

$$\bigcup\{ln(r, q) | q = pln(r, l_1) \cap pl_2, r \in l_0\}$$

Furthermore, as shown in Fig. 1 (d), we can always find 2 edges of PL_2, say $pe^r_{k_0}$ and $pe^r_{k_1}$, such that:

$$ln(r, q) \subseteq \bigcup_{t=0,1} cl\left[\check{Q}(l_0, l_1, [pe^r_{k_t}])\right], \tag{4}$$

where $cl[A]$ denotes the closure of the set A. Therefore, we have

$$Q(l_0, l_1, pl_2) \subseteq \bigcup_{r \in l_0} \bigcup_{t=0,1} cl\left[\check{Q}(l_0, l_1, [pe^r_{k_t}])\right]$$
$$= \bigcup_{j=0}^{s} cl\left[\check{Q}(l_0, l_1, [pe_{m_j}])\right]$$

where $pe_{m_j} \in E$ and $s \leqslant n - 1$.

Hence, we obtain:

$$\check{Q}(l_0, l_1, pl_2) \subseteq \bigcup_{j=0}^{s} cl\left[\check{Q}(l_0, l_1, [pe_{m_j}])\right].$$

Thus, for all $pl_2 \in PL_2$ we have:

$$\check{Q}(l_0, l_1, pl_2) \subseteq \bigcup_{j=0}^{n-1} cl\left[\check{Q}(l_0, l_1, [pe_{m_j}])\right],$$

and hence,

$$\bigcup_{pl_2 \in PL_2} \check{Q}(l_0, l_1, pl_2) \subseteq \bigcup_{j=0}^{n-1} cl\left[\check{Q}(l_0, l_1, [pe_{m_j}])\right],$$

Combining this with the Prop. 4.1 we have:

$$\bigcap_{pl_2 \in PL_2} \hat{Q}(l_0, l_1, pl) \supseteq \bigcap_{j=0}^{n-1} \hat{Q}(l_0, l_1, [pe_j]),$$

which finally implies:

$$\widehat{PQ}(l_0, l_1, PL_2) \supseteq \bigcap_{j=0}^{n-1} \hat{Q}(l_0, l_1, [pe_j]).$$

We can use a similar method when extending l_0 and l_1 into their partial counterparts PL_0 and PL_1, which will complete the proof of the theorem.

Theorem 5.6 *A conservative partial EEE surface $PEEE_c$ can be constructed from 3 PE_i partial edges as follows:*

$$PEEE_c^+|^{l_i^p}_{l_{i+1}^p} := Q(l_i^p, l_{i+1}^p, l_{i+2}^n)|^{l_i^p}_{l_{i+1}^p}; \tag{5}$$
$$PEEE_c^-|^{l_i^n}_{l_{i+1}^n} := Q(l_i^n, l_{i+1}^n, l_{i+2}^p)|^{l_i^n}_{l_{i+1}^n}. \tag{6}$$

Proof We have to show that the partial surface defined by the positive and negative classical surfaces in Equations 5 and 6 above is a partial EEE surface as defined in Definition 5.1 of the paper, i.e., that it contains all straight lines that intersect the three partial edges. Let l be such a line and fix a direction on the line.

The definition of $PEEE_c$ is equivalent to:

$$\widehat{PEEE}_c = \bigcap_{i=0}^{2} \hat{Q}(l_i^p, l_{i+1}^p, l_{i+2}^n); \tag{7}$$

$$\widetilde{PEEE}_c = \bigcap_{i=0}^{2} \check{Q}(l_i^n, l_{i+1}^n, l_{i+2}^p). \tag{8}$$

We first show that the intersection points of l with $Q(l_i^p, l_{i+1}^p, l_{i+2}^n)$ are outside the region $Q(l_i^p, l_{i+1}^p, l_{i+2}^n)|^{l_i^p}_{l_{i+1}^p}$. We will use the following two properties:

(i) There are at most 2 intersection points between a line and a quadratic surface if the line is not contained in the quadratic surface.

(ii) Each of the lines l_i^p and l_i^n is the intersection line between two planes parallel to the edges of $[PE_i]$, the infinite tube generated from PE_i, and, therefore, l_i^p and l_i^n are parallel.

Consider the plane $pln(l_{i+2}^p, l_{i+2}^n)$ formed by l_{i+2}^p and l_{i+2}^n. Then $p_{i+2} := l \cap pln(l_{i+2}^p, l_{i+2}^n)$ lies in between l_{i+2}^p and l_{i+2}^n on the plane $pln(l_{i+2}^p, l_{i+2}^n)$. We have

$$p_{i+2} \in \hat{Q}(l_i^p, l_{i+1}^p, l_{i+2}^n). \tag{9}$$

Similarly, for $p_{i+1} = l \cap pln(l_{i+1}^p, l_{i+1}^n)$, we have

$$p_{i+1} \in \check{Q}(l_i^p, l_{i+1}^p, l_{i+2}^n). \tag{10}$$

Therefore, the line l must have an intersection point with $Q(l_i^p, l_{i+1}^p, l_{i+2}^n)|_{l_{i+2}^p}^{l_{i+1}^p}$.

Similarly, we have

$$p_i \in \check{Q}(l_i^p, l_{i+1}^p, l_{i+2}^n), \tag{11}$$

with $p_i = l \cap pln(l_i^p, l_i^n)$. Combining this with Eq. (10), we deduce that the line l must have an even number of intersection points with $Q(l_i^p, l_{i+1}^p, l_{i+2}^n)|_{l_{i+1}^p}^{l_i^p}$. It now follows from property (i) above that there are actually no intersection points between l and $Q(l_i^p, l_{i+1}^p, l_{i+2}^n)|_{l_{i+1}^p}^{l_i^p}$. Therefore, we have the following property:

$$ls(p_i, p_{i+1}) \subseteq \check{Q}(l_i^p, l_{i+1}^p, l_{i+2}^n).$$

where $ls(p_i, p_{i+1})$ is the line segment of l between points p_i and p_{i+1} with the same direction as l. Note that for one value of $i = 0, 1$ or 2, the line segment $ls(p_i, p_{i+1})$ goes through the point at infinity. Similarly, we obtain:

$$ls(p_i, p_{i+1}) \subseteq \hat{Q}(l_i^p, l_{i+1}^p, l_{i+2}^n).$$

We conclude that

$$\begin{aligned} ls(p_i, p_{i+1}) &\subseteq \check{Q}(l_i^p, l_{i+1}^p, l_{i+2}^n) \cap \hat{Q}(l_i^p, l_{i+1}^p, l_{i+2}^n) \\ &\subseteq \partial_P PEEE_c, \end{aligned}$$

i.e., the segment of the line l between p_i and p_{i+1} is indeed contained in the partial surface $PEEE_c$. Since this is true of $i = 0, 1, 2$, the result follows.

A.3. Visibility Complex

The visibility complex [2] is a theoretical framework in which the visibility information of the 3D space can be efficiently represented and calculated. According to [5], the brute force method provided by Laurentini [4] can be further simplified with the visibility complex polytope structure. If the polygonal scene has n vertices and m visual event surfaces, it is claimed that the computational complexity can be reduced from $O(n^{12})$ to $O((n^4 + m^3)\log n)$, with $m = O(n^3)$ in the worst case.

However, with imprecision in the input data of the visibility complex, the intersection of different partial visual event surfaces and the degenerate situations (*e.g.* collinearity) needs to be handled carefully. How to generalize our framework to the visibility complex and thus to obtain a more practical algorithm for the visual hull with imprecise input is the subject of our future work.

References

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