# Computing Delaunay Triangulation with Imprecise Input Data

**A. A. Khanban** **A. Edalat**

Abstract. The key step in the construction of the Delaunay triangulation of a finite set of planar points is to establish correctly whether a given point of this set is inside or outside the circle determined by any other three points. We address the problem of formulating the in-circle test when the coordinates of the planar points are given only up to a given precision, which is usually the case in practice. By modelling imprecise points as rectangles, and using the idea of partial disc, we construct a reliable in-circle test that provides the best possible Delaunay triangulation with the imprecise input data given by rectangles.

## 1. Introduction

A Delaunay triangulation of a set of planar points is a triangulation with the property that the interior of the circumcircle of each triangle contains no points of the set [**1**]. If there are no four cocircular points in the set, then the Delaunay triangulation is unique. A divide and conquer algorithm [**4**] and an algorithm based on sweepline technique [**3**] compute the Delaunay triangulation in $O(n \log n)$, while a random incremental algorithm [**1**] has an average complexity of $O(n \log n)$. We present the simplest algorithm [**7**], which is $O(n^4)$, but illustrates the crucial role of the so-called in-circle test in computing the Delaunay triangulation:

**Algorithm:**

```
for each of the possible n choose 3 “triangles” of points do
{
    for each of the OTHER n - 3 points do
    {
        if all the OTHER points are outside of the circle
            circumscribing the triangle (i.e. the in-circle test)
        then this triangle IS in the unique Delaunay triangulation
        else this triangle is NOT in the unique Delaunay triangulation
    }
}
```

Any algorithm for Delaunay triangulation is based on correctly establishing whether a given point is inside or outside the circumcircle of three given points. When the coordinates of the $n$ points are infinitely precise, this in-circle test is determined in terms of finding the signs of various determinants. However, in practice, the input data is imprecise as the coordinates of the points are either the result of some measurements with certain error or are actually floating point numbers which represent rectangles rather than points in the plane. The in-circle test then becomes

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non-trivial. Ely and Leclerc [**2**] have formulated the following two essential properties for the in-circle test. By *reliable* it is meant that the test must never make an error if it does decide that the point is inside or outside the circle. By *sharp* it is meant that the test should only fail (to decide whether the point is inside or outside of the circle) because of the underlying geometry of the imprecise points.

The use of interval arithmetic [**6**] can provide *reliability* by ensuring that the successful result of the in-circle test is always correct, but this is almost always at the expense of sharpness as the test may frequently fail unjustifiably.

In [**2**], Ely and Leclerc model an imprecise point by a disc and show that any three discs representing three imprecise points determine eight circles in the plane, each tangent to the three discs, which are then used to formulate an in-circle test that is sharper than the naive interval arithmetic.

However, in practice, an imprecise point is almost always given by two imprecise $x$ and $y$ coordinates, representing two horizontal and vertical line segments respectively, which therefore give rise to a rectangle rather than a disc.

In this paper, we represent imprecise points by rectangles and, using the notion of partial disc established in [**5**], we construct a reliable in-circle test which is completely sharp: when the output is neither "inside" nor "outside", then there exist four cocircular points, one in each rectangle.

## 2. Partial Disc

We first recall the notion of a partial disc [**5**], which is the generalization of a disc. Any three noncollinear points $x, y, z$ in the plane determine a disc $D_{xyz}$ which has $x, y, z$ on its boundary. Now assume that all we know is that the three points $x, y, z$ lie respectively in the three rectangles $R_1, R_2, R_3$, where each rectangle $R = [a, b] \times [c, d]$ is the product of two rational intervals $[a, b]$ and $[c, d]$. We say that the three rectangles are *noncollinear* if there is no straight line which intersects all three rectangles. Assuming the noncollinearity condition, $R_1, R_2, R_3$ will determine a *partial disc* that is given by a pair of disjoint open subsets $(I(R_1, R_2, R_3), E(R_1, R_2, R_3))$ with the *interior*

$$I(R_1, R_2, R_3) = (\bigcap\{D_{xyz} \mid x \in R_1,\ y \in R_2,\ z \in R_3\})^\circ,$$

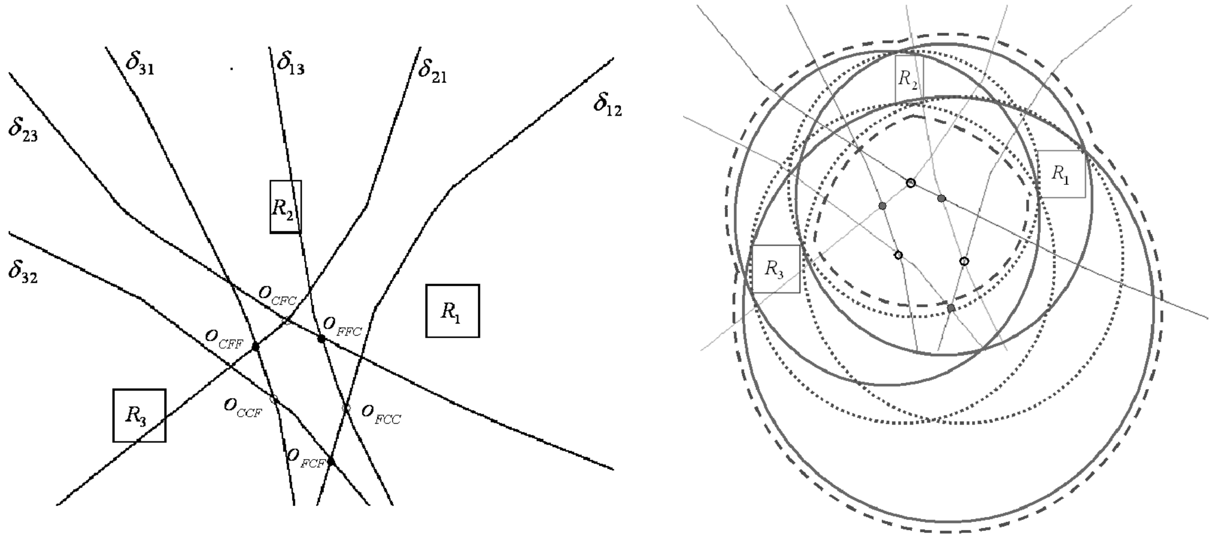
and the *exterior*

$$E(R_1, R_2, R_3) = (\bigcup\{D_{xyz} \mid x \in R_1,\ y \in R_2,\ z \in R_3\})^c,$$

where $A^\circ$ and $A^c$ are respectively the interior and complement of the set $A$. In words, $I := I(R_1, R_2, R_3)$ is the largest open set contained in the interior of any circle which intersects all three rectangles $R_1, R_2, R_3$, whereas $E := E(R_1, R_2, R_3)$ is the largest open set contained in the exterior of any such circle.

The sets $I$ and $E$ are computed as follows. Let $\delta_{ij}$ be the locus of all the points in $\mathbb{R}^2$ with the furthest distance from points in $R_i$ equal to the closest distance to $R_j$, i.e.

$$\delta_{ij} = \{x \in \mathbb{R}^2 \mid \max_{p \in R_i} d(x, p) = \min_{p \in R_j} d(x, p)\}.$$




(a) The Six Centres (b) Interior and Exterior

FIGURE 1. The centres and the interior and exterior of a partial disc.

Each $\delta_{ij}$ is a continuous curve made up of a number of linear and parabolic segments as in Figure 1(a).

PROPOSITION 2.1. *The intersections*

$$\delta_{21} \cap \delta_{23}, \quad \delta_{12} \cap \delta_{32}, \quad \delta_{21} \cap \delta_{31}, \quad \delta_{31} \cap \delta_{32}, \quad \delta_{13} \cap \delta_{23}, \quad \delta_{12} \cap \delta_{13}$$

*are all singletons iff the three rectangles are noncollinear. If the three rectangles are actually collinear then some of the above intersections will be empty.*

Assuming the noncollinearity condition, we define six centre points as follows:

$$\begin{aligned} \{o_{CFC}\} &= \delta_{21} \cap \delta_{23}, & \{o_{FCF}\} &= \delta_{12} \cap \delta_{32}, \\ \{o_{CFF}\} &= \delta_{21} \cap \delta_{31}, & \{o_{CCF}\} &= \delta_{31} \cap \delta_{32}, \\ \{o_{FFC}\} &= \delta_{13} \cap \delta_{23}, & \{o_{FCC}\} &= \delta_{12} \cap \delta_{13}, \end{aligned} \tag{2.1}$$

as in Figure 1(a). Note that $o_{CCF}$ is the centre of a circle with radius $r_{CCF}$, which passes through (i) the point of $R_1$ closest to $o_{CCF}$, (ii) the point of $R_2$ closest to $o_{CCF}$ and (iii) the point of $R_3$ furthest from $o_{CCF}$; hence the subscript in $o_{CCF}$. Similarly for the other vertices. If any one or more of the rectangles $R_1$, $R_2$ and $R_3$ are in fact singletons then some of the above centre points will coincide.

Let $D(o, r)$ denote the closed disc with centre $o$ and radius $r$. The interior $I$ and the exterior $E$ of the partial disc of $R_1, R_2, R_3$ are given by:

$$I = (D_1 \cap D_2 \cap D_3)^\circ, \qquad E = (D'_1 \cup D'_2 \cup D'_3)^c,$$

where $D_1 = D(o_{FCC}, r_{FCC})$, $D_2 = D(o_{CFC}, r_{CFC})$ and $D_3 = D(o_{CCF}, r_{CCF})$ on the one hand and $D'_1 = D(o_{CFF}, r_{CFF})$, $D'_2 = D(o_{FCF}, r_{FCF})$ and $D'_3 = D(o_{FFC}, r_{FFC})$ on the other hand. In Figure 1(b), the boundaries of the discs $D_1$, $D_2$ and $D_3$ are depicted with dotted lines, those of $D'_1$, $D'_2$ and $D'_3$ with solid lines and the boundaries of the sets $I$ and $E$ with dashed lines. The closed region bounded between $I$ and $E$, i.e. bounded between the two closed dashed curves, is the *boundary of the partial disc*.

## 3. The In-Circle Test

We first compute the $\delta_{ij}$ boundaries as in [**5**]. Then, we find the intersection points as in (2.1), which give us the centres of the circles. In a non-degenerate case, when all the three imprecise points are rectangles, there will be three centres for the interior circles and three centres for the exterior circles. The following formula gives the radius $r$ of any of the six circles, which make up the interior and the exterior of the partial disc, in terms of its centre $o$:

$$r = \min\left\{\{\max d(o,p) \mid p \text{ is a vertex of } R_i\} \mid i = 1,2,3\}\right\}.$$

Using the following *containment* predicate, we can verify that a given imprecise point or rectangle $R$ is inside, outside, or on the boundary of a partial disc given by three rectangles $R_1, R_2, R_3$.

$$\mathrm{Con}(R, (R_1, R_2, R_3)) = \begin{cases} \text{inside} & \text{if } R \subset I(R_1, R_2, R_3) \\ \text{outside} & \text{if } R \subset E(R_1, R_2, R_3) \\ \text{boundary} & \text{otherwise.} \end{cases}$$

Since the coordinates of the centres of the six interior and exterior circles are intersections of straight lines or parabolas with rational coefficients, the output of the containment predicate can be correctly computed using rational arithmetic.

THEOREM 3.1. *If the three rectangles are collinear, then the in-circle algorithm will verify it using Proposition 2.1. Otherwise, with the noncollinearity condition, the algorithm outputs the correct value of the containment predicate.*

We finally note that computing the partial disc passing through three imprecise points and applying the in-circle test for the fourth imprecise point is of order $O(1)$ and does not affect the complexity of any algorithm which uses this test.

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DEPARTMENT OF COMPUTING, IMPERIAL COLLEGE, LONDON, U.K.
*E-mail address*: `khanban@imperial.ac.uk` `ae@imperial.ac.uk`