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When Scott is Weak on the Top

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We construct an approximating chain of simple valuations on the upper space of a compact metric space whose lub is a given probability measure on the metric space. We show that whenever a separable metric space is homeomorphic to a G_δ subset of an ω-continuous dcpo equipped with its Scott topology, then the space of probability measures of the metric space equipped with the weak topology is homeomorphic with a subset of the maximal elements of the probabilistic power domain of the ω-continuous dcpo. Given an effective approximation of a probability measure by an increasing chain of normalised valuations on the upper space of a compact metric space, we show that the expected value of any Hölder continuous function on the space can be obtained up to any given accuracy. We present a novel application in computing integrals in dynamical systems. We obtain an algorithm to compute the expected value of any Hölder continuous function with respect to the unique invariant measure of the Feigenbaum map in the periodic doubling route to chaos.

1. Introduction

Domain theory was introduced by Dana Scott in 1970 as a mathematical foundation for the semantics of programming languages (Scott 1970). It is now a basic paradigm in denotational semantics in particular and in theoretical computer science in general.

A new direction for applications of domain theory has recently emerged. In particular, a domain-theoretic framework for measure and integration theory on locally compact second countable metric spaces has been developed in (Edalat 1995a; Edalat 1995b). The upper space of a locally compact second countable metric space, i.e. the set of non-empty compact subsets ordered by reverse inclusion, is an ω-continuous dcpo (directed complete partial order). On the other hand, the probabilistic power domain of any ω-continuous dcpo, i.e. the set of continuous valuations on the dcpo ordered pointwise, is an ω-continuous dcpo with a basis of simple valuations (Jones and Plotkin 1989). It was shown in (Edalat 1995a) that the set of probability measures of any locally compact second countable metric space can be injected into the set of maximal elements of the probabilistic power domain of the upper space of the metric space. This provides a new theory of approximation of measures and has led to a generalisation of the Riemann integral (Edalat 1995b; Edalat and Negri 1996). There have been diverse applications in fractal geometry (Edalat 1996), neural nets (Edalat 1995d), statistical physics (Edalat 1995c), stochastic processes and image compression (Edalat 1995e).

In this paper, we solve the following two basic problems in constructive and computational mathematics.

Suppose a probability measure on a compact metric space is given by its values on a countable base closed under finite unions and intersections. We construct an increasing chain of simple valuations on the upper space of the metric space, i.e. an approximating chain in the probabilistic power domain of the upper space, whose least upper bound is the probability measure.

In order to state the second problem, we first recall a fundamental feature of the various domain-theoretic models for classical Hausdorff spaces, e.g. the Cantor domain Σ^∞ (consisting of the set of finite and infinite sequences over a countable alphabet Σ with the prefix ordering), the dcpo of closed intervals of the unit interval ordered by reverse inclusion, the upper space of a second countable locally compact Hausdorff space (Edalat 1995a), the space of formal balls of a complete separable metric space (Edalat and Heckmann 1996) and Lawson's maximal hulls for Polish spaces (Lawson 1996). In all these models, an ω-continuous dcpo provides a computational framework for a separable metrizable space identified as the set of the maximal points of the dcpo equipped with the relative Scott topology. In other words, the relative Scott topology and the classical Hausdorff topology coincide. In this paper we examine this basic property for the spaces of probability measures equipped with the weak topology.

We show that whenever a separable metric space is homeomorphic to a G_δ subset of an ω-continuous dcpo equipped with its Scott topology, then the space of probability measures of the metric space equipped with the weak topology is homeomorphic with a subset of the maximal elements of the probabilistic power domain of the ω-continuous dcpo.

Since the weak topology is the most important topology in measure theory, the above coincidence of the weak and the relative Scott topologies highlights a satisfactory connection between classical measure theory and domain theory.

We give a necessary and sufficient condition that the least upper bound (lub) of an increasing chain of normalised simple valuations on the upper space of a compact metric space gives a probability measure on the metric space and show that when this condition is effectively satisfied one can compute the expected value of any Hölder continuous map with respect to the probability measure up to any given accuracy.

We illustrate the computational significance of these results in evaluating integrals with a novel application in chaos theory. Periodic doubling bifurcation to chaos is a universal route in which a one-parameter family of one-dimensional maps becomes chaotic. The Logistic family of maps on the unit interval is the prototype of such a system. As the parameter is increased, the family goes through an infinite sequence of periodic doubling bifurcations. At the limit of these parameter values, the map is at the edge of chaos and is an example of a Feigenbaum map, the prototype of an infinitely renormalizable map (de Mello and van Strien 1993). The unique ergodic measure of a C^2 Feigenbaum map with a non-flat critical point can be obtained in a natural way as the lub of an increasing chain of simple valuations on the dcpo of the closed intervals of the unit interval. We use this construction to obtain an algorithm to compute the expected value of any Hölder continuous function with respect to the ergodic measure of a Feigenbaum map.

2. The domain-theoretic framework

In this section, we briefly review from (Edalat 1995a; Edalat 1995b) the domain-theoretic framework for measure and integration theory on compact metric spaces which we need in this paper.

First, recall the basic notions in domain theory (Jung and Abramsky 1994). A non-empty subset $A \subseteq P$ of a poset $(P, \sqsubseteq)$ is *directed* if for any pair of elements $x, y \in A$ there is $z \in A$ with $x, y \sqsubseteq z$. A *directed complete partial order (dcpo)* is a partial order in which every directed subset A has a least upper bound (lub), denoted by $\bigsqcup A$. An open set $O \subseteq P$ of the Scott topology of a dcpo is a set which is upward closed (i.e. $x \in O \ \& \ x \sqsubseteq y \ \Rightarrow \ y \in O$) and is inaccessible by lubs of directed sets (i.e. if A is directed, then $\bigsqcup A \in O \ \Rightarrow \ \exists x \in A. x \in O$). The topology of a dcpo in this paper is always assumed to be the Scott topology. It can be shown that a function $f : D \to E$ from a dcpo D to another one E is continuous with respect to the Scott topology iff it is *monotone*, i.e. $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$, and preserves lubs of directed sets, i.e. $\bigsqcup_{i \in I} f(x_i) = f(\bigsqcup_{i \in I} x_i)$, where $\{x_i \mid i \in I\}$ is any directed subset of D. From this it follows that a continuous function $f : D \to D$ on a dcpo D with least element (or bottom) $\perp$ has a *least fixed point* given by $\bigsqcup_{n \geq 0} f^n(\perp)$. Given two elements x, y in a dcpo, we say x is *way-below* y or x *approximates* y, denoted by $x \ll y$, if whenever $y \sqsubseteq \bigsqcup A$ for a directed set A, then there is $a \in A$ with $x \sqsubseteq a$. We say that a subset $B \subseteq D$ is a *basis* for D if for each $d \in D$ the set A of elements of B way-below d is directed and $d = \bigsqcup A$. We say D is *continuous* if it has a basis; it is ω*-continuous* if it has a countable basis. We denote the set of maximal elements of a dcpo D by $\max(D)$.

Let X be a compact metric space and $\mathbf{U}X$ the upper space of X, consisting of the non-empty compact subsets of X ordered by reverse inclusion. Then $\mathbf{U}X$ is an ω-continuous dcpo and a basis of its Scott topology is given by collections of the form $\square a = \{C \in \mathbf{U}X \mid C \subseteq a\}$, for any open set $a \subseteq X$. The singleton map $s : X \to \mathbf{U}X$ defined by $s(x) = \{x\}$ embeds X onto the set of maximal elements of $\mathbf{U}X$. It sends any open subset of X to a G_δ subset (i.e. a countable intersection of open subsets) of $\mathbf{U}X$ and any Borel subset of X into a Borel subset of $\mathbf{U}X$ (Edalat 1995a).

We will consider the probabilistic power domain $\mathbf{PU}X$ of the upper space of X. Recall the basic definitions. A *valuation* (Saheb-Djahromi 1980; Lawson 1982; Jones 1989) on a topological space Y is a map $\nu : \Omega Y \to [0, \infty)$, where ΩY is the lattice of open subsets, which satisfies: (i) $\nu(a) + \nu(b) = \nu(a \cup b) + \nu(a \cap b)$ (modularity), (ii) $\nu(\emptyset) = 0$, and (iii) $a \subseteq b \Rightarrow \nu(a) \leq \nu(b)$. A *continuous* valuation (Lawson 1982; Jones and Plotkin 1989; Jones 1989) is a valuation such that whenever $A \subseteq \Omega(Y)$ is a directed set (wrt $\subseteq$) of open sets of Y, then $\nu(\bigcup_{O \in A} O) = \sup_{O \in A} \nu(O)$.

For any $b \in Y$, the *point valuation* based at b is the valuation $\delta_b : \Omega(Y) \to [0, \infty)$ defined by

$$\delta_b(O) = \begin{cases} 1 & \text{if } b \in O \\ 0 & \text{otherwise.} \end{cases}$$

Any finite linear combination $\sum_{i=1}^{n} r_i \delta_{b_i}$ of point valuations δ_{b_i} with constant coefficients $r_i \in [0, \infty)$, $(1 \leq i \leq n)$ is a continuous valuation on Y, which we call a *simple valuation*.

The *probabilistic power domain*, $\mathbf{P}Y$, of a topological space Y consists of the set of continuous valuations ν on Y with $\nu(Y) \leq 1$ and is ordered as follows: $\mu \sqsubseteq \nu$ iff, for all open sets O of Y, $\mu(O) \leq \nu(O)$. The partial order $(PY, \sqsubseteq)$ is a dcpo with bottom in which the lub of a directed set $\langle \mu_i \rangle_{i \in I}$ is given by $\bigsqcup_i \mu_i = \nu$, where for $O \in \Omega(Y)$ we have $\nu(O) = \sup_{i \in I} \mu_i(O)$. For an (ω)-continuous dcpo D equipped with its Scott topology, $\mathbf{P}D$ is also (ω)-continuous and has a basis consisting of simple valuations (Jones and Plotkin 1989). Moreover, any continuous valuation μ on an ω-continuous dcpo extends uniquely to a Borel measure on the dcpo (Norberg 1989). For convenience, we denote this unique extension by μ as well. If D has a bottom element $\perp$, then the *normalised probabilistic power domain*, $\mathbf{P}^1 D$, is the subdomain $\mathbf{P}^1 D = \{\mu \in \mathbf{P}D \mid \mu(D) = 1\}$ which is also an ω-continuous dcpo with bottom $\delta_\perp$ and a basis of normalised simple valuations (Edalat 1995b). It is easy to see that $\mathbf{P}D$ and $\mathbf{P}^1 D$ have the same set of maximal elements, i.e. $\max(\mathbf{P}D) = \max(\mathbf{P}^1 D)$.

Let $\mathbf{M}^1 X$ be the space of probability measures (normalised measures) on X with the weak topology, i.e. the coarsest topology on the set of normalised measures such that the functional

$$\begin{array}{cccc} F_g: & \mathbf{M}^1 X & \to & \mathbb{R} \\ & \mu & \mapsto & \int g \, d\mu \end{array}$$

is continuous for all bounded continuous maps $g : X \to \mathbb{R}$.

In (Edalat 1995b), it was shown that for any compact metric space X the embedding of X onto the set of maximal elements of the upper space $\mathbf{U}X$ of X by the singleton map $s : X \to \mathbf{U}X$ induces an injective map $e : \mathbf{M}^1 X \to \mathbf{P}^1 \mathbf{U}X$ given by $e(\mu) = \mu \circ s^{-1}$.

The image of e consists of continuous valuations on $\mathbf{U}X$ whose unique extension to a measure is supported on $s(X)$, the maximal elements of $\mathbf{U}X$. That is to say,

$$\mathrm{im}(e) = \{\nu \in \mathbf{PU}X \mid \nu(s(X)) = 1\} = \{\nu \in \mathbf{P}^1\mathbf{U}X \mid \nu(\mathbf{U}X \setminus s(X)) = 0\}.$$

This provides a domain-theoretic framework for classical measure theory. In fact, $\mathbf{P}^1\mathbf{U}X$ is ω-continuous with a basis of simple valuations. It follows that for any $\mu \in \mathbf{M}^1 X$ there exists an increasing chain of simple valuations $\langle \nu_m \rangle_{m \geq 0}$ in $\mathbf{P}^1\mathbf{U}X$ $(m \geq 0)$ with

$$\mu = \bigsqcup_{m \geq 0} \nu_m. \tag{1}$$

These simple valuations provide finite approximations to the measure μ. It was also shown in (Edalat 1995a), in fact for the more general case of a second countable locally compact Hausdorff space X, that $\mathrm{im}(e) \subseteq \max(\mathbf{PU}X)$ and it was conjectured that $\mathrm{im}(e) = \max(\mathbf{PU}X)$. This conjecture was later proved, in a more general setting, by J. Lawson (Lawson 1995).

This framework also leads to a generalisation of the Riemann theory of integration on a compact metric space (Edalat 1995b). Simple valuations are used to obtain generalised lower and upper Darboux sums and Riemann sums for any bounded real-valued function on X. For any simple valuation $\nu = \sum_{b \in B} r_b \delta_b \in PUX$, the *lower sum* and the *upper* *sum* of f with respect to ν are, respectively,

$$S^{\ell}(f,\nu) = \sum_{b \in B} r_b \inf f[b], \qquad S^{u}(f,\nu) = \sum_{b \in B} r_b \sup\, f[b].$$

Furthermore, for a *choice function* $\xi : B \to X$ with $\xi_b \in b$ for each $b \in B$, the sum

$$S_{\xi}(f,\nu) = \sum_{b \in B} r_b f(\xi_b)$$

is a *generalised Riemann sum* for f with respect to ν. It is then shown that, with respect to the chain of simple valuations ν_m ($m \geq 0$) with lub μ, the lower sums corresponding to ν_m ($m \geq 0$) increase, the upper sums decrease and the Riemann sums tend to the integral of f with respect μ, provided that f is bounded and continuous almost everywhere with respect to μ:

$$S^{\ell}(f,\nu_m) \nearrow \int f\, d\mu, \qquad S^{u}(f,\nu_m) \searrow \int f\, d\mu$$

$$S_{\xi_m}(f,\nu_m) \to \int f\, d\mu. \tag{2}$$

The fact that $\mathbf{P}^1\mathbf{U}X$ is an ω-continuous dcpo only guarantees the *existence* of the increasing chain of simple valuations ν_m in Equation (1) which approximate a finite measure μ on X. A basic question is how to explicitly construct such a chain of approximations. In the next section, we will give a solution to this problem.

3. Approximation of Measures

Assume $\mu \in \mathbf{M}^1X$ is a given probability measure on a compact metric space X. Let $\mathcal{A} = \langle A_1, A_2, \ldots, A_N \rangle$ be any ordered open covering of the compact metric space X, i.e., $A_i \subseteq X$ is open $i = 1, \ldots, N$ and $X = \bigcup_{i=1}^N A_i$. We first show that $\mathcal{A}$ induces a simple valuation below μ. Denoting the closure of a set A by $\overline{A}$, let

$$\mu_{\mathcal{A}} = \sum_{i=1}^{N} r_i \delta_{\overline{A_i}}, \tag{3}$$

where $r_i = \mu(A_i \setminus \bigcup_{j<i} A_j)$. Since the sets $A_i \setminus \bigcup_{j<i} A_j$ $(1 \leq i \leq N)$ are disjoint and their union is X, we have $\sum_{i=1}^N r_i = 1$, and therefore $\mu_{\mathcal{A}} \in \mathbf{P}^1\mathbf{U}X$.

Proposition 3.1. For any open subset $O \subseteq X$, we have:

$$\mu_{\mathcal{A}}(\Box O) = \sum_{\overline{A_i} \subseteq O} r_i \leq \mu(O) \leq \sum_{\overline{A_i} \cap O \neq \emptyset} r_i.$$

Corollary 3.1. For any $\mu \in \mathbf{M}^1X$ and any ordered open covering $\mathcal{A}$ of X we have $\mu_{\mathcal{A}} \sqsubseteq \mu \circ s^{-1}$ in $\mathbf{P}^1\mathbf{U}X$.

Proof. For any open set $V \subseteq \mathbf{U}X$, we have $V \subseteq \Box\, s^{-1}V$. Therefore, by Proposition 3.1 with $O = s^{-1}V$ we have:

$$\mu_{\mathcal{A}}(V) \leq \mu_{\mathcal{A}}(\Box\, s^{-1}V) \leq \mu(s^{-1}V).$$

□

Definition 3.1. For two ordered open coverings $\mathcal{A} = \langle A_1, \ldots, A_N \rangle$ and $\mathcal{B} = \langle B_1, \ldots, B_M \rangle$, the *refinement $\mathcal{A} \wedge \mathcal{B}$ of $\mathcal{A}$ by $\mathcal{B}$* is the ordered open covering with subsets of the form $C_{(i,j)} = A_i \cap B_j$, $1 \leq i \leq N$ and $1 \leq j \leq M$, ordered lexicographically, i.e., $(i,j) < (i',j')$ iff either $i < i'$ or $i = i'$ and $j < j'$.

Put $r_i = \mu(A_i \setminus \bigcup_{j<i} A_j)$ as before, and let

$$r_{(i,t)} = \mu(C_{(i,t)} \setminus \bigcup_{(i',t')<(i,t)} C_{(i',t')}).$$

Proposition 3.2. $\mu_{\mathcal{A}} \sqsubseteq \mu_{\mathcal{A}\wedge\mathcal{B}}$.

Proof.
Note that

$$\begin{aligned}
& \bigcup_{1\leq t\leq M} C_{(i,t)} \setminus \bigcup_{(i',t')<(i,t)} C_{(i',t')} \\
= & \bigcup_{1\leq t\leq M} [C_{(i,t)} \setminus \bigcup_{(i,1)\leq(i,t')<(i,t)} C_{(i,t')})] \setminus \bigcup_{(i',t')<(i,1)} C_{(i',t')} \\
= & \bigcup_{1\leq t\leq M} [A_i \cap (B_t \setminus \bigcup_{t'<t} B_{t'})] \setminus \bigcup_{i'<i} A_{i'} \\
= & (A_i \cap X) \setminus \bigcup_{j<i} A_j = A_i \setminus \bigcup_{j<i} A_j.
\end{aligned}$$

Since, for $1 \leq t \leq M$, the sets $C_{(i,t)} \setminus \bigcup_{(i',t')<(i,t)} C_{(i',t')}$ are disjoint, it follows that: $r_i = \sum_{1\leq t\leq M} r_{(i,t)}$. Therefore, for any open subset O of $\mathbf{U}X$ we have:

$$\begin{aligned}
\mu_{\mathcal{A}\wedge\mathcal{B}}(O) &= \sum_{\overline{A_i\cap B_t}\in O} r_{(i,t)} \geq \sum_{1\leq t\leq M} \sum_{\overline{A_i}\in O} r_{(i,t)} \\
&= \sum_{\overline{A_i}\in O} \sum_{1\leq t\leq M} r_{(i,t)} = \sum_{\overline{A_i}\in O} r_i = \mu_{\mathcal{A}}(O).
\end{aligned}$$

□

Lemma 3.1. Let μ and ν be continuous valuations on a topological space Y. Suppose $B \subseteq \Omega Y$ is a basis which is closed under finite intersections. If $\mu(O) = \nu(O)$ for all $O \in B$, then $\mu = \nu$.

Proof. Given any valuation λ on Y, we can deduce from modularity and a simple induction that

$$\begin{aligned}
& \lambda(\bigcup_{i=1}^{n} O_i) \\
= & \sum_{1\leq i\leq n} \lambda(O_i) - \sum_{1\leq i_1<i_2\leq n} \lambda(O_{i_1} \cap O_{i_2}) + \sum_{1\leq i_1<i_2<i_3\leq n} \lambda(O_{i_1} \cap O_{i_2} \cap O_{i_3}) - \\
& \cdots + \cdots\cdots (-1)^{n+1} \lambda(\bigcap_{1\leq i\leq n} O_i),
\end{aligned}$$

for any finite collection of open subsets $\langle O_i \rangle_{1\leq i\leq n}$. Let $B' \subseteq \Omega Y$ be the set of all finite unions of open sets in B. It follows by the above relation that μ and ν coincide on all elements of B'. Since any open set in Y is the directed union of elements of B', the result follows. □

Assume $\mathcal{B}_n$ is an ordered covering of open subsets of X with diameters less than $1/n$ for $n \geq 1$. Define $\mathcal{A}_n$ for $n \geq 1$ inductively by $\mathcal{A}_1 = \mathcal{B}_1$ and $\mathcal{A}_{n+1} = \mathcal{A}_n \wedge \mathcal{B}_{n+1}$.

Theorem 3.1. $\mu = \bigsqcup_{m \geq 1} \mu_{\mathcal{A}_m}$.

Proof. For any open set $O \subseteq X$, the set $\Box O = \{C \in \mathbf{U}X \mid C \subseteq O\}$ is a basic open subset of $\mathbf{U}X$. Also we have $(\Box O_1) \cap \ldots \cap (\Box O_n) = \Box(O_1 \cap \ldots \cap O_n)$, and therefore these basic open sets are closed under finite intersections. By Lemma 3.1, it is sufficient therefore to show that $\sup_{m \geq 1} \mu_{\mathcal{A}_m}(\Box O) = \mu(O)$, for all open subsets $O \subseteq X$. Let $O \subseteq X$ be open. We know by Corollary 3.1 and Proposition 3.2 that $\langle \mu_{\mathcal{A}_m} \rangle_{m \geq 1}$ is an increasing chain with $\bigsqcup_{m \geq 1} \mu_{\mathcal{A}_m} \sqsubseteq \mu \circ s^{-1}$. It is therefore sufficient to check that $\sup_{m \geq 1} \mu_{\mathcal{A}_m}(\Box O) \geq \mu(O)$, for all open subsets $O \subseteq X$. Let $\epsilon > 0$ be given. As X is a normal space, O is the directed union of all open subsets whose closures are contained in O. Therefore, there exists an open set $U \subseteq X$ with $\overline{U} \subseteq O$ such that $\mu(O) - \mu(U) < \epsilon$. It follows that there exists $M \geq 1$ such that the distance between $\overline{U}$ and the complement of O is at least $1/M$. Assume $\mathcal{A}_M = \langle A_1, \ldots, A_N \rangle$ and $\mu_{\mathcal{A}_M} = \sum_{i=1}^{N} r_i \delta_{\overline{A_i}}$, where r_i is given by the formula below Equation (3). Since A_i has diameter less than $1/M$ for $1 \leq i \leq N$, it follows that $\overline{A_i} \cap U \neq \emptyset$ implies $\overline{A_i} \subseteq O$. Therefore,

$$\begin{aligned} \mu_{\mathcal{A}_M}(\Box O) &= \sum_{\overline{A_i} \subseteq O} r_i \\ &\geq \sum_{\overline{A_i} \cap U \neq \emptyset} r_i \quad \text{by the above remark} \\ &\geq \mu(U) \quad \text{by Proposition 3.1} \\ &> \mu(O) - \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the proof is complete. □

4. The Scott topology versus the weak topology

In this section, we assume that X is a separable metric space, D is an ω-continuous dcpo equipped with the Scott topology and $s : X \to D$ is a topological embedding such that $s(X) \subseteq D$ is a G_δ subset.

Proposition 4.1.
(i) $s(X) \subseteq \max(D)$.
(ii) s takes open sets and closed sets to G_δ subsets.
(iii) s takes Borel sets to Borel sets.

Proof. (i) Let $x, y \in X$ with $s(x) \sqsubseteq s(y)$ and let $O \subseteq X$ be an open neighbourhood of x. Then $s(O) = O' \cap s(X)$ for some open $O' \subseteq D$. Since O' is an upper set we get $s(y) \in s(O)$. This holds for any open neighbourhood O of x in the Hausdorff space X. Hence, $x = y$. On the other hand, $s(X)$, being a G_δ subset, is an upper set. Therefore, $s(X) \subseteq \max(D)$.
(ii) If $O \subseteq X$ is open then $s(O) = O' \cap s(X)$ where $O' \subseteq D$ is open and $s(X)$ is a G_δ set. Hence, $s(O)$ is a G_δ set. Let $C \subseteq X$ be closed. Since any closed subset of a metric space is a G_δ subset, we have $C = \cap_{i \geq 0} O_i$ where $O_i \subseteq X$ are open. Then, $s(C) = \cap_{i \geq 0} s(O_i)$ is a G_δ set since each $s(O_i)$ is a G_δ set.

(iii) This follows from (ii); see (Edalat 1995b, Corollary 5.11). □

Recall from Section 1 that any continuous valuation $\nu \in \mathbf{P}D$ extends uniquely to a measure on D. Let $e : \mathbf{M}^1 X \to \mathbf{P}D$ be defined by $e(\mu) = \mu \circ s^{-1}$. The next proposition is the generalisation of the corresponding result in (Edalat 1995b) with respect to the embedding of a second countable locally compact metric space into the set of maximal elements of it upper space. Recall that any probability measure μ on a metric space is regular (Billingsley 1979), i.e. for any Borel subset B,

$$\mu(B) = \inf\{\mu(O) | O \text{ open}, B \subseteq O\} = \sup\{\mu(C) | C \text{ closed}, B \supseteq C\}.$$

Proposition 4.2.

(i) e is one to one.
(ii) $\mathrm{im}(e) = \{\mu \in \mathbf{P}D \mid \mu(s(X)) = 1\}$ and $e : \mathbf{M}^1 X \to \mathrm{im}(e)$ has inverse $j : \mathrm{im}(e) \to \mathbf{M}^1 X$ given by $j(\nu) = \nu \circ s$.
(iii) $\mathrm{im}(e) \subseteq \max(D)$.

Proof.
(i) For any open subset $O \subseteq X$, there exists, by Proposition 4.1(ii), open subsets $O_i \subseteq D$ $(i \geq 0)$ with $s(O) = \cap_{i \geq 0} O_i$. Hence, $O = \cap_{i \geq 0} s^{-1}(O_i)$ and we have

$$\mu(O) = \inf_{i \geq 0} \mu(s^{-1}(O_i)) = \inf_{i \geq 0} (e(\mu))(O_i).$$

It follows that if $e(\mu) = e(\nu)$ then $\mu(O) = \nu(O)$ for all open sets $O \subseteq X$, and therefore $\mu = \nu$ by regularity.
(ii) We have $s(X) = \cap_{i \geq 0} O_i$, for some open sets $O_i \subseteq D$. Then, for any $\mu \in \mathbf{M}^1 X$,

$$(e(\mu))(s(X)) = \inf_{i \geq 0} (e(\mu))(O_i) = \inf_{i \geq 0} \mu(s^{-1}(O_i)) = \inf_{i \geq 0} \mu(X) = 1.$$

On the other hand if $\nu \in \mathbf{P}D$ with $\nu(s(X)) = 1$, then $j(\nu) = \nu \circ s$ is a probability measure on X and for any open $O \subseteq D$ we have $(e(j(\nu)))(O) = \nu(s(s^{-1}(O))) = \nu(O \cap s(X)) = \nu(O)$ since ν is supported on $s(X)$. Hence, $e(j(\nu)) = \nu$ and $\nu \in \mathrm{im}(e)$. Finally, for any $\mu \in \mathbf{M}^1 X$, we have $j(e(\mu)) = (\mu \circ s^{-1}) \circ s = \mu$.
(iii) Let $\mu, \nu \in \mathbf{P}D$ with $\mu \in \mathrm{im}(e)$ and $\mu \sqsubseteq \nu$. We have $\mu(B) \leq \nu(B)$ for all G_δ subsets $B \subseteq D$. In particular, it follows that $\nu(s(X)) = 1$. Suppose $\nu \neq \mu$. Then $\mu(O) < \nu(O)$, i.e. $\mu(O \cap s(X)) < \nu(O \cap s(X))$, for some open $O \subseteq D$. The set $X \setminus s^{-1} O$ is a closed and therefore a G_δ subset of the metric space X. Hence, $s(X \setminus s^{-1} O) = s(X) \setminus (O \cap s(X))$ is, by Proposition 4.1(ii) a G_δ subset of D. Therefore,

$$\begin{aligned} \nu(s(X)) &= \nu(O \cap s(X)) + \nu(s(X) \setminus O \cap s(X)) \\ &> \mu(s(X) \cap O) + \mu(s(X) \setminus O \cap s(X)) = 1, \end{aligned}$$

which is a contradiction. □

We will show below that e is a topological embedding. This means that the topology on $\mathrm{im}(e)$ induced from the weak topology by $e : \mathbf{M}^1 X \to \mathbf{P}^1 D$ and the relative Scott topology on $\mathrm{im}(e) \subseteq \mathbf{P}D$ coincide. We first state a suitable characterisation of convergence of a sequence of measures on X in the weak topology.

Proposition 4.3. (Stroock 1993) For a separable metric space X and a sequence $\langle\mu_m\rangle_{m\geq 0}$ in $\mathbf{M}^1X$, we have: $\lim_{m\to\infty}\mu_m=\mu$ in the weak topology iff

$$\liminf_{m\to\infty}\mu_m(O)\geq\mu(O)$$

for all open subsets $O\subseteq X$.

The theorem below gives a similar criterion for the convergence of a sequence of valuations in $\mathbf{P}D$.

Lemma 4.1. (Kirch 1993, page 46) Let $\nu=\sum_{a\in A}r_a\delta_a$ be a simple valuation and μ a continuous valuation on a continuous dcpo D. Then $\nu\ll\mu$ iff for all $B\subseteq A$ we have $\sum_{a\in B}r_a<\mu(\Uparrow B)$, where $\Uparrow B=\{x\in D\mid\exists b\in B.\ b\ll x\}$.

Theorem 4.1. Suppose D is an ω-continuous dcpo and $\langle\mu_m\rangle_{m\geq 0}$ is a sequence in $\mathbf{P}D$. Then, $\mu_m\to\mu$ as $m\to\infty$ in the Scott topology iff

$$\liminf_{m\to\infty}\mu_m(O)\geq\mu(O)$$

for all Scott open subsets $O\subseteq D$.

Proof. Suppose $\mu_m\to\mu$ as $m\to\infty$ and let $O\subseteq D$ be Scott open. Assume $\epsilon>0$ is given. Since $\mathbf{P}D$ is continuous with a base of simple valuations, μ is the directed lub of simple valuations way below it. As $\mu(O)=\sup_{\nu\ll\mu}\nu(O)$, there exists $\nu\ll\mu$ such that $\nu(O)>\mu(O)-\epsilon$. Since $\mu_m\to\mu$ and $\Uparrow\nu$ is Scott open, there exists $N\geq 0$ such that for all $m\geq N$ we have $\nu\ll\mu_m$. Therefore, $\mu_m(O)\geq\nu(O)>\mu(O)-\epsilon$ for all $m\geq N$. This shows that $\liminf_{m\to\infty}\mu_m(O)\geq\mu(O)$.
Conversely, suppose $\langle\mu_m\rangle_{m\geq 0}$ does not converge to μ in the Scott topology. Since the simple valuations below μ induce a neighbourhood basis for μ, there exists a simple valuation $\nu=\sum_{a\in A}r_a\delta_a$ with $\nu\ll\mu$ and a subsequence $\langle\mu_{m_i}\rangle_{i\geq 0}$ such that $\nu\not\ll\mu_{m_i}$ for all $i\geq 0$. By Lemma 4.1, for each $i\geq 0$ there exists $B\subseteq A$ with $\mu_{m_i}(\Uparrow B)\leq\nu(\Uparrow B)<\mu(\Uparrow B)$. Since the set of subsets of A is finite, there exist $B\subseteq A$ and a subsequence $\langle\mu_{m_{i_j}}\rangle_{j\geq 0}$ such that $\mu_{m_{i_j}}(\Uparrow B)\leq\nu(\Uparrow B)<\mu(\Uparrow B)$ for all $j\geq 0$. Therefore, $\liminf_{m\to\infty}\mu_m(\Uparrow B)\leq\nu(\Uparrow B)<\mu(\Uparrow B)$ which contradicts our assumption. $\square$

The above theorem can be easily generalised to converging nets in $\mathbf{P}D$ for any continuous dcpo D.

Corollary 4.1. The mapping $e:\mathbf{M}^1X\to\mathbf{P}D$ is a topological embedding.

Proof. To show that e is continuous, let $\langle\mu_m\rangle_{m\geq 0}$ be a sequence in $\mathbf{M}^1X$ with $\lim_{m\to\infty}\mu_m=\mu$. We need to check that $e(\mu_m)\to e(\mu)$ in $\mathbf{P}D$ as $m\to\infty$. For any open subset $O\subseteq D$ we have:

$$\begin{aligned}
&\liminf_m(e(\mu_m))(O)\\
&=\liminf_m\mu_m(s^{-1}(O))\\
&\geq\mu(s^{-1}(O)) &&\text{by Proposition 4.3}\\
&=(e(\mu))(O).
\end{aligned}$$

It follows by Theorem 4.1 that $e(\mu_m)\to e(\mu)$ as $m\to\infty$.
To show that $j:\mathrm{im}(e)\to\mathbf{M}^1X$ is continuous, let $\langle\nu_m\rangle_{m\geq 0}$ be a sequence in $\mathrm{im}(e)$ with $\lim_{m\to\infty}\nu_m=\nu$. Since $s:X\to D$ is an embedding, for any open subset $O\subseteq X$ we have

$s(O) = O' \cap s(X)$ for some open subset $O' \subseteq D$. Therefore,

$$\begin{array}{ll} \liminf_m (j(\nu_m))(O) & \\ = \liminf_m \nu_m(sO) & \\ = \liminf_m \nu_m(O' \cap sX) & \\ = \liminf_m \nu_m(O') & \nu_m \text{ is supported in } sX \\ \geq \nu(O') & \text{by Theorem 4.1} \\ = \nu(O' \cap sX) & \nu \text{ is supported in } sX \\ = (j(\nu))(O). & \end{array}$$

It follows by Proposition 4.3 that $j(\nu_m) \to j(\nu)$ as $m \to \infty$. □

We conclude that in the domain-theoretic framework for measure theory, the Scott topology is indeed an extension of the classical weak topology on measures, the most important topology used in measure theory.

We conclude this section with an application in iterated function system (IFS) theory. An IFS with probabilities on a compact metric space X is given by a finite collection of continuous maps $f_i : X \to X$ each with a probability weight $p_i > 0$, $i \in \Sigma_N = \{1, 2, \cdots, N\}$ with $\sum_{i=1}^{N} p_i = 1$. The *Markov operator* $T : \mathbf{M}^1 X \to \mathbf{M}^1 X$ is given by

$$T(\mu)(B) = \sum_{i=1}^{N} p_i \mu(f_i^{-1}(B))$$

for any Borel subset $B \subseteq X$. An IFS with probabilities is *hyperbolic* if the maps f_i are contracting for $i = 1, \ldots, N$. For such an IFS, Hutchinson (Hutcjinson 1981) showed, by defining a metric on $\mathbf{M}^1 X$ with respect to which the Markov operator is a contracting map, that this operator has a unique fixed point, the so-called the *invariant measure* of the IFS.

A generalisation of the above notion of an IFS was introduced in (Edalat 1996). An IFS with probabilities is *weakly hyperbolic* if for each infinite sequence $i_1 i_2 \ldots \in \Sigma_N^\omega$, the intersection $\bigcap_{n \geq 1} f_{i_1} f_{i_2} \ldots f_{i_n} X$ is a singleton set. Note that this definition also makes sense for an IFS with probabilities on a compact metrizable space. It was shown in (Edalat 1996) that for a weakly hyperbolic IFS with probabilities the Markov operator has a unique fixed point μ^*. In fact $\mu^* \circ s^{-1}$, where $s : X \to \mathbf{U}X$ is the singleton map, is the unique fixed point of the map

$$\begin{array}{rccc} H : & \mathbf{P}^1\mathbf{U}X & \to & \mathbf{P}^1\mathbf{U}X \\ & \mu & \mapsto & H(\mu) \end{array}$$

defined by $H(\mu)(O) = \sum_{i=1}^{N} p_i \mu(f_i^{-1}(O))$. We can now obtain the full generalization of Hutchinson's result.

Theorem 4.2. For any weakly hyperbolic IFS with probabilities, the sequence $\langle T^m \mu \rangle_{m \geq 0}$ of the iterates of the Markov operator T converges in the weak topology, for any initial $\mu \in \mathbf{M}^1 X$, to the unique invariant measure μ^* of the IFS.

Proof. Since δ_X is the least element of $\mathbf{P}^1\mathbf{U}X$, we have $\delta_X \sqsubseteq \mu \circ s^{-1}$. By monotonicity of H we obtain $H^m \delta_X \sqsubseteq H^m(\mu \circ s^{-1}) = (T^m \mu) \circ s^{-1}$. Since the sequence $\langle H^m \delta_X \rangle_{m \geq 0}$

converges in the Scott topology to $\mu^* \circ s^{-1} = \bigsqcup_{m\geq 0} H^m \delta_X$, it follows that the same is true for the sequence $(T^m\mu) \circ s^{-1}$. By Corollary 4.1, $\langle T^m\mu\rangle_{m\geq 0}$ converges in the weak topology to μ^*. □

5. Computation of Integrals

In Section 3, we showed how a normalised measure μ on a compact metric space X can be obtained, via the embedding of $\mathbf{M}^1X$ onto the set of maximal elements of $\mathbf{P}^1\mathbf{U}X$, as the lub of an increasing chain of simple valuations on $\mathbf{U}X$. In this section we will examine the converse problem. Suppose $\mathcal{A} \subseteq \mathbf{P}^1\mathbf{U}X$ is a directed set of simple valuations, and we would like to know when the lub $\bigsqcup\mathcal{A}$ of $\mathcal{A}$ determines a Borel measure on X. We obtain a necessary and sufficient condition such that $\bigsqcup\mathcal{A}$ is a maximal element of $\mathbf{P}^1\mathbf{U}X$ and therefore corresponds to an element $\mu \in \mathbf{M}^1X$. We then show that when this condition is effectively satisfied, we can compute the expected value of any Hölder continuous function with respect to μ up to any desired accuracy. We denote the diameter of any compact subset $c \subseteq X$ by $|c|$.

Proposition 5.1. We have $\bigsqcup\mathcal{A} \in \mathbf{M}^1X$ iff for all $\alpha > 0$ and all $\beta > 0$, there exists $\sum_{c\in C} r_c\delta_c \in \mathcal{A}$ with $\sum_{|c|\geq\beta} r_c < \alpha$.

Proof. The 'only if' part is proved in (Edalat 1995b, Proposition 4.14). For the 'if' part, assume $\mu = \bigsqcup\mathcal{A} \notin \mathbf{M}^1X$. Then $\mu(s(X)) < 1$. For each $n \geq 1$, let $\{b_i \mid i \in I_n\}$ be the collection of all open balls of X with radius less than $\frac{1}{n}$. Put $O_n = \bigcup_{i\in I_n} \Box b_i$. We have $s(X) = \bigcap_{n\geq 1} O_n$. Therefore, $\mu(s(X)) = \inf_n \mu(O_n) < 1$ and there exists $n \geq 1$ such that $1 - \mu(O_n) = \alpha > 0$. By assumption there exists $\nu = \sum_{c\in C} r_c\delta_c \in \mathcal{A}$ with $\sum_{|c|\geq 1/n} r_c < \alpha$. It follows that $\mu(O_n) \geq \nu(O_n) > 1 - \alpha$ which is a contradiction. □

Corollary 5.1. If for all $\beta > 0$ there exists $\sum_{c\in C} r_c\delta_c \in \mathcal{A}$ with $|c| \leq \beta$ for all $c \in C$, then $\bigsqcup\mathcal{A} \in \mathbf{M}^1X$.

We say an increasing chain $\langle\mu_i\rangle_{i\geq 0}$ of simple valuations in $\mathbf{P}^1\mathbf{U}X$ with lub $\mu \in \mathbf{M}^1X$ is an *effective approximation* of μ if for all positive integers m and n there exists $i \geq 0$, recursively given in terms of m and n, such that $\mu_i = \sum_{c\in C} r_c\delta_c$ satisfies $\sum_{|c|\geq 1/m} r_c < 1/n$.

Example 5.1. For any $\mu \in \mathbf{M}^1X$, which is given by its values on a countable basis of X closed under finite unions and intersections, the chain of simple valuations $\mu_{\mathcal{A}_i}$, $i \geq 1$, in Section 3 is an effective approximation of μ since $\mu_{\mathcal{A}_i}$ is made up of compact subsets with diameters less than $1/i$.

Suppose $\mu \in \mathbf{M}^1X$ has an effective approximation by a chain of simple valuations $\langle\nu_i\rangle$, $i \geq 0$. Assume that we have a Hölder continuous function $f : X \to \mathbb{R}$, i.e. there are constants $k \geq 0$ and $h > 0$ such that $|f(x) - f(y)| \leq k(d(x,y))^h$ for all $x, y \in X$, and let $|X|$ be the diameter of X. We can then compute the expected value of f with respect to μ up to any given accuracy as follows. Let $\epsilon > 0$ be given. Choose the positive integers m and n with $1/m < (\epsilon/2k)^{1/h}$ and $1/n < \epsilon/(2k|X|^h)$, and let the integer i be such that $\nu_i = \sum_{c\in C} r_c\delta_c$ satisfies $\sum_{|c|\geq 1/m} r_c < 1/n$. We have

$$S^\ell(f,\mu_i) \leq \int f\,d\mu \leq S^u(f,\mu_i), \qquad S^\ell(f,\mu_i) \leq S_\xi(f,\mu_i) \leq S^u(f,\mu_i)$$

where $S_\xi(f,\mu_i)$ is any generalised Riemann sum for μ_i. For any $c \in C$ we have $\sup f[c] - \inf f[c] \leq k|X|^h$; whereas for $c \in C$ with $|c| < 1/m$ we have $\sup f[c] - \inf f[c] < \epsilon/2$. Hence,

$$\begin{array}{rcl} |\int f\,d\mu - S_\xi(f,\mu_{\mathcal{A}})| & \leq & S^u(f,\mu_i) - S^\ell(f,\mu_i) = \sum_{c\in C}^{N} r_c(\sup f[c] - \inf f[c]) \\ & = & \sum_{|c|\geq 1/m} r_c(\sup f[c] - \inf f[c]) + \sum_{|c|<1/m} r_c(\sup f[c] - \inf f[c]) \\ & \leq & \epsilon/2 + \epsilon/2 = \epsilon. \end{array}$$

Therefore any Riemann sum for μ_i gives the value of the integral up to ϵ accuracy. We have then shown:

Theorem 5.1. The expected value of any Hölder continuous function on a compact metric space can be obtained up to any given accuracy with respect to any normalised measure with an effective approximation by an increasing chain of normalised valuations on the upper space of the metric space.

6. Computation at the Edge of Chaos

Feigenbaum's discovery of the periodic doubling route to chaos is one of the major scientific achievements of the recent decades. We first give a brief account of the subject (Devaney 1989; de Mello and van Strien 1993).

The prototype of a dynamical system following this universal route to chaos is provided by the *Logistic family*,

$$\begin{array}{rccc} f_c: & [0,1] & \to & [0,1] \\ & x & \mapsto & cx(1-x) \end{array}$$

where c is a real number which increases from 1 to 4.

For $1 = c_0 < c < 3$, the orbit $\langle f_c^n(x)\rangle_{n\geq 0}$ of any $x \in (0,1)$ converges to the unique attracting fixed point $\frac{c-1}{c}$ of f_c. At $c = c_1 = 3$, a periodic doubling bifurcation takes place: The attracting fixed point loses its stability and becomes repelling; at the same time an attracting periodic orbit of period two is born nearby. For $c_1 < c < c_2$, where $c_2 \approx 3.499$, the ω-limit set † of the orbit of any point $x \in (0,1) \setminus \{\frac{c-1}{c}\}$ is the period-two orbit. At $c = c_2$, the family undergoes another period doubling bifurcation. The period-two orbit becomes repelling and at the same time an attracting period-four orbit is created nearby.

This periodic doubling scenario is repeated ad infinitum at $c_1 < c_2 < c_3 < \ldots < c_n < \ldots$, such that at c_n $(n \geq 1)$ the attracting orbit of period 2^{n-1} becomes repelling, but in its neighbourhood an attracting orbit of period 2^n is created. For each $n \geq 1$, the dynamics of f_c for $c_{n-1} < c < c_n$ is quite simple: the ω-limit set of the orbit of almost all points in $[0,1]$ is the periodic orbit of period 2^{n-1}. In the limit, we have $c_\infty = \lim_{n\to\infty} c_n \approx 3.569$. For $c > c_\infty$, the system can exhibit chaotic behaviour. This

† The ω-limit set of a sequence is the set of limits of all its convergent subsequences.

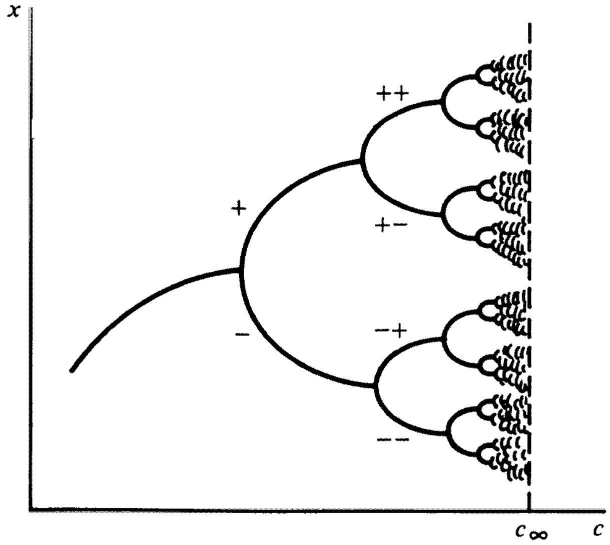

Fig. 1. Periodic doubling bifurcation of the Logistic family

means that the ω-limit set of the orbit of a typical point is a *strange attractor*: the orbit wanders around an attracting infinite set and the orbits of two close points will eventually diverge from each other. Figure 1 depicts the attractor of the system as c increases from 1 to c_∞.

At $c = c_\infty$ the system is at the edge of chaos, which we will now study.

The dynamics of f_{c_∞} is determined by the orbit $x_n = f^n_{c_\infty} t$ $(n \geq 0)$ of the turning point $t = .5$ where the derivative of f_c vanishes. See Figure 2.

In fact f_{c_∞} is an example of a *Feigenbaum map*. Feigenbaum maps are the prototype of *infinitely renormalizable* maps, which are of basic importance in discrete dynamical systems. We will not define these terms here and refer to (de Mello and van Strien 1993) for the precise definitions. Here, we need to use some basic properties of a Feigenbaum map which we will now state.

A Feigenbaum map $f : [0,1] \to [0,1]$ with $f(0) = f(1) = 0$ is piecewise monotone, continuous and has precisely one turning point t which is therefore a maximum. Furthermore, for each $n \geq 0$, the 2^{n+1} points $\langle x_n \rangle_{n=1}^{2^{n+1}}$ of the orbit of the turning point t induce 2^n disjoint closed intervals I^n_j, with end points $x_j = f^j(t)$ and $x_{j+2^n} = f^{j+2^n}(t)$ $(1 \leq j \leq 2^n)$ such that

(i) $fI^n_j = I^n_{j+1}$ $(j \geq 1, \bmod 2^n)$. The orbit of any $x \in (0,1)$ is eventually trapped in $I^n = \bigcup_{1 \leq j \leq 2^n} I^n_j$ for each $n \geq 0$.

(ii) The intervals I^{n+1}_j $(1 \leq j \leq 2^{n+1})$ are nested in the intervals I^n_j $(1 \leq j \leq 2^n)$ for each $n \geq 0$, as shown for f_{c_∞} in Figure 3.

This is similar to the way the Cantor set is constructed. In fact, assume from now on that f is C^2 (i.e. has a continuous second derivative) and that t is a non-flat point (i.e. there exists a C^2 diffeomorphism $\phi : \mathbb{R} \to [0,1]$ with $\phi(0) = t$ such that $f \circ \phi$ is a

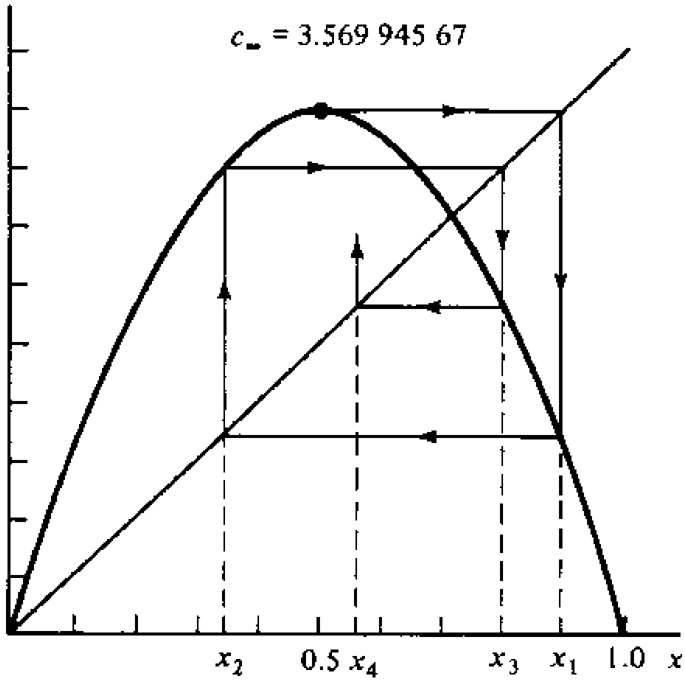

Fig. 2. The graph of f_{c_∞} and the orbit of 0.5

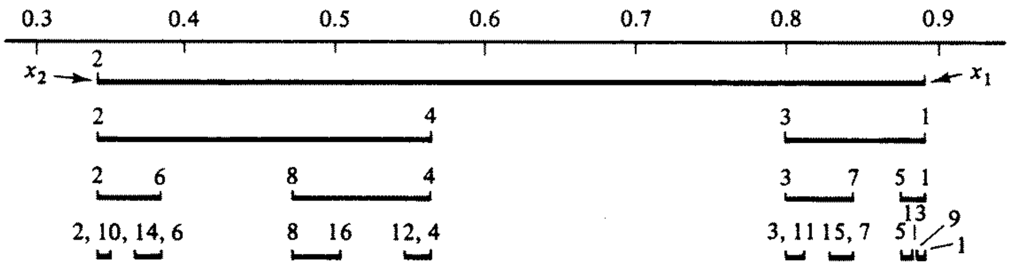

Fig. 3. The construction of the attractor of the Feigenbaum map

polynomial near the origin (de Mello and van Strien 1993, page 156)) then the length of the longest interval among I_j^n $(j = 1, \ldots, 2^n)$ tends to zero as $n \to \infty$. In this case, the intersection $A = \bigcap_{n \geq 0} I^n$ is indeed a Cantor set which is the strange attractor of the system. (See (de Mello and van Strien 1993, pages 113 and 350).)

Furthermore, it is also known that there exists a unique probability measure $\mu^* \in \mathbf{M}^1[0,1]$ which is invariant with respect to f, i.e. it is a fixed point of the map

$$\begin{array}{rccc} \mathbf{M}^1 f: & \mathbf{M}^1[0,1] & \to & \mathbf{M}^1[0,1] \\ & \mu & \mapsto & \mu \circ f^{-1}. \end{array}$$

The support of μ^* is the strange attractor A and μ^* is the unique *Bowen-Ruelle-Sinai* measure for f, i.e. it satisfies,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\phi(f^i x)=\int\phi\,d\mu^*,$$

for any continuous function $\phi:[0,1]\to\mathbb{R}$ and almost all $x\in[0,1]$. Indeed μ^* is an *ergodic* measure: The time average of any continuous function ϕ (LHS of the above equation) is the same as its space average with respect to μ^* (RHS of the equation). Moreover, if for $x_n\in I^n$ we put

$$\mu_n=\frac{1}{2^n}\sum_{j=1}^{2^n}\delta_{x(j,n)},$$

where $x(j,n)=f^j x_n$, then $\lim_{n\to\infty}\mu_n=\mu^*$ in the weak topology. Therefore, for any continuous function $\phi:[0,1]\to\mathbb{R}$ we have:

$$\lim_{n\to\infty}\int\phi\,d\mu_n=\lim_{n\to\infty}\frac{1}{2^n}\sum_{j=1}^{2^n}\phi(f^j x_n)=\int\phi\,d\mu^*.$$

We obtain an algorithm to compute $\int\phi\,d\mu^*$ for a Hölder continuous function ϕ up to a given threshold of accuracy $\epsilon>0$. This can be achieved by moving from the space $\mathbf{M}^1[0,1]$ equipped with the weak topology into the larger space $\mathbf{P}^1\mathbf{I}[0,1]$ equipped with the Scott topology. Here, $\mathbf{I}[0,1]$ is the ω-continuous dcpo of closed subintervals of the unit interval ordered by reverse inclusion. Alternatively, we can work with $\mathbf{P}^1\mathbf{U}[0,1]$.

Put

$$\nu_n=\frac{1}{2^n}\sum_{j=1}^{2^n}\delta_{I_j^n}.$$

We have the following domain-theoretic construction of the invariant measure of f.

Theorem 6.1. The unique invariant measure of a C^2 Feigenbaum map with a non-flat turning point is $\mu^*=\bigsqcup_{n\geq 0}\nu_n$.

Proof. As the intervals I_j^{n+1} are nested in I_j^n, it follows, by the Splitting Lemma (Edalat 1995b), that $\nu_n\sqsubseteq\nu_{n+1}$ for each $n\geq 0$. Since the length of the longest interval among I_j^n $(j=1,\ldots,2^n)$ tends to zero as $n\to\infty$, it follows by Corollary 5.1 that $\bigsqcup_{n\geq 0}\nu_n\in\mathrm{im}(e)$ and it therefore gives a measure on $[0,1]$. Furthermore, each ν_n is a fixed point of

$$\begin{array}{rccc}\mathbf{P}^1\mathbf{I}f: & \mathbf{P}^1\mathbf{I}[0,1] & \to & \mathbf{P}^1\mathbf{I}[0,1]\\ & \mu & \mapsto & \mu\circ f^{-1},\end{array}$$

because

$$\nu_n\circ f^{-1}=\frac{1}{2^n}\sum_{j=1}^{2^n}\delta_{fI_j^n}=\nu_n$$

as f permutes the 2^n intervals I_j^n $(j\geq 1,\ \mathrm{mod}\ 2^n)$. Since $\mathbf{P}^1\mathbf{I}f$ is Scott continuous, the lub $\bigsqcup_{n\geq 0}\nu_n$ is a fixed point of $\mathbf{P}^1\mathbf{I}f$ and hence a fixed point of $\mathbf{M}^1 f$. By the uniqueness of the invariant measure μ^*, it follows that $\mu^*=\bigsqcup_{n\geq 0}\nu_n$. □

Note that $\nu_n\sqsubseteq\mu_n$ in $\mathbf{P}^1\mathbf{I}[0,1]$ for each $n\geq 0$: Without loss of generality assume

$x_n \in I_1^n$, then $f^j x_n \in I_{j+1}^n$ for $j \geq 1$, mod 2^n. By the Splitting Lemma, we obtain $\nu_n \sqsubseteq \mu_n$ for each $n \geq 0$.

Assume that $\phi : [0,1] \to \mathbb{R}$ is a Hölder continuous function satisfying $|\phi(x) - \phi(y)| \leq c(|x-y|)^h$ for all $x, y \in [0,1]$ for some $c \geq 0$ and $h > 0$. Let $\epsilon > 0$. To compute $\int \phi \, d\mu^*$ up to ϵ accuracy, we obtain the least $n \geq 0$, say n_ϵ, such that the length of the longest interval among I_j^n $(1 \leq j \leq 2^n)$ is less than $(\epsilon/c)^{1/h}$, i.e. $|f^j(t) - f^{j+2^n}(t)| \leq (\epsilon/c)^{1/h}$ for all $j = 1, 2, \ldots, 2^n$. A Riemann sum for ν_{n_ϵ} is given by

$$S_\epsilon = \frac{1}{2^{n_\epsilon}} \sum_{j=1}^{2^{n_\epsilon}} \phi(f^j(t)).$$

It follows as in Section 5 that $|S_\epsilon - \int \phi \, d\mu^*| \leq \epsilon$. Therefore S_ϵ is the required approximation.

The above domain-theoretic technique can also be applied to *Fibonacci maps* which are the prototype of non-renormalizable maps (de Mello and van Strien 1993) and one can compute the expected value of any Hölder continuous map with respect to the unique invariant measure of a Fibonacci map.

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