Moments of the Fluid Level in a Fluid Queue

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Abstract

First Moment

Assume the fluid level in a queue changes at a fixed rate in the interval $a \leq t < b$ as shown in Figure 1, i.e. that the fluid arrival rate and drain rate are both constant in the interval. Define the interval width $\Delta = b - a$.

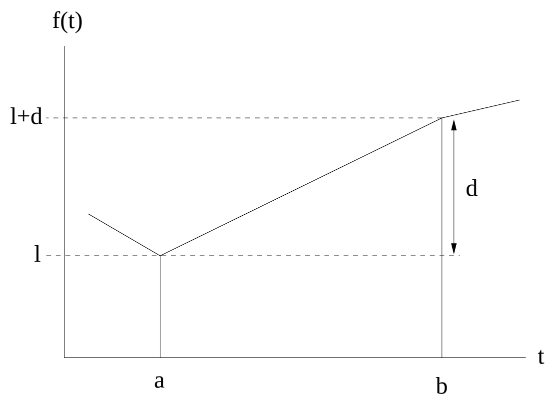

Figure 1: Constant rate of change of fluid in (a,b)

The first moment (i.e. the mean) of the fluid level in this interval is, by inspection, $l + d/2$ where l is the fluid level at time a and d is the change in level over the interval (i.e. the level is $l + d$ at time b). Formally, this comes from:

$$\begin{aligned} \frac{1}{\Delta}\int_a^b f(t)\,dt &= \frac{1}{\Delta}\int_a^b l + (t-a)\frac{d}{\Delta}\,dt \\ &= \frac{1}{\Delta}\left| lt + \frac{d}{\Delta}(t^2/2 - at)\right|_a^b \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Delta}\left(l\Delta + \frac{d}{2\Delta}(b-a)^2\right) \\
&= l + d/2
\end{aligned}
$$

Now consider a sample path of the fluid level over the observation interval $(0, T)$. Assume the arrival/drain rate changes at times $t_1, t_2, ..., t_n$, $0 < t_i < T$ and define $t_0 = 0, t_{n+1} = T$. The sample path comprises a set of piecewise linear components like the above over $n+1$ sub-intervals with widths $\Delta_i = t_i - t_{i-1}, 1 \leq i \leq n+1$, collectively spanning the observation period (0,T). To compute the mean over $(0, T)$, it is necessary to extend the above integral over the whole domain, viz.

$$
\frac{1}{T}\sum_{i=1}^{n+1}\int_{t_{i-1}}^{t_i} f(t)dt = \frac{1}{T}\sum_{i=1}^{n+1}\Delta_i(l_i + d_i/2)
$$

Here, l_i is the fluid level at the start of interval i and d_i is the change in level over interval i. Note that $T = \sum_{i=1}^{n}\Delta_i$.

Higher Moments

The n^{th} moment of the fluid level is a simple generalisation of the above. For the single interval case, it is given by:

$$
\frac{1}{\Delta}\int_a^b f(t)^n \, dt
$$

For example, the second moment is given by:

$$
\begin{aligned}
\frac{1}{\Delta}\int_a^b f(t)^2 \, dt &= \frac{1}{\Delta}\int_a^b \left(l + (t-a)\frac{d}{\Delta}\right)^2 dt \\
&= l^2 + d\,l + d^2/3
\end{aligned}
$$

The third moment is similarly found to be:

$$
\frac{1}{\Delta}\int_a^b f(t)^3 \, dt = l^3 + 3d\,l^2/2 + l\,d^2 + d^3/4
$$

By inspection, it is apparent that the k^{th} moment can be expressed in closed form thus:

$$
\frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i} l^i \, d^{k-i}
$$

Thus, over the whole domain $(0, T)$, the k^{th} moment can be written:

$$
\frac{1}{T}\sum_{i=1}^{n+1}\Delta_i\left(\frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j} l_i^j \, d_i^{k-j}\right)
$$

Implementation

To implement this, one needs to maintain one accumulator for each moment. This accumulator is updated each time the rate of change of fluid level changes. For example, for three moments, we see code like this:

```
acc1[ node ] += dt * ( level[ node ] + dLevel / 2.0 ) ;
acc2[ node ] += dt * ( sq( level[ node ] ) + level[ node ] * dLevel +
                         sq( dLevel ) / 3.0 ) ;
acc3[ node ] += dt * ( level[ node ] * sq( level[ node ] ) +
                         3.0/2.0 * sq( level[ node ] ) * dLevel +
                         level[ node ] * sq( dLevel ) +
                         dLevel * sq( dLevel ) / 4.0 ) ;
```

for a given node in a network of fluid queues. Here, `dt` is the width of the interval (just ended), `level` is the level at the start of that interval and `dLevel` is the change in level during the interval.

Note that if T is large enough, the last interval (t_n, T) can be omitted with negligible loss of accuracy; this may simplfy the code.