On the Computational Content of Classical Logic

Alex Summers
Transfer Presentation
Imperial College
Spring Term 2006

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- The $\mathcal{X}$ calculus

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- The $\mathcal{X}$ calculus
- Aims

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- Type Assignment for $\mathcal{X}$

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λ-calculus

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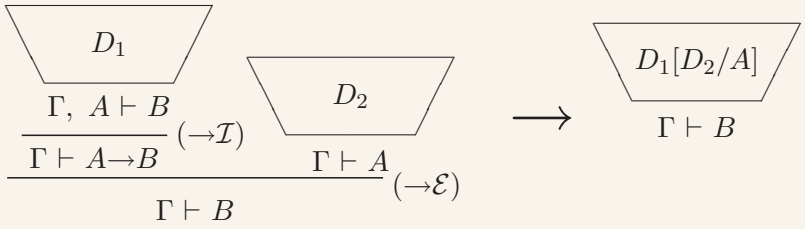

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  - $((A \to B) \to A) \to A$ (Peirce's Law).
  - Reading B as $\bot$, "If assuming $\neg A$ lets us deduce A (contradiction) then A must be true".

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  - "If A_i are all true then at least one of B_j is true".
  - Abbreviate to $\Gamma \vdash \Delta$.

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- Subformula property (apart from the cut rule).
- Cuts are useful, but make meta-theoretical results awkward.

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- The cut-free sequent calculus has the subformula property.
  - Easy consistency proof.
  - Proof-search is straightforward.
- What does this notion of reduction mean computationally?

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- *Exports* Reminiscent of function abstractions; binding an input and an output of P, and exporting the corresponding 'function'.
- *Mediators* Two subterms P and Q, which require a term to 'mediate' in between.

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  - This is called a *logical rule*

The $\mathcal{X}$-calculus

Two kinds of *connectors*: sockets ($x, y, z, \ldots$) and plugs ($\alpha, \beta, \gamma, \ldots$). Four basic syntax constructs:

$$\begin{array}{rcll} P, Q & ::= & \langle x.\alpha \rangle & (\text{capsule}) \\ & | & \widehat{x} P \widehat{\alpha} \cdot \beta & (\text{export}) \\ & | & P \widehat{\alpha} \, [y] \, \widehat{x} Q & (\text{mediator}) \\ & | & P \widehat{\alpha} \dagger \widehat{x} Q & (\text{cut}) \end{array}$$

- *Cuts* An active computation, attempting to connect each α in P with each x in Q.

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- *Cuts* An active computation, attempting to connect each α in P with each x in Q.
- If a logical rule cannot be applied, the cut must *propagate* into the structure of P and/or Q, to 'single out' the αs and xs.
- Sometimes cuts may propagate into P (left propagation) or Q (right propagation).

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  - Otherwise, the cut is destroyed (the capsule remains).

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$$\textit{either} \begin{cases} Q\widehat{\gamma} \dagger \widehat{y}(P\widehat{\beta} \dagger \widehat{z}R) \\ (Q\widehat{\gamma} \dagger \widehat{y}P)\widehat{\beta} \dagger \widehat{z}R \end{cases} \qquad \alpha \notin fs(P), x \notin fs(Q,R)$$

Reduction Rules

Activation Rules

$(act\text{-}\mathrm{L}) : P\widehat{\alpha} \dagger \widehat{x}Q \to P\widehat{\alpha} \not\prec \widehat{x}Q$ *if* P *does not freshly introduce* α

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  - Creates a cut between P and each x found.
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- Left cuts propagate inside P.

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  - 'Dual' notion of substitution ($P[Q/\alpha]$?).
  - Take an evaluation context Q to the 'calls' in P.

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  - Polymorphism
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  - Other logical connectives (continuations).

Polymorphism (on the whiteboard)

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Questions

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Are there any?

Sequent Calculus

$$\dfrac{\dfrac{\dfrac{}{\langle y.\pi\rangle \mathrel{\vdots} y:\varphi \vdash_{\text{SP}} \pi:\varphi}\,(cap)}{\widehat{y}\langle y.\pi\rangle\widehat{\pi}\cdot\theta \mathrel{\vdots} \vdash_{\text{SP}} \theta:\varphi\to\varphi}\,(exp)}{\widehat{y}\langle y.\pi\rangle\widehat{\pi}\cdot\theta \mathrel{\vdots} \vdash_{\text{SP}} \theta:\forall X.(X\to X)}\,(\forall\mathcal{R}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\langle x.\gamma\rangle \mathrel{\vdots} x:A\to A \vdash_{\text{SP}} \gamma:A\to A,\alpha:A\to A}\,(cap) \qquad \dfrac{}{\langle p.\alpha\rangle \mathrel{\vdots} p:A\to A, x:A\to A \vdash_{\text{SP}} \alpha:A\to A}\,(cap)}{\langle x.\gamma\rangle\widehat{\gamma}\,[x]\,\widehat{p}\langle p.\alpha\rangle \mathrel{\vdots} x:(A\to A)\to(A\to A), x:A\to A \vdash_{\text{SP}} \alpha:A\to A}\,(med)}{\langle x.\gamma\rangle\widehat{\gamma}\,[x]\,\widehat{p}\langle p.\alpha\rangle \mathrel{\vdots} x:(A\to A)\to(A\to A), x:\forall X.(X\to X) \vdash_{\text{SP}} \alpha:A\to A}\,(\forall\mathcal{L})}{\langle x.\gamma\rangle\widehat{\gamma}\,[x]\,\widehat{p}\langle p.\alpha\rangle \mathrel{\vdots} x:\forall X.(X\to X) \vdash_{\text{SP}} \alpha:A\to A}\,(\forall\mathcal{L})$$

$$\dfrac{\widehat{y}\langle y.\pi\rangle\widehat{\pi}\cdot\theta \mathrel{\vdots} \vdash_{\text{SP}} \theta:\forall X.(X\to X) \qquad \langle x.\gamma\rangle\widehat{\gamma}\,[x]\,\widehat{p}\langle p.\alpha\rangle \mathrel{\vdots} x:\forall X.(X\to X) \vdash_{\text{SP}} \alpha:A\to A}{(\widehat{y}\langle y.\pi\rangle\widehat{\pi}\cdot\theta)\widehat{\theta}\,\dagger\,\widehat{x}(\langle x.\gamma\rangle\widehat{\gamma}\,[x]\,\widehat{p}\langle p.\alpha\rangle) \mathrel{\vdots} \vdash_{\text{SP}} \alpha:A\to A}\,(cut)$$

Sequent Calculus

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y : \varphi \vdash_{\mathrm{ML}} y : \varphi}\,(ax)}{\vdash_{\mathrm{ML}} \lambda y.y : \varphi \to \varphi}\,(\to\mathcal{I})}{\vdash_{\mathrm{ML}} \lambda y.y : \forall Z.Z \to Z}\,(\forall\mathcal{I}) \qquad \dfrac{\dfrac{\dfrac{}{x : \forall Z.Z \to Z \vdash_{\mathrm{ML}} x : \forall Z.Z \to Z}\,(ax)}{x : \forall Z.Z \to Z \vdash_{\mathrm{ML}} x : (A \to A) \to (A \to A)}\,(\forall\mathcal{E}) \qquad \dfrac{\dfrac{}{x : \forall Z.Z \to Z \vdash_{\mathrm{ML}} x : \forall Z.Z \to Z}\,(ax)}{x : \forall Z.Z \to Z \vdash_{\mathrm{ML}} x : A \to A}\,(\forall\mathcal{E})}{x : \forall Z.Z \to Z \vdash_{\mathrm{ML}} x\ x : A \to A}\,(\to\mathcal{E})}{\vdash_{\mathrm{ML}} \mathtt{let}\ x = \lambda y.y\ \mathtt{in}\ x\ x : A \to A}\,(\mathtt{let})$$

Sequent Calculus

$$\frac{}{\Gamma, A \vdash A, \Delta}\,(\mathrm{Ax}) \qquad \frac{}{\langle x.\alpha\rangle :\cdot \Gamma, x : A \vdash \alpha : A, \Delta}\,(cap)$$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A{\to}B, \Delta}\,(\to\mathcal{R}) \qquad \frac{P :\cdot \Gamma, x : A \vdash \alpha : B, \Delta}{\widehat{x}P\widehat{\alpha}\cdot\beta :\cdot \Gamma \vdash \beta : A{\to}B, \Delta}\,(exp)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A{\to}B \vdash \Delta}\,(\to\mathcal{L}) \qquad \frac{P :\cdot \Gamma \vdash \alpha : A, \Delta \quad Q :\cdot \Gamma, y : B \vdash \Delta}{P\widehat{\alpha}\,[x]\,\widehat{y}Q :\cdot \Gamma, x : A{\to}B \vdash \Delta}\,(med)$$

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