Discovering World with Fuzzy Logic: Perspectives and Approaches to Formalization of Human-Consistent Logical Systems

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A Unified Compilation Style Labelled Deductive System for Modal, Substructural and Fuzzy Logics

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1 Introduction

The use of logic in areas such as computer science and artificial intelligence has led to the proliferation of a large number of various logical systems, often characterised by different notions of derivability relation, different sets of logical connectives as well as different underlying semantics. Logics belonging to the *same family* usually differ in "small" ways, either in their proof theory or in their semantics. For example, normal modal logics differ from each other only in the set of properties of their related semantic accessibility relation [14,18]. On the other hand, logics belonging to *different families* differ in more fundamental ways. They are often characterised by different notions of derivability relations, different semantics and different notions of semantic entailment. Results in [10,4,2] have already shown that the Labelled Deductive System (LDS) approach [15] facilitates uniform labelled proof systems, using, respectively, tableaux and natural deduction systems. This paper goes a step further. It describes a proof theoretic and semantic approach in which logics belonging to different families can be given common notions of derivability relation and semantic entailment. This approach builds upon the LDS methodology and is called the *Compilation approach for Labelled Deductive Systems* (CLDS).

Three sample logics are considered, the modal logic of elsewhere [9], the multiplicative fragment of substructural linear logic [11], and Łukasiewicz fuzzy logic [13]. Classical logic and its extensions, such as the modal logic of elsewhere [12], are logics that present their notion of derivability relations in terms of a relation between *sets* of formulae (or assumptions) and single formulae. Assumptions can be used in any order and an arbitrary (possibly none) number of times. Substructural logics, on the other hand, are logics whose derivability relations are often described as relations between *sequences* of formulae (or assumptions) and single formulae [11]. According to the type of substructural logic, assumptions can be used only in certain specific order and a certain number of times (e.g., in linear logic assumptions have all to be used exactly once, whereas in relevance logic assumptions are all used at least once). Similarly for Łukasiewicz fuzzy logic, in which assumptions are used at most once. The results in this paper anchor this logic firmly as a substructural logic, but with the underlying semantic notions of resources interpreted as degrees of truth (i.e. numerical values) of formulae [15].

The approach developed in this paper provides a general presentation of derivability relation which is equally applicable to each of these three logics. In the CLDS, a logical theory, written in a given *logical language*, is combined with a *labelling algebra*, written in a first-order *labelling language*, which axiomatises the properties — semantical or proof theoretical — that uniquely identify the underlying logic. In the case of the logic of elsewhere, the labelling algebra is a binary first-order theory that axiomatises the Kripke semantic accessibility relation as the inequality relation between "possible worlds". In the case of substructural logics, both linear and fuzzy, the labelling algebra is a binary first-order theory axiomatising standard structural rules (e.g., exchange and permutation) in terms of properties on labels. The two languages (logical language and labelling language) are combined via the LDS's notion of *declarative unit* [15]. A declarative unit $\alpha : \lambda$ expresses that the formula α is *true* or *verified* at the label (point) λ. Depending on the logic, labels are interpreted in different ways. In modal logic, labels are interpreted as possible worlds, in substructural logic as combination of resources, and in fuzzy logic as degrees of truth. Inference rules are defined to act on both syntactic components of declarative units, logical formulae and labels, according to the desired properties of the connectives and of the labelling algebra.

In this paper, the CLDS approach is applied to the logic of elsewhere, to the multiplicative fragment of linear logic and to Łukasiewicz fuzzy logic, giving rise to three systems denoted respectively with $\mathrm{E}_{\mathrm{CLDS}}$, $\mathrm{L}_{\mathrm{CLDS}}$ and $\mathrm{F}_{\mathrm{CLDS}}$. In each of these systems, a theory, called *configuration*, is a set of declarative units and R-literals, where the R-literals specify a structure of points (actual worlds, resources or degrees of truth) and the declarative units describe which formulae are assumed to be verified at each point in the structure. R-literals are of the form $R(\lambda_i, \lambda_j)$ or $\neg R(\lambda_i, \lambda_j)$, where λ_i and λ_j are labels. In the case of modal logic, they express worlds which are or are not in relation with each other; in substructural logic, resources which are or are not "included" within each other[1], whereas in Łukasiewicz fuzzy logic R-literals denote the standard greater-than-or-equal relation on real numbers. The proof theory of each of these three CLDS systems is a *uniform* natural deduction system in that for each logic the natural deduction rules for connectives can be equally applied to other logics belonging to the same family. So, for instance, the set of inference rules for the logic of elsewhere can also be used for any other normal modal logic, and the rules for substructural logic are equally applicable to relevance, linear and, as it is shown in Section 5, Łukasiewicz fuzzy logic. The difference between one modal logic and another or between one substructural logic and another is captured entirely by the labelling algebra.

[1] The notion of inclusion is with respect to interpretation of a label as the set of formulas that it verifies.

The combined feature (i.e. logical theory and labelling algebra) of the three CLDS systems provides the underlying logics with some additional advantages. For the logic of elsewhere, it retains the advantages of both implicit (e.g. [14]) and explicit (e.g. [23]) traditional formalisations. Statements such as "necessary α" can be captured succinctly, using the modal operator $\Box$, by simply writing the single declarative unit $s_0 : \Box\alpha$ (where s_0 is the labelling algebra representation of the actual world). Like the explicit approach, the language is rich enough to allow explicit syntactic reference to particular possible worlds and to specific inequality or equality relationships between possible worlds. As for the substructural logic, both linear and fuzzy, the combined feature of the $\mathrm{L}_{\mathrm{CLDS}}$ and $\mathrm{F}_{\mathrm{CLDS}}$ systems facilitates an "object-level" formalisation of operational and structural properties of their proof theories, the former by means of the logical operators and the latter by means of the labelling algebra. Label conditions expressed in the rules, together with the labelling algebra, provide the proof theory with the same features as the standard structural rules of substructural logic and Łukasiewicz fuzzy logic [11,10,28], but facilitating a presentation of the derivability relation in terms of a relation between sets of formulae and formulae. Moreover, each of the three CLDS systems facilitates a *proper generalisation* of respective standard formalisms, in that it facilitates reasoning about what is true or verified at different points in a (possibly singleton) structure of actual worlds, actual resources, actual degrees of truth.

The paper is organised as follows. In Section 2 the language and syntax of a CLDS are defined together with the notion of a *configuration* — a CLDS system's equivalent to a theory. A general natural deduction style proof system for a CLDS is given, together with a general model-theoretic semantics, based on a translation method into classical logic, and a notion of semantic entailment. The three specific $\mathrm{E}_{\mathrm{CLDS}}$, $\mathrm{L}_{\mathrm{CLDS}}$ and $\mathrm{F}_{\mathrm{CLDS}}$ systems are defined in Sections 3, 4 and 5 respectively, together with their soundness, completeness and correspondence results. The paper ends with a general discussion in Section 6.

Some remarks may be helpful regarding notation. Throughout the paper predicate symbols begin with an upper-case letter, whereas constants, variables and function symbols begin with a lower-case letter. Greek letters meta-variables are used to refer in general to terms and expressions in the system. Larger entities such as structures, sets, theories and languages are symbolised in calligraphic font, $\mathcal{A}, \mathcal{B}, \mathcal{C}$, etc. The power set of a given set $\mathcal{A}$ is denoted by $PW(\mathcal{A})$.

2 The CLDS Approach

In this section the CLDS approach is formally described. Basic definitions of a CLDS language and syntax are given together with the notion of a *configuration* — the CLDS system's equivalent to a modal or substructural theory.

2.1 Languages and Syntax

A CLDS language is defined as an ordered pair $\langle \mathcal{L}_P, \mathcal{L}_L \rangle$, where $\mathcal{L}_P$ is a *propositional language* composed of a countable set of propositional letters, $\{p, q, r, \dots\}$, and a set of unary and binary connectives, and $\mathcal{L}_L$, called a *labelling language*, is a binary fragment of a first-order language composed of a countable set of constant symbols $\{s_0, s_1, s_2, \dots\}$, a countable set of variables $\{x, y, z, \dots\}$, a binary predicate symbol R, called R-predicate, a (possibly empty) finite set of function symbols $\{f_1, f_2, \dots\}$, the set of logical connectives $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ and the quantifiers $\forall$ and $\exists$. The first-order language $Func(\mathcal{L}_P, \mathcal{L}_L)$ is an extension of $\mathcal{L}_L$ defined as follows.

Definition 1. Let $\mathcal{L}_P$ be a propositional language and $\{\alpha_1, \alpha_2, \dots\}$ be the set of all wffs of $\mathcal{L}_P$. The semi-extended labelling language $Func(\mathcal{L}_P, \mathcal{L}_L)$ is defined as the language $\mathcal{L}_L$ extended with a set of *skolem* function symbols $\{sk^n_{\alpha_1}, sk^n_{\alpha_2}, \dots\}$, with arity $n \geq 0$.

The ground terms of $Func(\mathcal{L}_P, \mathcal{L}_L)$, called *labels*, are interpreted differently according to the family of logics that is under consideration. In the case of modal logics, they refer to possible worlds, in the case of linear logics they denote "resources", whereas in the case of Łukasiewicz fuzzy logic they refer to degrees of truth. The binary R-predicate represents, in the case of modal logics, the accessibility relation between possible worlds, in the case of linear logic, a partial ordering of "inclusion" between resources, and in fuzzy logic the greater-than-or-equal relation on real numbers. Labels constructed using skolem function symbols of $Func(\mathcal{L}_P, \mathcal{L}_L)$ have specific roles in a CLDS proof system. As shown in Section 3, the skolem symbols in the $\mathrm{E_{CLDS}}$ system are unary function symbols of the form f_α and box_α used to denote specific possible worlds. For each wff α and possible world (label) λ, the ground term $f_\alpha(\lambda)$ names a *particular* possible world specifically associated with α which formalises the Kripke semantic notion "there exists a possible world...". In contrast, ground terms of the form box_α can be thought of as referring to any *arbitrary* world specifically associated with α, and are used to express Kripke semantic notions of the form "for all possible worlds ...". In the case of the $\mathrm{L_{CLDS}}$ system, as shown in Section 4, the skolem symbols are instead constant symbols, called *parameters*, of the form c_α. For each wff α of $\mathcal{L}_P$, the parameter c_α denotes the smallest resource needed to verify α, and is sometimes referred to as the *characteristic* label of α. Analogously for the $\mathrm{F_{CLDS}}$ system, but with the parameters c_α denoting, instead, the maximum degree of truth of a formula α.

To capture different classes of logics within the CLDS approach, an appropriate first-order theory, written in the language $\mathcal{L}_L$, called a *labelling algebra* and denoted by $\mathcal{A}$, needs to be defined. For example, any normal modal logic can be captured by defining the labelling algebra of a CLDS system as the first-order theory axiomatising its semantic accessibility relation [27]. (Examples of such CLDS systems are largely described in [24].) For the logic of elsewhere, the notion of "elsewhere" expresses that worlds are accessible from each another if and only if they are different. This notion is captured in the $\mathrm{E_{CLDS}}$ system by defining the labelling algebra as a binary first-order theory given by the equality theory and the axiom $\forall x, y(R(x,y) \leftrightarrow x \neq y)$, so that the R-predicate R is equivalent to the inequality relation. In the case of linear logic, the labelling algebra is a binary first-order theory which axiomatises (i) the R-predicate R as a pre-ordering relation and (ii) four main properties of the function symbol $\circ$, called "resource composition", of the $\mathrm{L_{CLDS}}$ labelling language. The labelling algebra of the $\mathrm{F_{CLDS}}$ system, instead, axiomatises the R-predicate R as a total pre-ordering relation, and extends the set of the properties of the $\mathrm{L_{CLDS}}$ function symbol $\circ$, given in this case by an arithmetic expression on degrees of truth, with monotonicity. Sections 3, 4 and 5 provide, respectively, formal definitions of the $\mathrm{E_{CLDS}}$, $\mathrm{L_{CLDS}}$ and $\mathrm{F_{CLDS}}$ labelling algebra.

Syntax. The CLDS language facilitates the formalisation of two types of information, (i) what holds at particular points and (ii) which points are in relation with each other and which are not. These two types of information are captured within the syntax of a CLDS system by two different types of syntactic entities, the *declarative units* and the *R-literals*. A declarative unit is defined as a pair *formula:label*, where the label component is a ground term of the semi-extended labelling language $Func(\mathcal{L}_P, \mathcal{L}_L)$ and the formula is a wff of the language $\mathcal{L}_P$. An R-literal is any ground literal in the semi-extended labelling language involving the R-predicate, usually of the form $R(\lambda_1, \lambda_2)$ and $\neg R(\lambda_1, \lambda_2)$, where λ_1 and λ_2 are labels, expressing that λ_2 is or is not related to λ_1. For each R-literal Δ, the *conjugate* of Δ, written $\overline{\Delta}$, is the opposite in sign of Δ (i.e. $\neg R(\lambda_1, \lambda_2)$ if $\Delta = R(\lambda_1, \lambda_2)$ and $R(\lambda_1, \lambda_2)$ if $\Delta = \neg R(\lambda_1, \lambda_2)$).

This combined aspect of the CLDS syntax yields a definition of a CLDS theory more general than the traditional notion of a modal, substructural or fuzzy theory ([18,11,13]). Informally, a CLDS theory, called a *configuration*, is composed of two sets, a set of R-literals and a set of declarative units. An example of a $\mathrm{E_{CLDS}}$ theory is the pair of sets $\{R(s_0, s_1), R(s_0, s_2), \neg R(s_1, f_p(s_1))\}$ and $\{\Box(p \rightarrow q) : s_0, \Box r : s_0, \Diamond p : s_1, p : f_p(s_1)\}$, whereas examples of $\mathrm{L_{CLDS}}$ and $\mathrm{F_{CLDS}}$ theories are the two pairs of sets $\{R(\circ(c_p, c_q), s_1)\}$ and $\{p \otimes q : s_1, q : c_q, p : c_p\}$, and $\{R(c_q, c_p)\}$ and $\{p \rightarrow q : 1, p : c_p, \neg q : s_1\}$, respectively. The formal definition of a configuration is given below.

Definition 2. Given a CLDS language, a configuration is a tuple $\langle \mathcal{D}, \mathcal{F} \rangle$ where $\mathcal{D}$, called a *diagram*, is a finite set of R-literals and $\mathcal{F}$ is a function from the set of ground terms of *Func*$(\mathcal{L}_P, \mathcal{L}_L)$ to the set PW(wff$(\mathcal{L}_P)$) of sets of wffs of $\mathcal{L}_P$.

In the next section, a "basic" natural deduction style proof system for an arbitrary CLDS is given, in which inference rules and the notion of a derivability relation are defined between configurations. A set $\mathcal{R}$ of such inference rules, together with a CLDS language $\langle \mathcal{L}_P, \mathcal{L}_L \rangle$ and a labelling algebra $\mathcal{A}$, uniquely define a CLDS system (i.e. for any CLDS system S, $S = \langle \langle \mathcal{L}_P, \mathcal{L}_L \rangle, \mathcal{A}_\mathrm{S}, \mathcal{R}_S \rangle$).

2.2 A "basic" natural deduction system

The "structural" aspect of a CLDS theory has led to the idea of defining deductive processes that describe how configurations can "evolve" by reasoning within and between the local theories associated with each point in the configuration or by reasoning about the diagram of a configuration. Inference rules and derivability relation are defined between configurations. An inference rule of a CLDS is generally defined as follows.

Definition 3. An inference rule $\mathcal{I}$ is a set of pairs of configurations, where each such pair is written as $\mathcal{C}/\mathcal{C}'$. If $\mathcal{C}/\mathcal{C}' \in \mathcal{I}$ then we say $\mathcal{C}$ is an *antecedent* configuration of $\mathcal{I}$, and $\mathcal{C}'$ is an *inferred* (or *consequence*) configuration of $\mathcal{I}$ with respect to $\mathcal{C}$.

All the rules except one have the effect of expanding the antecedent configuration. These rules can extend an antecedent configuration $\mathcal{C}$ with either a declarative unit, or with an R-literal or with both. However, configurations equal or smaller than the antecedent one can also be inferred. This is facilitated by an inference rule called the $\mathcal{C}$-Reduction (C-R) rule. A graphical representation of the inference rules is given throughout the paper. The reader is referred to [24,6] for a complete formal definition of a CLDS proof system. Tables 1 and 2 illustrate the inference rules for the $\rightarrow$ and $\neg$ connectives[2], and for the R-literals respectively. Please note that, in both these tables, $\mathcal{C}\langle\alpha : \lambda\rangle$ (respectively $\mathcal{C}\langle\Delta\rangle$) denotes that $\mathcal{C}$ includes a declarative unit $\alpha : \lambda$ (respectively R-literal Δ). Declarative units and R-literals contained in square brackets are assumptions introduced within a derivation that are subsequently discharged. $\mathcal{C}'\langle\psi\rangle$, where ψ is a declarative unit or an R-literal, represents that the inferred configuration $\mathcal{C}'$ is $\mathcal{C}$ extended with ψ. $\tilde{\mathcal{C}}$ are the configurations derived in subderivations after adding temporary assumptions to the antecedent configuration $\mathcal{C}$. The rules in Table 1 have the same format in the $\mathrm{E}_{\mathrm{CLDS}}$, $\mathrm{L}_{\mathrm{CLDS}}$ and $\mathrm{F}_{\mathrm{CLDS}}$ systems. The standard semantic difference between the classical, substructural and fuzzy connectives $\rightarrow$ and $\neg$ is fully captured by the labels used in the rules. Specifically, in the $\mathrm{E}_{\mathrm{CLDS}}$ system, the labels are defined to be the same, i.e. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, for any arbitrary

[2] Only these connectives are considered here since, these are the only ones to be in common to the three logics.

Table 1. Natural deduction rules for $\rightarrow$ and $\neg$ connectives.

$$\frac{\mathcal{C}\langle \alpha \rightarrow \beta : \lambda_1, \alpha : \lambda_2 \rangle}{\mathcal{C}'\langle \beta : \lambda_3 \rangle} \ (\rightarrow\mathcal{E}) \qquad \frac{\begin{array}{c}\mathcal{C}\langle [\alpha : \lambda_1] \rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle \beta : \lambda_2 \rangle\end{array}}{\mathcal{C}'\langle \alpha \rightarrow \beta : \lambda_3 \rangle} \ (\rightarrow\mathcal{I})$$

$$\frac{\mathcal{C}\langle \neg\neg\alpha : \lambda \rangle}{\mathcal{C}'\langle \alpha : \lambda \rangle} \ (\neg\neg) \qquad \frac{\begin{array}{c}\mathcal{C}\langle [\alpha : \lambda_1] \rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle \bot : \lambda_2 \rangle\end{array}}{\mathcal{C}'\langle \neg\alpha : \lambda_3 \rangle} \ (\neg\mathcal{I})$$

ground term λ of the semi-extended labelling language, whereas in the $\mathrm{L_{CLDS}}$ and $\mathrm{F_{CLDS}}$ systems they are defined to be of a more specific form (see Sections 4 and 5 respectively). In Sections 3, 4 and 5 this set of rules is further extended with elimination and introduction rules for the modal operators $\Box$ and $\Diamond$, the structural operator $\otimes$ and an additional elimination rule for the $\mathrm{F_{CLDS}}$ $\rightarrow$, respectively.

The notion of contradiction (or inconsistency) in the CLDS approach strictly depends on the type of logic. Modal logics (and therefore the logic of elsewhere) include a classical notion of inconsistency, for which the symbol $\bot$ used in the $(\neg\mathcal{I})$ rule of the $\mathrm{E_{CLDS}}$ system is a short-hand for any $\mathcal{L}_P$ wff of the form $\alpha \wedge \neg\alpha$. Linear logic instead respects a different notion of contradiction, according to which the declarative unit $\bot : \lambda$ denotes an inconsistency only when the label λ is a "consistent resource". This is further explained in Section 4. A similar notion of inconsistency is used in the $\mathrm{F_{CLDS}}$ system. However, in the $\mathrm{F_{CLDS}}$ system this notion of substructural inconsistency, together with the additional (monotonicity) property of the $\mathrm{F_{CLDS}}$ labelling algebra, gives to the inconsistency extra classical properties such as the *ex falsum quod libet* property, which does not hold in linear logic.

The R-literals rules, in Table 2, facilitate instead reasoning about the diagram of a configuration, using the particular underlying labelling algebra $\mathcal{A}$, and infering R-literals and declarative units which would not be inferred using only the logical connectives. For logics of the same family (i.e. different substructural logics or different modal logics), the (R-A) rule captures *entirely* the difference between one CLDS system and another, allowing all other inference rules to be equally applicable to any CLDS system of the same family (e.g., [24,10]). For logics belonging to different families, (R-A) reflects the different underlying semantics, capturing only *partially* the logics since additional rules are often needed. In the $\mathrm{E_{CLDS}}$ system, where the labelling algebra axiomatises the symmetry property of the R predicate,

Table 2. Rules for the R-literals

$(\bot\mathcal{E})\ \dfrac{\mathcal{C}\langle\Delta,\overline{\Delta}\rangle}{\mathcal{C}'\langle\alpha:\lambda\rangle}$	$(C\text{-R})\ \dfrac{\mathcal{C}}{\mathcal{C}'}$ where $\mathcal{C}'\subseteq\mathcal{C}$
$(R\mathcal{I})\ \dfrac{\begin{matrix}\mathcal{C}\langle[\ \overline{\Delta}\ ]\rangle\\ \vdots\\ \tilde{\mathcal{C}}\langle\bot:\lambda\rangle\end{matrix}}{\mathcal{C}'\langle\Delta\rangle}$	$(R\text{-A})\ \dfrac{\mathcal{C}}{\mathcal{C}'\langle\Delta\rangle}$ if $\mathcal{A}\cup\mathcal{D}\vdash_{FOL}\Delta$

the $(R$-A) rule allows, for instance, the inference of R-literals of the form $R(\lambda_2,\lambda_1)$ whenever the antecedent configuration includes R-literals of the form $R(\lambda_1,\lambda_2)$, thus embedding the symmetry property of the accessibility relation in the derivation process. This enables the derivation of a declarative unit of the form $\alpha\rightarrow\Box\Diamond\alpha:\lambda$, for an arbitrary label λ. In the $\mathrm{L_{CLDS}}$ system, the $(R$-A) rule allows, for instance, the inference of R-literals of the form $R(\circ(\lambda_1,\lambda_2),\circ(\lambda_2,\lambda_1))$, where $\circ$ is the resource composition function, thus embedding the structural "commutativity" property of linear logic [10,4] in the derivation process. Such a property enables the derivations of declarative units of the form $\alpha\otimes\beta\rightarrow\beta\otimes\alpha:\lambda$, for the particular label $\lambda=1$, which is a theorem of linear logic. For Łukasiewicz fuzzy logic, the $(R$-A) rule allows not only the inference of R-literals given by the $\mathrm{L_{CLDS}}$ system, but also the inference of R-literals of the form $R(\lambda_1,\circ(\lambda_1,\lambda_2))$, thus capturing a "monotonicity property" on the degrees of truth. Such a property facilitates the derivation for example of declarative units of the form $\alpha\rightarrow(\beta\rightarrow\alpha):1$, which corresponds to a theorem of Łukasiewicz fuzzy logic. It is important to emphasise that the diagram of any configuration is a ground binary theory. Therefore, for any configuration it is always possible to define an upper limit to the domain on which the first-order labelling algebra $\mathcal{A}$ is applied, in order to perform an $(R$-A) inference step.

The rules $(\bot\mathcal{E})$ and $(R\mathcal{I})$ in Table 2 express additional forms of interactions between R-literals and declarative units. The $(\bot\mathcal{E})$ rule allows the inference of falsity (i.e. $\bot:\lambda$) whenever R-literals and its negations are present in a configuration. This is necessary because since no compound classical formulae with R-literals can be inferred in a configuration, inconsistency of this form would not otherwise be captured. The $(R\mathcal{I})$ rule enables instead the derivation of R-literals in the presence of a logical inconsistency. The set of rules given in Table 2 is also extended in Sections 4 and 5 to include additional forms of interactions between R-literals and declarative units.

Derivability. Informally, a proof is a non empty sequence of configurations, $\mathcal{C}_0, \ldots, \mathcal{C}_n$, where $\mathcal{C}_0$ is the initial configuration and, for each $0 < i \leq n$, $\mathcal{C}_i$ is obtained from $\mathcal{C}_{i-1}$ by the application of an inference rule. A configuration $\mathcal{C}'$ is said to be *derivable* from a configuration $\mathcal{C}$ in a CLDS system S, written $\mathcal{C} \vdash_{\mathrm{S}} \mathcal{C}'$, if and only if there exists a proof $\mathcal{C}, \ldots, \mathcal{C}'$. This is formally defined below.

Definition 4. Given a CLDS system S, a *proof* is a pair $\langle \mathcal{P}, m \rangle$, where $\mathcal{P}$ is a sequence of configurations $\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}$, with $n > 0$, and m is a mapping from the set $\{0, \ldots, n-1\}$ to $\mathcal{R}_S$ such that for each i, $0 \leq i < n$, $\mathcal{C}_i / \mathcal{C}_{i+1} \in m(i)$.

Definition 5 (Derivability). Given a CLDS system S, and two configurations $\mathcal{C}$ and $\mathcal{C}'$, $\mathcal{C}'$ is *derivable* from $\mathcal{C}$ in S, written $\mathcal{C} \vdash_{\mathrm{S}} \mathcal{C}'$, if there exists a proof $\langle \{\mathcal{C}, \ldots, \mathcal{C}'\}, m \rangle$.

It is easy to show that the derivability relation $\vdash_{\mathrm{S}}$ of a CLDS system S is reflexive, transitive and monotonic, (for a proof see [24]). Notation 6 captures the standard notion of a derivability relation between theories (configurations) and formulae (declarative units or R-literals) in terms of the more general derivability relation given in Definition 5. A "vice-versa" characterisation can be shown — a configuration $\mathcal{C}'$ is derivable from a configuration $\mathcal{C}$ if each unit of information (declarative units and R-literals) of $\mathcal{C}'$ is derivable from $\mathcal{C}$. This is proved in Lemma 7.

Notation 6. Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$ be a configuration and π be either a declarative unit or an R-literal. Then $\mathcal{C} \vdash_{\mathrm{S}} \pi$ if there exists a configuration $\mathcal{C}'$ such that $\mathcal{C} \vdash_{\mathrm{S}} \mathcal{C}'$ and $\pi \in \mathcal{C}'$. Moreover, if π is a declarative unit of the form $\alpha : \lambda$ then $\mathcal{C} + [\alpha : \lambda]$ is the configuration $\langle \mathcal{D}, \mathcal{F}' \rangle$, such that $\mathcal{F}'(\lambda) = \mathcal{F}(\lambda) \cup \{\alpha\}$ and for any λ' different from λ, $\mathcal{F}'(\lambda') = \mathcal{F}(\lambda')$. If π is an R-literal Δ, then $\mathcal{C} + [\Delta]$ is the configuration $\langle \mathcal{D}', \mathcal{F} \rangle$ such that $\mathcal{D}' = \mathcal{D} \cup \{\Delta\}$.

Lemma 7. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations of a CLDS system S, such that $\mathcal{C}' - \mathcal{C}$ is finite*[3]*. $\mathcal{C} \vdash_{\mathrm{S}} \mathcal{C}'$ if and only if for each $\pi \in \mathcal{C}' - \mathcal{C}$, where π is a declarative unit or an R-literal, $\mathcal{C} \vdash_{\mathrm{S}} \pi$.*

Proof: The "only if" part is trivial, whereas the "if" part is proved by induction on the size of $\mathcal{C}' - \mathcal{C}$. A formal description of this proof is given in [24]. □

[3] $\mathcal{C}' - \mathcal{C}$ (formally defined in [24]) is basically the set of declarative units and R-literals in $\mathcal{C}'$ but not in $\mathcal{C}$.

2.3 Semantics

A propositional CLDS can be considered to be a "semi-translated" approach to a given logic. In the case of E_{CLDS} system a Kripke-like accessibility relation is syntactically expressed as R-literals, but without requiring the full translation of modal formulae into first-order sentences. In the L_{CLDS} and F_{CLDS} systems, on the other hand, the CLDS approach facilitates the "meta-level" features of the underlying logic to be formalised as part of the object-level proof system. This semi-translated approach to linear logic and Łukasiewicz fuzzy logic still preserves the concise aspect of their respective logical languages. The semantics of a CLDS system is given in terms of a first order semantics using a translation method. This enables the development of a model-theoretic approach equally applicable to any logics, including those belonging to different families, whose semantics can be axiomatised in first order logic. In this section, the translation method underlying a CLDS system is defined and the notions of model, satisfiability and semantic entailment are given in terms of classical semantics.

As mentioned above, a declarative unit $\alpha : \lambda$ represents that the formula α is verified (or holds) at the point λ, whose interpretation is strictly related to the underlying logic. In what follows, this semantic notion is expressed in terms of first-order statements of the form $[\alpha]^*(\lambda)$, where $[\alpha]^*$ is a predicate symbol. The relationships between these monadic predicate symbols are constrained by a set of first-order axiom schemas which capture the satisfiability conditions of each type[4] of formula α. An *extended labelling language* $Mon(\mathcal{L}_P, \mathcal{L}_L)$ is then defined, which extends the language $Func(\mathcal{L}_P, \mathcal{L}_L)$ with a countable set of monadic predicate symbols $[\alpha]^*$, one for each wff α of $\mathcal{L}_P$.

Definition 8. Let $Func(\mathcal{L}_P, \mathcal{L}_L)$ be a semi-extended labelling language. Let $\alpha_1, \ldots, \alpha_n, \ldots$, be the ordered set of wffs of $\mathcal{L}_P$. The *extended labelling language* $Mon(\mathcal{L}_P, \mathcal{L}_L)$ is defined as the language $Func(\mathcal{L}_P, \mathcal{L}_L)$ extended with the following set of unary predicate symbols

$$\{[\alpha_1]^*, \ldots, [\alpha_n]^*, \ldots\}$$

An *extended algebra* $\mathcal{A}^+$ is a first-order theory, written in $Mon(\mathcal{L}_P, \mathcal{L}_L)$, which extends a labelling algebra $\mathcal{A}$ with axiom schemas on the monadic predicates. These schemas strictly depend on the underlying logic. For example, in the E_{CLDS} system the extended labelling algebra $\mathcal{A}^+$ includes the two axiom schemas $\forall x([\Box\alpha]^*(x) \rightarrow (\forall y(R(x,y) \rightarrow [\alpha]^*(y))))$ and $\forall x((R(x, box_\alpha(x)) \rightarrow [\alpha]^*(box_\alpha(x))) \rightarrow [\Box\alpha]^*(x))$ which, together with the axiom $\forall x, y(R(x,y) \leftrightarrow x \neq y)$, capture the Kripke semantic meaning of the elsewhere $\Box$ operator. Formal definitions of the extended algebras $\mathcal{A}^+$ for the E_{CLDS}, L_{CLDS} and F_{CLDS} systems are given in Sections 3, 4 and 5 respectively.

[4] The type of a wff is given by the main connective of the wff itself, e.g., the wff $\Diamond(p \rightarrow q)$ is a $\Diamond$-formula, whereas the formulae $\alpha \otimes (\beta \rightarrow \gamma)$ is a $\otimes$-formula.

The notions of satisfiability and semantic entailment are common to any CLDS system. These are based on a translation method, which associates syntactic expressions of the CLDS system with sentences of the first-order language $Mon(\mathcal{L}_P, \mathcal{L}_L)$, and hence associates configurations with first-order theories in the language $Mon(\mathcal{L}_P, \mathcal{L}_L)$. Each declarative unit $\alpha : \lambda$ is translated into the sentence $[\alpha]^*(\lambda)$ and R-literals are translated as themselves. Therefore, the first-order translation of a configuration is a first-order theory including the R-literals, which are present in the diagram of the configuration, and the set of monadic formulae $[\alpha]^*(\lambda)$ that correspond to the declarative units present in the configuration.

Definition 9. Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$ be a configuration. The *first-order translation* of $\mathcal{C}$, written $FOT(\mathcal{C})$, is a theory written in $Mon(\mathcal{L}_P, \mathcal{L}_L)$ and defined by the expression: $FOT(\mathcal{C}) = \mathcal{D} \cup \mathcal{DU}$, where $\mathcal{DU} = \{[\alpha]^*(\lambda) \mid \alpha \in \mathcal{F}(\lambda),\ \lambda$ is a ground term of $Func(\mathcal{L}_P, \mathcal{L}_L)\}$.

Note that labels can only be ground terms of the language $Func(\mathcal{L}_P, \mathcal{L}_L)$, so the first-order translation of a configuration is a set of *ground literals* of the language $Mon(\mathcal{L}_P, \mathcal{L}_L)$. Notions of model, satisfiability and semantic entailment are given in terms of classical semantics (where "$\mathcal{M} \Vdash_{FOL} \psi$" signifies that the classical formula ψ is true in the classical model $\mathcal{M}$, according to the standard definition).

Definition 10. Given a CLDS system S, the associated extended algebra $\mathcal{A}_S^+$, a declarative unit $\alpha : \lambda$ and a R-literal Δ,

$$\mathcal{M} \text{ is a semantic structure of } S \quad \Leftrightarrow_{\text{def}} \quad \mathcal{M} \text{ is a model of } \mathcal{A}_S^+ \tag{1}$$

$$\mathcal{M} \Vdash_S \alpha : \lambda \quad \Leftrightarrow_{\text{def}} \quad \mathcal{M} \Vdash_{FOL} [\alpha]^*(\lambda) \tag{2}$$

$$\mathcal{M} \Vdash_S \Delta \quad \Leftrightarrow_{\text{def}} \quad \mathcal{M} \Vdash_{FOL} \Delta \tag{3}$$

In the above definition, (1) defines the class of models of a CLDS system S in terms of models of the extended algebra $\mathcal{A}_S^+$ associated with S. (2) and (3) define the satisfiability of declarative units and R-literals in terms of classical satisfiability of their associated first-order translations. A semantic structure $\mathcal{M}$ satisfies a configuration $\mathcal{C}$, written $\mathcal{M} \Vdash_S \mathcal{C}$, if and only if for each $\pi \in \mathcal{C}$ (where π is a declarative unit or an R-literal), $\mathcal{M} \Vdash_S \pi$. The notion of semantic entailment in a CLDS system is given here as a relation between configurations.

Definition 11. Let $S = \langle \langle \mathcal{L}_P, \mathcal{L}_L, \rangle, \mathcal{A}, \mathcal{R} \rangle$ be a CLDS and let $\mathcal{A}_S^+$ be the extended algebra of S. Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$ and $\mathcal{C}' = \langle \mathcal{D}', \mathcal{F}' \rangle$ be two configurations of S and $FOT(\mathcal{C}) = \mathcal{D} \cup \mathcal{DU}$ and $FOT(\mathcal{C}') = \mathcal{D}' \cup \mathcal{DU}'$ their respective first-order translations. The configuration $\mathcal{C}$ semantically entails $\mathcal{C}'$, written $\mathcal{C} \models_S \mathcal{C}'$, iff for each $\Delta \in \mathcal{D}'$, $\mathcal{A}_S^+ \cup FOT(\mathcal{C}) \models_{FOL} \Delta$, and for each $[\alpha]^*(\lambda) \in \mathcal{DU}'$, $\mathcal{A}_S^+ \cup FOT(\mathcal{C}) \models_{FOL} [\alpha]^*(\lambda)$.

In Sections 3, 4 and 5 the above definition expresses the notion of semantic entailment for the $\mathrm{E_{CLDS}}$, $\mathrm{L_{CLDS}}$ and $\mathrm{F_{CLDS}}$ systems, which are denoted by $\models_{\mathrm{E}}$, $\models_{\mathrm{L}}$ and $\models_{\mathrm{F}}$ respectively.

Proving soundness. Given that the semantics is based on a first-order translation method, the proof of the soundness property of the $\vdash_{\mathrm{S}}$ for a CLDS system $\mathcal{S}$ is based on the soundness property of the first-order classical derivability relation $\vdash_{\mathrm{FOL}}$. A diagrammatic representation of the soundness theorem of a CLDS system $\mathcal{S}$ is given in Figure 1. The soundness statement, which

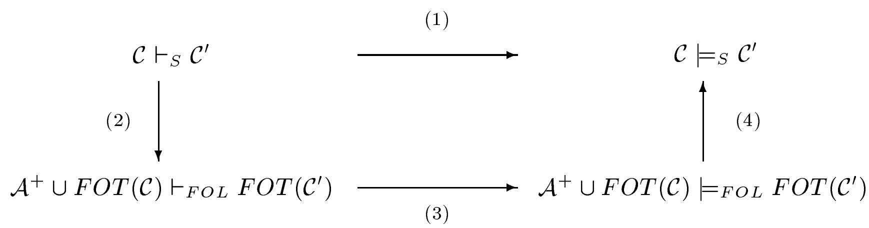

Fig. 1. Proof of the soundness property of a CLDS system $\mathcal{S}$.

corresponds to the arrow labelled with (1), is proved by the composition of three main steps, arrows (2), (3) and (4) respectively. The first step (arrow (2)) proves that the hypothesis, $\mathcal{C} \vdash_S \mathcal{C}'$, for a CLDS system $\mathcal{S}$, implies that $\mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{\mathrm{FOL}} FOT(\mathcal{C}')$. This trivially implies (by soundness of first-order logic) that $\mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{\mathrm{FOL}} FOT(\mathcal{C}')$, which gives the second step of the proof (arrow (3)). Arrow (4) is given by the notion of the semantic entailment between configurations given in Definition 11. This methodology is generally applicable to any CLDS system.

The first step is the only one that needs to be proved for each specific logic formalised in the CLDS approach. This is proved by using a technique that differs from the standard technique used in the literature for proving soundness of a natural deduction proof system. In general (see [14] for an example of standard soundness proof for natural deduction proof systems) soundness is proved by induction on the number of inference steps in a given derivation, taking into account the specific *context* of each inference rule. In this paper instead we define the notions of size of an inference rule and size of a proof, and apply induction on the size of a given derivation. In this way there is no difference (apart from the size) between the inference rules that introduce new assumptions and those which do not introduce new assumptions. These notions are formally defined below, using the following additional notation.

Notation 12. Given a CLDS system S, its set $\mathcal{R}_\mathrm{S}$ of inference rules is classified into four categories. The first category denoted with $\mathcal{I}^{00}$ includes just the (C-R) rule, as being the only rule which does not infer new declarative units or new R-literals. The second category, denoted by $\mathcal{I}^0$, consists of the inference rules that infer new declarative units and/or new R-literals without using any subderivations as conditions. The third category, denoted by $\mathcal{I}^+$, is the set of inference rules that require one subderivation as a condition. Finally, the fourth category, denoted by $\mathcal{I}^{++}$, is the set of inference rules that use two subderivations as conditions.

Definition 13. Let S be a CLDS system, let $\mathcal{I}_i \in \mathcal{R}_\mathrm{S}$ and let $\mathcal{C}/\mathcal{C}' \in \mathcal{I}_i$. The *size* of $\mathcal{C}/\mathcal{C}'$ with respect to $\mathcal{I}_i$, written $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_i)$, is defined as follows:

- If $\mathcal{I}_i \in \mathcal{I}^{00}$ then $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_i) = 0$.
- If $\mathcal{I}_i \in \mathcal{I}^{0}$ then $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_i) = 1$.
- If $\mathcal{I}_i \in \mathcal{I}^{+}$ then $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_i) = 1 + l_1$, where l_1 is the smallest of the sizes of all subderivations that can be used as condition of the rule.
- If $\mathcal{I}_i \in \mathcal{I}^{++}$ then $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_i) = 1 + l_1 + l_2$, where l_1 and l_2 are the smallest of the sizes of all the two subderivations that can be used as conditions of the rule.

Definition 14. Let S be a CLDS system, the *size of a proof* $\langle\{\mathcal{C} \ldots \mathcal{C}_n\}, m\rangle$, written $l(\langle\{\mathcal{C} \ldots \mathcal{C}_n\}, m\rangle)$, is defined as follows

$$l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle) = \sum_{k=0}^{n-1} l(\mathcal{C}_k/\mathcal{C}_{k+1}, m(k))$$

Proving completeness. The completeness property of a CLDS system with respect to the notion of semantic entailment given in Definition 11 can be proved using standard Henkin-style methodology for classical logic [18]. The theorem states that, given a CLDS system S and two configurations $\mathcal{C}$ and $\mathcal{C}'$ such that $\mathcal{C}' - \mathcal{C}$ is finite[5], if $\mathcal{C}'$ is semantically entailed from $\mathcal{C}$ then $\mathcal{C}'$ is also derived from $\mathcal{C}$. The methodology adopted to prove the completeness of a CLDS system is diagrammatically represented in Figure 2 and it can be informally described as follows. The proof is of the contrapositive statement (arrow (1)), which states that, given a CLDS system S and two configurations $\mathcal{C}$ and $\mathcal{C}'$ such that $\mathcal{C}' - \mathcal{C}$ is a finite, if $\mathcal{C} \nvdash_{\mathrm{CLDS}} \mathcal{C}'$ then $\mathcal{C} \not\models_{\mathrm{CLDS}} \mathcal{C}'$. This is proved by the composition of two main steps, arrows (2) and (3). Arrow (3) is already given by Definition 11, while arrow (2) represents the main part of the theorem. The proof of arrow (2) is based on the property *if $\mathcal{C}$ is a consistent configuration then $\mathcal{C}$ is satisfiable*, known as the "Model Existence Lemma". The lemma consists of the following reasoning steps. (Note that the definition of a consistent configuration strictly depends on the CLDS system.)

[5] Obviously, if the configuration difference $\mathcal{C}' - \mathcal{C}$ were infinite, an infinite proof sequence would be required to prove $\mathcal{C}'$ from $\mathcal{C}$.

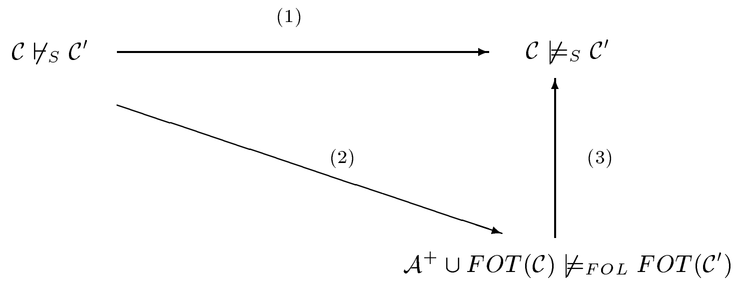

Fig. 2. Proof of the completeness property of a CLDS system S.

1. The hypothesis that $\mathcal{C}'$ is not derivable from $\mathcal{C}$, $\mathcal{C} \nvdash_{\mathrm{CLDS}} \mathcal{C}'$, implies that there exists a $\pi \in \mathcal{C}' - \mathcal{C}$ (where π is a declarative unit or an R-literal) such that $\mathcal{C} \nvdash_{\mathrm{CLDS}} \pi$.
2. The above step implies that the configuration $\mathcal{C}$ extended with $\neg\pi$ (written $\mathcal{C} + [\neg\pi]$) is a consistent configuration.
3. The second step implies that the configuration $\mathcal{C} + [\neg\pi]$ is satisfiable. Therefore, there exists a semantic structure $\mathcal{M}$ of the CLDS system S which satisfies $\mathcal{C}$ and which also satisfies $\neg\pi$. It is then shown that $\mathcal{M}$ does not satisfy π. Thus, since $\pi \in \mathcal{C}'$, by definition of satisfiability of a configuration, $\mathcal{M}$ does not satisfy $\mathcal{C}'$.

3 The $\mathrm{E}_{\mathrm{CLDS}}$ system

In this section, the $\mathrm{E}_{\mathrm{CLDS}}$ system is formally described on the basis of the CLDS approach defined in Section 2. The $\mathrm{E}_{\mathrm{CLDS}}$ language is defined as the ordered pair $\langle \mathcal{L}_P, \mathcal{L}_L \rangle$, where $\mathcal{L}_P$ is a propositional language composed of a countable set of propositional letters $\{p, q, r, \dots\}$, the set of classical connectives $\{\vee, \wedge, \neg, \rightarrow\}$ and the set of modal operators $\{\Box, \Diamond\}$. The labelling language $\mathcal{L}_L$ is a first-order language composed of a countable set of constant symbols $\{s_0, s_1, s_2, \dots\}$, a countable set of variables $\{x, y, z, \dots\}$, the R-predicates R and $=$ and the set of classical connectives and quantifiers. The *semi-extended labelling language* includes two sets of special unary function symbols.

Definition 15. Let $\mathcal{L}_P$ be the $\mathrm{E}_{\mathrm{CLDS}}$ modal language and $\{\alpha_1, \alpha_2, \dots\}$ be the set of all wffs of $\mathcal{L}_P$. The *semi-extended labelling language* $Func(\mathcal{L}_P, \mathcal{L}_L)$ is defined as the language $\mathcal{L}_L$ extended with the sets of unary function symbols $\{f_{\alpha_1}, f_{\alpha_2}, \dots\}$ and $\{box_{\alpha_1}, box_{\alpha_2}, \dots\}$.

As mentioned in Section 2, terms of the form $f_\alpha(\lambda)$ are used to express Kripke semantic notions of the form "there exists a possible world …", whereas terms of the form $box_\alpha(\lambda)$ are used to expressed Kripke semantic notions of the form "for all possible worlds ...". However, formally speaking, $f_\alpha(\lambda)$ and $box_\alpha(\lambda)$ are just terms of $\mathcal{L}_L$ and within a particular model might even refer to the same possible world. The whole set of ground terms of $Func(\mathcal{L}_P, \mathcal{L}_L)$ defines the set of *labels* in the $\mathrm{E_{CLDS}}$ system.

Syntax. The predicate $=$ is introduced in the labelling language $\mathcal{L}_L$ in order to capture the meaning of the Kripke semantic accessibility relation of the logic of elsewhere. Within this logic, possible worlds are accessible from each other if and only if they are not equal [12,9]. Syntactically, the $\mathrm{E_{CLDS}}$ language facilitates the formalisation of three types of information, (i) what holds at particular possible worlds, (ii) which worlds are in relation with each other and which are not and (iii) which worlds are equal to each other and which are not. Whereas the first type of information is captured by the *declarative units*, the last two types of information are captured by the following definition of R-literals.

Definition 16 (R-literals). Let $\langle \mathcal{L}_P, \mathcal{L}_L \rangle$ be the $\mathrm{E_{CLDS}}$ language. An R-literal is any literal of the form $R(\lambda_1, \lambda_2)$, $\neg R(\lambda_1, \lambda_2)$, $\lambda_1 = \lambda_2$ and $\lambda_1 \neq \lambda_2$, where λ_1 and λ_2 are ground terms of the language $Func(\mathcal{L}_P, \mathcal{L}_L)$.

$\mathrm{E_{CLDS}}$ theories are as described in Definition 2, but with respect to the above definition of R-literals. The syntax of the $\mathrm{E_{CLDS}}$ system allows arbitrary sets of modal formulae to be associated with (different) labels, describing not only one initial set of local assumptions (as in the implicit approach of the logic of elsewhere [12]) but allowing for several (distinct) local initial modal theories to be specified. With the addition of R-literals, these local theories can be stated to be related to the same possible world or to different possible worlds and therefore interacting with each other. This yields to a definition of a $\mathrm{E_{CLDS}}$ theory more general than the traditional notion of a modal theory given in [18,14,9].

The essential component which uniquely characterises the $\mathrm{E_{CLDS}}$ system is the *labelling algebra*. This is given by the standard equality theory, which is the reflexivity axiom, $\forall x(x = x)$, and the equality substitution axiom schemas for box_α and f_α functional terms, $\forall x, y(x = y \rightarrow box_\alpha(x) = box_\alpha(y))$, $\forall x, y(x = y \rightarrow f_\alpha(x) = f_\alpha(y))$, the equality predicate, $\forall x, y, z(x = y \wedge x = z \rightarrow y = z)$, together with the special axiom schema (**E**) defined below.

Definition 17 (Labelling algebra $\mathcal{A}_\mathrm{E}$). The *labelling algebra* $\mathcal{A}_\mathrm{E}$ is the first-order theory, written in the language $Func(\mathcal{L}_P, \mathcal{L}_L)$, given by the s-tandard equality theory together with the following axiom:

(E) $\forall x, y(R(x, y) \leftrightarrow (x \neq y))$

Axiom (**E**) expresses the meaning of the Kripke accessibility relation in the specific case of the logic of elsewhere, for which only the possible worlds different from a given possible world are accessible from it. The $\mathrm{E_{CLDS}}$ system is then defined by the tuple $\langle \langle \mathcal{L}_P, \mathcal{L}_L \rangle, \mathcal{A}_\mathrm{E}, \mathcal{R}_\mathrm{E} \rangle$, where the set of inference rules $\mathcal{R}_\mathrm{E}$ is described in the following section.

3.1 Proof theory and Semantics of the $\mathrm{E}_{\mathrm{CLDS}}$

To give a full definition of the $\mathrm{E}_{\mathrm{CLDS}}$ system it is necessary to specify the set of inference rules for the classical and modal operators of the language $\mathcal{L}_P$, as well as the set of inference rules for reasoning about relationships, equality and inequality between possible worlds. In the $\mathrm{E}_{\mathrm{CLDS}}$ system, given an antecedent configuration $\mathcal{C}$ three types of reasoning step can occur. Those of the first type are "classical", and occur within any particular local modal theory included in $\mathcal{C}$, respecting standard notions of inference for classical connectives. These are given in Table 3 together with a special rule, called $(\mathcal{I}_{Sub})$. Note that rules for $\rightarrow$ and $\neg$ connectives are a specialised version

Table 3. Natural deduction rules for classical connectives.

$$\frac{\begin{array}{c}\mathcal{C}\langle[\alpha:\lambda]\rangle\\ \vdots\\ \tilde{\mathcal{C}}\langle\bot:\lambda'\rangle\end{array}}{\mathcal{C}'\langle\neg\alpha:\lambda\rangle}\ (\neg\mathcal{I}) \qquad \frac{\begin{array}{c}\mathcal{C}\langle[\alpha:\lambda]\rangle\\ \vdots\\ \tilde{\mathcal{C}}\langle\beta:\lambda\rangle\end{array}}{\mathcal{C}'\langle\alpha\rightarrow\beta:\lambda\rangle}\ (\rightarrow\mathcal{I})$$

$$\frac{\mathcal{C}\langle\neg\neg\alpha:\lambda\rangle}{\mathcal{C}'\langle\alpha:\lambda\rangle}\ (\neg\neg) \qquad \frac{\mathcal{C}\langle\alpha\rightarrow\beta:\lambda,\alpha:\lambda\rangle}{\mathcal{C}'\langle\beta:\lambda\rangle}\ (\rightarrow\mathcal{E})$$

$$\frac{\mathcal{C}\langle\alpha\vee\beta:\lambda\rangle \quad \begin{array}{c}\mathcal{C}\langle[\alpha:\lambda]\rangle\\ \vdots\\ \tilde{\mathcal{C}}\langle\gamma:\lambda\rangle\end{array} \quad \begin{array}{c}\mathcal{C}\langle[\beta:\lambda]\rangle\\ \vdots\\ \overline{\mathcal{C}}\langle\gamma:\lambda\rangle\end{array}}{\mathcal{C}'\langle\gamma:\lambda\rangle}\ (\vee\mathcal{E}) \qquad \frac{\mathcal{C}\langle\alpha:\lambda\rangle}{\mathcal{C}'\langle\alpha\vee\beta:\lambda\rangle}\ (\vee\mathcal{I})$$

$$\frac{\mathcal{C}\langle\alpha\wedge\beta:\lambda\rangle}{\mathcal{C}'\langle\alpha:\lambda\rangle}\ (\wedge\mathcal{E}) \qquad \frac{\mathcal{C}\langle\alpha\wedge\beta:\lambda\rangle}{\mathcal{C}'\langle\beta:\lambda\rangle}\ (\wedge\mathcal{E})$$

$$\frac{\mathcal{C}\langle\alpha:\lambda,\beta:\lambda\rangle}{\mathcal{C}'\langle\alpha\wedge\beta:\lambda\rangle}\ (\wedge\mathcal{I}) \qquad \frac{\mathcal{C}\langle\alpha:\lambda,\lambda=\lambda'\rangle}{\mathcal{C}'\langle\alpha:\lambda'\rangle}\ (\mathcal{I}_{Sub})$$

of the introduction and elimination rules given in Table 1. The $(\mathcal{I}_{Sub})$ rule expresses a specific form of interaction between the R-literals constructed from the $=$ predicate and the declarative units. This interaction is similar to an equality substitution property for declarative units. With respect to the notion of length of an inference rule given in Section 2, $(\mathcal{I}_{Sub})$ has length equal to 1, i.e. $(\mathcal{I}_{Sub})\in\mathcal{I}^0$.

Rules of the second type are "modal" and concern the interaction between different modal theories in $\mathcal{C}$. These are given in Table 4. The $(\Diamond\mathcal{E})$ rule can be seen (informally) as a "skolemization" of the existential quantifier

Table 4. Natural deduction rules for modal operators.

$$\frac{\mathcal{C}\langle\lambda : \Diamond\alpha\rangle}{\mathcal{C}'\langle f_\alpha(\lambda) : \alpha, R(\lambda, f_\alpha(\lambda))\rangle}\ (\Diamond\mathcal{E}) \qquad \frac{\mathcal{C}\langle\lambda_2 : \alpha, R(\lambda_1, \lambda_2)\rangle}{\mathcal{C}'\langle\lambda_1 : \Diamond\alpha\rangle}\ (\Diamond\mathcal{I})$$

$$\frac{\mathcal{C}\langle\lambda_1 : \Box\alpha, R(\lambda_1, \lambda_2)\rangle}{\mathcal{C}'\langle\lambda_2 : \alpha\rangle}\ (\Box\mathcal{E}) \qquad \frac{\begin{array}{c}\mathcal{C}\langle[R(\lambda, box_\alpha(\lambda))]\rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle box_\alpha(\lambda) : \alpha\rangle\end{array}}{\mathcal{C}'\langle\lambda : \Box\alpha\rangle}\ (\Box\mathcal{I})$$

over possible worlds which is semantically implied by the formula $\Diamond\alpha$ in the premise. The term $f_\alpha(\lambda)$ defines a particular possible world uniquely associated with the formula α, and inferred to be accessible from the possible world λ (i.e. $R(\lambda, f_\alpha(\lambda))$). It is clear from the definition that this rule has the effect of expanding both components (diagram and set of declarative units) of the antecedent configuration. In the $(\Box\mathcal{I})$ rule, the temporary assumption $R(\lambda, box_\alpha(\lambda))$ should be read as "given an arbitrary world accessible from λ", using then the term $box_\alpha(\lambda)$ not to name particular objects (possible worlds) but to refer to an arbitrary object.

Both classical and modal reasoning steps are based on the logical (classical and modal) information (wffs) incorporated in the declarative units that belong to $\mathcal{C}$. The third type of reasoning step is instead related to the diagram information in $\mathcal{C}$ and to the "interaction" between the diagram and the declarative units. In this case, inferred configurations are often "structural expansions" of (i.e. additions of R-literals to) the antecedent configurations. These are identical to those given in Table 2, but with the meta-variables Δ and $\overline{\Delta}$ referring respectively to the notions of R-literal given in Definition 16 and its conjugate. The rule (R-A) in particular, facilitates reasoning about the diagram of a configuration, using the specific labelling algebra $\mathcal{A}_{\mathrm{E}}$. For instance, the (R-A) rule and the (**E**) axiom of the labelling algebra $\mathcal{A}_{\mathrm{E}}$ give the symmetry property of the accessibility relation R. This is proved using the symmetry property of the $\neq$ predicate, which is given by the symmetry of the equality predicate $=$ included in $\mathcal{A}_{\mathrm{E}}$. An example graphical representation of a derivation is given in Figure 3, which shows the proof of the axiom $(\Box\alpha \wedge \alpha) \rightarrow \Box\Box\alpha$ at any arbitrary world λ, that characterises the logic of elsewhere.

Semantics. The semantics of the $\mathrm{E_{CLDS}}$ system is based on the model theoretic semantics defined in Section 2.3 for a general CLDS system. The *extended algebra* $\mathcal{A}_{\mathrm{E}}^{+}$ of the $\mathrm{E_{CLDS}}$ system is formally given below.

$\mathcal{C}_\emptyset\langle\rangle$	
$\mathcal{C}_1\langle[\Box\alpha \wedge \alpha : \lambda]\rangle$	(assumption)
$\mathcal{C}_2\langle\Box\alpha : \lambda, \alpha : \lambda\rangle$	$(\wedge\mathcal{E})$
$\mathcal{C}_3\langle[R(\lambda, box_{\Box\alpha}(\lambda))]\rangle$	(assumption)
$\mathcal{C}_4\langle[R(box_{\Box\alpha}(\lambda), box_\alpha(box_{\Box\alpha}(\lambda)))]\rangle$	(assumption)
$\mathcal{C}_5\langle[\neg\alpha : box_\alpha(box_{\Box\alpha}(\lambda))]\rangle$	(assumption)
$\mathcal{C}_6\langle[R(\lambda, box_\alpha(box_{\Box\alpha}(\lambda)))]\rangle$	(assumption)
$\mathcal{C}_7\langle\alpha : box_\alpha(box_{\Box\alpha}(\lambda))\rangle$	$(\Box\mathcal{E})$
$\mathcal{C}_8\langle\bot : box_\alpha(box_{\Box\alpha}(\lambda))\rangle$	$(\wedge\mathcal{I})$
$\mathcal{C}_9\langle\neg R(\lambda, box_\alpha(box_{\Box\alpha}(\lambda)))\rangle$	(R-I)
$\mathcal{C}_{10}\langle\lambda = box_\alpha(box_{\Box\alpha}(\lambda))\rangle$	(R-A)
$\mathcal{C}_{11}\langle\alpha : box_\alpha(box_{\Box\alpha}(\lambda))\rangle$	$(\mathcal{I}_{Sub})$
$\mathcal{C}_{12}\langle\bot : box_\alpha(box_{\Box\alpha}(\lambda))\rangle$	$(\wedge\mathcal{I})$
$\mathcal{C}_{13}\langle\alpha : box_\alpha(box_{\Box\alpha}(\lambda))\rangle$	$(\neg\mathcal{I})$
$\mathcal{C}_{14}\langle\Box\alpha : box_{\Box\alpha}(\lambda)\rangle$	$(\Box\mathcal{I})$
$\mathcal{C}_{15}\langle\Box\Box\alpha : \lambda\rangle$	$(\Box\mathcal{I})$
$\mathcal{C}_{16}\langle(\Box\alpha \wedge \alpha) \rightarrow \Box\Box\alpha : \lambda\rangle$	$(\rightarrow\mathcal{I})$

Fig. 3. $\mathrm{E}_{\mathrm{CLDS}}$ derivation of the (E3) axiom

Definition 18 (Extended algebra $\mathcal{A}_{\mathrm{E}}^{+}$). Given the extended labelling language $Mon(\mathcal{L}_P, \mathcal{L}_L)$ and the labelling algebra $\mathcal{A}_{\mathrm{E}}$, the *extended algebra* $\mathcal{A}_{\mathrm{E}}^{+}$ is the theory in $Mon(\mathcal{L}_P, \mathcal{L}_L)$, given by $\mathcal{A}_{\mathrm{E}}$ extended with the following axiom schemas (Ax1)–(Ax9). For any wffs α and β of $\mathcal{L}_P$:

(Ax1) $\forall x([\alpha \wedge \beta]^*(x) \leftrightarrow ([\alpha]^*(x) \wedge [\beta]^*(x)))$
(Ax2) $\forall x([\neg\alpha]^*(x) \leftrightarrow \neg[\alpha]^*(x))$
(Ax3) $\forall x([\alpha \vee \beta]^*(x) \leftrightarrow ([\alpha]^*(x) \vee [\beta]^*(x)))$
(Ax4) $\forall x([\alpha \rightarrow \beta]^*(x) \leftrightarrow ([\alpha]^*(x) \rightarrow [\beta]^*(x)))$
(Ax5) $\forall x([\Diamond\alpha]^*(x) \rightarrow (R(x, f_\alpha(x)) \wedge [\alpha]^*(f_\alpha(x))))$
(Ax6) $\forall x(\exists y(R(x,y) \wedge [\alpha]^*(y)) \rightarrow [\Diamond\alpha]^*(x))$
(Ax7) $\forall x((R(x, box_\alpha(x)) \rightarrow [\alpha]^*(box_\alpha(x))) \rightarrow [\Box\alpha]^*(x))$
(Ax8) $\forall x([\Box\alpha]^*(x) \rightarrow (\forall y(R(x,y) \rightarrow [\alpha]^*(y))))$
(Ax9) $\forall x, y([\alpha]^*(x) \wedge x = y \rightarrow [\alpha]^*(y))$

The first four axiom schemas express the distributive properties of the logical connectives among the monadic predicates of $Mon(\mathcal{L}_P, \mathcal{L}_L)$. The axiom schemas (Ax1)–(Ax8) reflect the traditional Kripke semantic definition of satisfiability of modal wffs[6]. The axiom schemas (Ax5)–(Ax8), together with the

[6] This is easily seen by interpreting the truth of $[\alpha]^*(x)$ as the truth of the modal formula α in the possible world x.

axiom (**E**) of the labelling algebra $\mathcal{A}_{\mathrm{E}}$, express the *specific* semantic meaning of the modal operators $\Diamond$ and $\Box$ for the logic of elsewhere, and (Ax9) is the equality substitution schema extended to each predicate symbol $[\alpha]^*$.

The notions of satisfiability and semantic entailment of the $\mathrm{E}_{\mathrm{CLDS}}$ system, denoted with $\models_{\mathrm{E}}$, are as specified in Definitions 10 and 11, but based on the extended algebra $\mathcal{A}_{\mathrm{E}}^{+}$. Soundness and completeness of the $\mathrm{E}_{\mathrm{CLDS}}$ proof system with respect to the semantic entailment $\models_{\mathrm{E}}$ are proved in the following section.

3.2 Main results about the $\mathrm{E}_{\mathrm{CLDS}}$ system

Soundness. The soundness and completeness proofs, based respectively on the two methodologies described in Section 2.3, take advantage of the soundness and completeness of first-order logic. Most of the theorems, lemmas and propositions used to prove these two properties extend those given in [24], in order to cover additional cases corresponding to the equality and inequality between possible worlds and to the special rule $(\mathcal{I}_{Sub})$. Hence, the proofs described in this paper will consider only these extended cases. The reader is referred to [24] for the remaining parts of the proofs.

Theorem 19 (Soundness of $\mathrm{E}_{\mathrm{CLDS}}$). *Let $E = \langle\langle\mathcal{L}_P, \mathcal{L}_L\rangle, \mathcal{A}_{\mathrm{E}}, \mathcal{R}_{\mathrm{E}}\rangle$ be the E_{CLDS} system and let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations. If $\mathcal{C} \vdash_{\mathrm{E}} \mathcal{C}'$ then $\mathcal{C} \models_{\mathrm{E}} \mathcal{C}'$.*

The proof of Theorem 19, represented diagrammatically in Figure 1, is mainly based on the proof of the following lemma.

Lemma 20. *Let $\mathcal{A}_{\mathrm{E}}^{+}$ be the extended algebra of the E_{CLDS} system, let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations and let $FOT(\mathcal{C})$ and $FOT(\mathcal{C}')$ be their respective first-order translations. If $\mathcal{C} \vdash_{\mathrm{E}} \mathcal{C}'$ then $\mathcal{A}_{\mathrm{E}}^{+}, FOT(\mathcal{C}) \vdash_{\mathrm{FOL}} FOT(\mathcal{C}')$.*

Proof: The proof is by induction on the size of smallest derivations of the form $\langle\{\mathcal{C}_0, \dots, \mathcal{C}_n\}, \overline{m}\rangle$, where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_n = \mathcal{C}'$. In what follows $\langle\{\mathcal{C}_0, \dots, \mathcal{C}_n\}, m\rangle$ is a proof of this smallest size with length $l \geq 0$. The base case is when $l = 0$. This means, by Definition 14, that $\mathcal{C}' \subseteq \mathcal{C}$, hence the theorem trivially follows. The inductive step is proved by cases on the inference rule (different from the ($\mathcal{C}$-R) rule[7]) applied on the last step $\mathcal{C}_{n-1}/\mathcal{C}_n$ of the derivation. The theorem holds by inductive hypothesis for the first part of the derivation. It is sufficient to show that $\mathcal{A}_{\mathrm{E}}^{+}, FOT(\mathcal{C}_{n-1}) \vdash_{\mathrm{FOL}} FOT(\mathcal{C}_n)$. For Cases 1–16, corresponding to the inference rules given in Tables 2, 3 and 4 except the $(\mathcal{I}_{Sub})$, the reader is referred to [24].
Case17: $(\mathcal{I}_{Sub})$.
In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{Sub}$. Then there exist a declarative unit $\alpha : \lambda$ and an R-literal of the form $\lambda = \lambda'$ such that $\{\alpha : \lambda, \lambda = \lambda'\} \subseteq \mathcal{C}_{n-1}$. Therefore, the

[7] Note that the lemma can trivially be proved to hold for proofs obtained by extending those considered here with a ($\mathcal{C}$-R) rule on the last step.

set $\{[\alpha]^*(\lambda), \lambda = \lambda'\} \subseteq FOT(\mathcal{C}_{n-1})$. This implies, by applying the equality substitution axiom of $\mathcal{A}_\mathrm{E}^+$ to the predicate $[\alpha]^*$, that $\mathcal{A}_\mathrm{E}^+, FOT(\mathcal{C}_{n-1}) \vdash_\mathrm{FOL} [\alpha]^*(\lambda')$. Since $\mathcal{C}_n = \mathcal{C}_{n-1} + [\alpha : \lambda']$, $\mathcal{A}_\mathrm{E}^+, FOT(\mathcal{C}_{n-1}) \vdash_\mathrm{FOL} FOT(\mathcal{C}_n)$. □

Proof of Theorem 19:
By hypothesis $\mathcal{C} \vdash_\mathrm{E} \mathcal{C}'$. By Lemma 20, $\mathcal{A}_\mathrm{E}^+, FOT(\mathcal{C}) \vdash_\mathrm{FOL} FOT(\mathcal{C}')$. By soundness of first-order logic $\mathcal{A}_\mathrm{E}^+, FOT(\mathcal{C}) \models_\mathrm{FOL} FOT(\mathcal{C}')$. Hence, by definition of semantic entailment, $\mathcal{C} \models_\mathrm{E} \mathcal{C}'$.

Completeness. The completeness theorem states that, given the $\mathrm{E_{CLDS}}$ system and two configurations $\mathcal{C}$ and $\mathcal{C}'$, whose difference is finite, if $\mathcal{C}'$ is semantically entailed from $\mathcal{C}$ then $\mathcal{C}'$ is also derived from $\mathcal{C}$. This is formally defined below.

Theorem 21 (Completeness of $\mathrm{E_{CLDS}}$). *Let $E = \langle\langle \mathcal{L}_P, \mathcal{L}_L\rangle, \mathcal{A}_\mathrm{E}, \mathcal{R}_\mathrm{E}\rangle$ be a $\mathrm{E_{CLDS}}$ system and let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations such that $\mathcal{C}' - \mathcal{C}$ is a finite configuration. If $\mathcal{C} \models_\mathrm{E} \mathcal{C}'$ then $\mathcal{C} \vdash_\mathrm{E} \mathcal{C}'$.*

As discussed in Section 2.3, the Henkin-style methodology for classical logic is used here to prove Theorem 21. Basic notions in this proof are those of *consistent* and *maximal consistent* configurations.

Definition 22 (Consistent configuration). Given the $\mathrm{E_{CLDS}}$ system and a configuration $\mathcal{C}$, $\mathcal{C}$ is *inconsistent* if $\mathcal{C} \vdash_\mathrm{E} \bot : \lambda$ for some ground term λ of $Func(\mathcal{L}_P, \mathcal{L}_L)$. $\mathcal{C}$ is *consistent* if it is not inconsistent.

Definition 23 (Maximal Consistent Configuration). Given the $\mathrm{E_{CLDS}}$ system, a configuration $\mathcal{C}_\mathrm{mcc}$ is a *maximal consistent* configuration of $\mathrm{E_{CLDS}}$, if (i) it is consistent and (ii) for any $\pi \notin \mathcal{C}_\mathrm{mcc}$ (where π is a declarative unit or an R-literal), the configuration $\mathcal{C}_\mathrm{mcc} + [\pi]$ is not consistent.

The proof of Theorem 21 is by contraposition. It is shown, using the construction of a *canonical* model of the $\mathrm{E_{CLDS}}$ system, that

$$\mathcal{C} \not\vdash_\mathrm{E} \mathcal{C}' \text{ implies } \mathcal{C} \not\models_\mathrm{E} \mathcal{C}' \tag{4}$$

In standard Henkin-style proofs of completeness for modal logics, the canonical model is obtained by progressively building maximal consistent sets (see for example [18]) where consistency is locally checked according to the properties of the underlying accessibility relation. In the CLDS approach, the explicit declarative representation of possible worlds and of relationships between possible worlds facilitates the construction of a canonical model for the $\mathrm{E_{CLDS}}$ system by simply extending a given consistent configuration into a single maximal consistent configuration where consistency is then checked globally. This is shown in the following lemma.

Lemma 24. *Given the $\mathrm{E_{CLDS}}$ system, every consistent configuration $\mathcal{C}$ can be extended to a maximal consistent configuration $\mathcal{C}_\mathrm{mcc}$.*

PROOF: Let $\mathcal{C}$ be a consistent configuration and let $\pi_1, \pi_2, \pi_3, \ldots, \pi_n, \ldots$ be an ordering on the set of all declarative units and R-literals of $\langle \mathcal{L}_P, \mathcal{L}_L \rangle$. Starting from $\mathcal{C}_0 = \mathcal{C}$, a sequence of consistent configurations $\mathcal{C}_i$ is constructed by inductively defining, for each element π, $\mathcal{C}_i$ to be

$$\mathcal{C}_i = \begin{cases} \mathcal{C}_{i-1} + [\pi] & \text{if } \mathcal{C}_{i-1} + [\pi_i] \text{ is consistent} \\ \mathcal{C}_{i-1} & \text{otherwise} \end{cases}$$

It is easy to show that, for each $i \geq 0$, $\mathcal{C}_i$ is a consistent configuration. Now, let $\mathcal{C}_{\mathrm{mcc}}$ be the configuration obtained from the union of all $\mathcal{C}_i$, $\mathcal{C}_{\mathrm{mcc}} = \cup_{i \geq 0} \mathcal{C}_i$. It is easy to show that $\mathcal{C}_{\mathrm{mcc}}$ is maximal and consistent (see Proposition 3.8 in [24]). Hence $\mathcal{C}_{\mathrm{mcc}}$ is a maximal consistent configuration. □

Within a maximal consistent configuration declarative units and R-literals satisfy particular properties. These are listed in the following lemma. Only the properties related to R-literals constructed from the = predicate are proved here. The reader is referred to [24] for formal proofs of the remaining cases.

Lemma 25. *Let $\mathcal{C}_{\mathrm{mcc}}$ be a maximal consistent configuration of E_{CLDS}. Then for any R-literal π, any wffs α and β and any label λ,*

1. *π and $\overline{\pi}$ are not both in $\mathcal{C}_{\mathrm{mcc}}$.*
2. *Either $\pi \in \mathcal{C}_{\mathrm{mcc}}$ or $\overline{\pi} \in \mathcal{C}_{\mathrm{mcc}}$.*
3. *$\alpha : \lambda$ and $\neg\alpha : \lambda$ are not both in $\mathcal{C}_{\mathrm{mcc}}$.*
4. *Either $\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ or $\neg\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$.*
5. *$\alpha \wedge \beta : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ if and only if $\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ and $\beta : \lambda \in \mathcal{C}_{\mathrm{mcc}}$.*
6. *$\alpha \vee \beta : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ if and only if $\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ or $\beta : \lambda \in \mathcal{C}_{\mathrm{mcc}}$.*
7. *$\alpha \to \beta : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ if and only if if $\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ then $\beta : \lambda \in \mathcal{C}_{\mathrm{mcc}}$.*
8. *If $\Box\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ and $R(\lambda, \lambda') \in \mathcal{C}_{\mathrm{mcc}}$ then $\alpha : \lambda' \in \mathcal{C}_{\mathrm{mcc}}$.*
9. *If $\neg R(\lambda, box_\alpha(\lambda)) \in \mathcal{C}_{\mathrm{mcc}}$ or $\alpha : box_\alpha(\lambda) \in \mathcal{C}_{\mathrm{mcc}}$, then $\Box\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$.*
10. *If $\Diamond\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ then $R(\lambda, f_\alpha(\lambda)) \in \mathcal{C}_{\mathrm{mcc}}$ and $\alpha : f_\alpha(\lambda) \in \mathcal{C}_{\mathrm{mcc}}$.*
11. *If $R(\lambda, \lambda') \in \mathcal{C}_{\mathrm{mcc}}$ and $\alpha : \lambda' \in \mathcal{C}_{\mathrm{mcc}}$, then $\Diamond\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$.*
12. *$R(\lambda, \lambda') \in \mathcal{C}_{\mathrm{mcc}}$ if and only if $\lambda \neq \lambda' \in \mathcal{C}_{\mathrm{mcc}}$.*
13. *$\lambda = \lambda \in \mathcal{C}_{\mathrm{mcc}}$*
14. *If $\lambda = \lambda_1 \in \mathcal{C}_{\mathrm{mcc}}$ and $\lambda = \lambda' \in \mathcal{C}_{\mathrm{mcc}}$ then $\lambda_1 = \lambda' \in \mathcal{C}_{\mathrm{mcc}}$.*
15. *If $\lambda = \lambda_1 \in \mathcal{C}_{\mathrm{mcc}}$ then $box_\alpha(\lambda) = box_\alpha(\lambda_1) \in \mathcal{C}_{\mathrm{mcc}}$.*
16. *If $\lambda = \lambda_1 \in \mathcal{C}_{\mathrm{mcc}}$ then $f_\alpha(\lambda) = f_\alpha(\lambda_1) \in \mathcal{C}_{\mathrm{mcc}}$.*
17. *If $\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ and $\lambda = \lambda' \in \mathcal{C}_{\mathrm{mcc}}$ then $\alpha : \lambda' \in \mathcal{C}_{\mathrm{mcc}}$.*

PROOF: Property (1) is proved only for the case of π equal to $\lambda = \lambda'$ for arbitrary labels λ, λ'. Suppose that both $\lambda = \lambda'$ and $\lambda \neq \lambda'$ are in $\mathcal{C}_{\mathrm{mcc}}$. Then by definition of the $(\bot\mathcal{E})$ rule $\mathcal{C}_{\mathrm{mcc}} \vdash_{\mathrm{E}} \bot : \lambda$ which contradicts the hypothesis $\mathcal{C}_{\mathrm{mcc}}$ being a maximal consistent configuration.

Property (2) is also proved only for the case of π equal to $\lambda = \lambda'$ for arbitrary labels λ, λ'. Suppose that neither $\lambda = \lambda'$ nor $\lambda \neq \lambda'$ is in $\mathcal{C}_{\mathrm{mcc}}$. Then by definition of maximality $\mathcal{C}_{\mathrm{mcc}} + [\lambda = \lambda'] \vdash_{\mathrm{E}} \bot : \lambda$ and $\mathcal{C}_{\mathrm{mcc}} + [\lambda \neq \lambda'] \vdash_{\mathrm{E}}$

$\bot:\lambda'$. Then it is easy to show (see [24]) that there exist two configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\mathcal{C}_1 \subseteq \mathcal{C}_{\rm mcc}$, $\mathcal{C}_1 + [\lambda = \lambda'] \vdash_{\rm E} \bot:\lambda$, $\mathcal{C}_2 \subseteq \mathcal{C}_{\rm mcc}$ and $\mathcal{C}_2 + [\lambda \neq \lambda'] \vdash_{\rm E} \bot:\lambda'$. By monotonicity of the $\rm E_{CLDS}$ derivability relation, also the configuration $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is such that $\mathcal{C} \subseteq \mathcal{C}_{\rm mcc}$, $\mathcal{C} + [\lambda = \lambda'] \vdash_{\rm E} \bot:\lambda$ and $\mathcal{C} + [\lambda \neq \lambda'] \vdash_{\rm E} \bot:\lambda'$. It is therefore possible, using $(R\mathcal{I})$ rule, to construct a proof showing that $\mathcal{C} \vdash_{\rm E} \bot:\lambda$, which, by the monotonicity of $\vdash_{\rm E}$, contradicts the hypothesis $\mathcal{C}_{\rm mcc}$ be consistent.

Properties (13), (14), (15) and (16), which express the closure of the maximal consistent configuration with respect to reflexivity and equality substitution, are also proved by contradiction using the (R-A) rule. Similarly for property (12). Property (17) is proved by contradiction using $(\mathcal{I}_{Sub})$ rule. For all the other cases, the reader is referred to [24]. □

To prove the *Model Existence Lemma* for the $\rm E_{CLDS}$ system (arrow (2) in Figure 2), it is essential to define the notion of a $\rm E_{CLDS}$ canonical model. This is given with respect to a maximal consistent configuration and the classical notion of a canonical interpretation.

Definition 26 (Canonical Interpretation). Let $\mathcal{C}_{\rm mcc} = \langle \mathcal{D}_{\rm mcc}, \mathcal{F}_{\rm mcc} \rangle$ be a maximal consistent configuration of $\rm E_{CLDS}$ and let $FOT(\mathcal{C}_{\rm mcc})$ be its first-order translation. A canonical model of $\rm E_{CLDS}$ is the pair $(\mathcal{U}, \mathcal{I}_{\rm mcc})$, where $\mathcal{U}$ is the Herbrand universe of the language $Mon(\mathcal{L}_P, \mathcal{L}_L)$ and $\mathcal{I}_{\rm mcc}$ is an interpretation function on the language $Mon(\mathcal{L}_P, \mathcal{L}_L)$ defined as follows.

- For each ground term λ,
  $\| \lambda \|_{I_{\rm mcc}} = \lambda$
- For the binary predicate R,
  $\| R \|_{I_{\rm mcc}} = \{\langle \lambda_i, \lambda_j \rangle \mid R(\lambda_i, \lambda_j) \in FOT(\mathcal{C}_{\rm mcc})\}$ [8]
- For the binary predicate $=$,
  $\|{=}\|_{I_{\rm mcc}} = \{\langle \lambda_i, \lambda_j \rangle \mid \lambda_i = \lambda_j \in FOT(\mathcal{C}_{\rm mcc})\}$
- For each monadic predicate $[\alpha]^*$,
  $\| [\alpha]^* \|_{I_{\rm mcc}} = \{\lambda_i \mid [\alpha]^*(\lambda_i) \in FOT(\mathcal{C}_{mcc})\}$

The following lemma shows that the canonical interpretation constructed in the above definition is a canonical model of the $\rm E_{CLDS}$ system.

Lemma 27 (Canonical model). *Let $\mathcal{C}_{\rm mcc}$ be a maximal consistent configuration of $\rm E_{CLDS}$ and let $(\mathcal{U}, \mathcal{I}_{\rm mcc})$ be a canonical interpretation. Then $(\mathcal{U}, \mathcal{I}_{\rm mcc})$ is a canonical model of $\rm E_{CLDS}$.*

PROOF: To show that $(\mathcal{U}, \mathcal{I}_{\rm mcc})$ is a canonical model of $\rm E_{CLDS}$ it is needed to show, by Definition 10, that $(\mathcal{U}, \mathcal{I}_{\rm mcc})$ is a model of the extended algebra $\mathcal{A}^+_{\rm E}$. This means to show that $(\mathcal{U}, \mathcal{I}_{\rm mcc})$ is a model of the axioms (Ax1)–(Ax9) as well as of the equality theory axioms included in $\mathcal{A}^+_{\rm E}$. This is easy to prove by

[8] Notice that $FOT(\mathcal{C}_{\rm mcc})$ contains only ground literals

using Lemma 25 and the fact that, by definition of canonical interpretation, for each declarative unit and R-literal π, $(\mathcal{U}, \mathcal{I}_{\rm mcc}) \Vdash FOT(\pi)$ if and only if $FOT(\pi) \in FOT(\mathcal{C}_{\rm mcc})$. □

It is now possible to prove the *Model Existence Lemma* for the $\mathrm{E_{CLDS}}$ system.

Lemma 28 (Model Existence Lemma). *Let $\mathcal{C}_{\rm mcc}$ be a maximal consistent configuration, and let $\mathcal{M}_{\rm mcc} = (\mathcal{U}, \mathcal{I}_{\rm mcc})$ be a canonical model of $E_{\rm CLDS}$. Then for any π (where π is a declarative unit or an R-literal) of $E_{\rm CLDS}$, $\mathcal{M}_{\rm mcc} \models_{\rm E} \pi$ if and only if $\pi \in \mathcal{C}_{\rm mcc}$.*

Proof: Only the case of π equal to an R-literal constructed from the $=$ predicate is considered here. For the other cases, the reader is referred to [24]. Let π be of the form $\lambda = \lambda'$. If $\lambda = \lambda' \in \mathcal{C}_{\rm mcc}$ then $\lambda = \lambda' \in FOT(\mathcal{C}_{\rm mcc})$. This implies, by Definition 26, that $\mathcal{M}_{\rm mcc} \models_{\rm E} \lambda = \lambda'$. If $\lambda = \lambda' \notin \mathcal{C}_{\rm mcc}$ then by Lemma 25 $\lambda \neq \lambda' \in \mathcal{C}_{\rm mcc}$, which implies that $\mathcal{M}_{\rm mcc} \models_{\rm E} \lambda \neq \lambda'$ and hence $\mathcal{M}_{\rm mcc} \not\models_{\rm E} \lambda = \lambda'$. The case for π of the form $\lambda \neq \lambda'$ is similar. □

Corollary 29. *Let $\mathcal{C}$ be a consistent configuration of the $E_{\rm CLDS}$ system. Then $\mathcal{C}$ is satisfiable.*

Proof: The proof trivially follows from Lemmas 24 and 28. □

The following proposition is the final result needed to prove the completeness theorem.

Proposition 30. *Let $\mathcal{C}$ be a configuration of the $E_{\rm CLDS}$ system and let π be a declarative unit or an R-literal such that $\pi \notin \mathcal{C}$. If $\mathcal{C} \not\vdash_{\rm E} \pi$ then $\mathcal{C} + [\pi]$ is a consistent configuration.*

Proof: Only the case of π equal to a R-literal constructed from the $=$ predicate is considered. For the other cases, the reader is referred to [24]. Let π be of the form $\lambda = \lambda'$. The contrapositive of the proposition is proved. Suppose that $\mathcal{C} + [\lambda \neq \lambda']$ is not consistent. Then $\mathcal{C} + [\lambda \neq \lambda'] \vdash_{\rm E} \bot : \lambda_1$. By definition of $(R\mathcal{I})$, the configuration $\mathcal{C}' = \mathcal{C} + [\lambda = \lambda']$ is derivable from $\mathcal{C}$. Hence, $\mathcal{C} \vdash_{\rm E} \lambda = \lambda'$. Similarly for π equal to $\lambda \neq \lambda'$. □

The proof of Theorem 21 can now be given.

Proof of Theorem 21:

The proof is by contrapositive. Assume that $\mathcal{C} \not\vdash_{\rm E} \mathcal{C}'$. Then by Lemma 7 there exists a $\pi \in \mathcal{C}' - \mathcal{C}$, where π is a declarative unit or an R-literal, such that $\mathcal{C} \not\vdash_{\rm E} \pi$. Then by Proposition 30, $\mathcal{C} + [\neg\pi]$ is a consistent configuration. By Corollary 29, $\mathcal{C} + [\neg\pi]$ is satisfiable. Let $\mathcal{M}$ be the canonical model that satisfies the configuration $\mathcal{C} + [\neg\pi]$. So $\mathcal{M} \models_{\rm E} \mathcal{C}$ and $\mathcal{M} \models_{\rm E} \neg\pi$. There are three cases to consider, according to the form of π. Only the case of π equal to an R-literal of the form $\lambda = \lambda'$ is considered here. For the other two cases the reader is referred to [24]. Let π be of the form $\lambda = \lambda'$. By Definition 10, $\mathcal{M} \Vdash_{\mathrm{FOL}} \lambda \neq \lambda'$, which implies that $\mathcal{M} \nVdash_{\mathrm{FOL}} \lambda = \lambda'$. Then by Definitions 10 and 11 $\mathcal{A}^+, FOT(\mathcal{C}) \nVdash_{\mathrm{FOL}} \lambda = \lambda'$. Hence $\mathcal{C} \not\models_{\mathrm{E}} \mathcal{C}'$.

Correspondence result. In Section 1, it has been stated that the $\mathrm{E}_{\mathrm{CLDS}}$ system is a *generalisation* of the standard implicit formalisation of the modal logic of elsewhere, in that it facilitates reasoning about structures of actual worlds, which may or may not be singleton structures. This claim is substantiated here by showing (i) that there exists a correspondence between the $\mathrm{E}_{\mathrm{CLDS}}$ system and the Hilbert system for the logic of elsewhere, whenever certain restrictions are imposed on initial configurations, and (ii) that the correspondence clearly fails if no restriction is imposed. As far as the first result is concerned, the restriction consists of allowing initial configurations only of the form $\mathcal{C}_0 = \langle \{\}, \mathcal{F}_0 \rangle$ where for any label λ, $\mathcal{F}_0(\lambda) = \{\}$. In particular, the following theorem shows that any declarative unit of the form $\alpha : \lambda$, for any label λ, can be derived from an empty initial configuration $\mathcal{C}_0$ if and only if its formula α is a theorem within a sound and complete Hilbert system for modal logic [9]. A definition of the Hilbert system taken into consideration is first given.

Definition 31. Let $\mathcal{L}_P$ be the propositional modal logic considered in the $\mathrm{E}_{\mathrm{CLDS}}$ system. The Hilbert system for the logic of elsewhere, written $\mathcal{E}_{Ax}$, is a standard propositional logic axiomatisation [18] extended with the following schemas:

(E1) $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$
(E2) $\alpha \rightarrow \Box\Diamond\alpha$
(E3) $\Diamond\Diamond\alpha \rightarrow (\alpha \vee \Diamond\alpha)$

together with the (MP) and (Nec) rules:

(Nec) If $\vdash_{\mathcal{E}_{Ax}} \alpha$ then $\vdash_{\mathcal{E}_{Ax}} \Box\alpha$

Theorem 32 (Simple correspondence). *Consider the E_{CLDS} system, the Hilbert system $\mathcal{E}_{Ax}$, and the initial empty configuration $\mathcal{C}_{\{\}} = \langle \{\}, \mathcal{F} \rangle$, given by $\mathcal{F}(\lambda) = \{\}$, for any label λ. Let α be a formula of $\mathcal{L}_P$. Then:*

$\vdash_{\mathcal{E}_{Ax}} \alpha$ if and only if
for all ground terms $\lambda \in Mon(\mathcal{L}_P, \mathcal{L}_L)$ $\mathcal{C}_{\{\}} \vdash_{\mathrm{E}} \alpha : \lambda$

PROOF: ("Only if") part: The proof is by induction on the number of steps of the shortest derivation $\vdash_{\mathcal{E}_{Ax}} \alpha$. The formal proof is given in [24] but with the following three extra cases on the base case of the induction. The base case is when there are zero number of steps, i.e. α is an instantiation of the schemas given in Definition 31. It is sufficient to prove that $\mathcal{C}_{\{\}} \vdash_{\mathrm{E}} \alpha : \lambda$, for α equal to (E1), (E2) and (E3). Only the case of the (E2) schema is considered here, since the proof of the (E3) is already given in Figure 3 and the proof of (E1) schema is fully given in [24]. Let λ be a ground term of $Mon(\mathcal{L}_P, \mathcal{L}_L)$. Then $\mathcal{C}_{\{\}} \vdash_{\mathrm{E}} \Box\alpha \rightarrow \Box\Diamond\alpha : \lambda$ as given by Figure 4.

$\mathcal{C}_\emptyset\langle\rangle$	
$\quad\mathcal{C}_1\langle[\alpha:\lambda]\rangle$	(assumption)
$\quad\quad\mathcal{C}_2\langle[R(\lambda, box_{\Diamond\alpha}(\lambda))]\rangle$	(assumption)
$\quad\quad\mathcal{C}_3\langle\lambda \neq box_{\Diamond\alpha}(\lambda)\rangle$	(R-A)
$\quad\quad\mathcal{C}_4\langle box_{\Diamond\alpha}(\lambda) \neq \lambda\rangle$	(R-A)
$\quad\quad\mathcal{C}_5\langle R(box_{\Diamond\alpha}(\lambda), \lambda)\rangle$	(R-A)
$\quad\quad\mathcal{C}_6\langle\Diamond\alpha : box_{\Diamond\alpha}(\lambda)\rangle$	($\Diamond\mathcal{I}$)
$\quad\mathcal{C}_7\langle\Box\Diamond\alpha:\lambda\rangle$	($\Box\mathcal{I}$)
$\mathcal{C}_8\langle\alpha \rightarrow \Box\Diamond\alpha:\lambda\rangle$	($\rightarrow\mathcal{I}$)

Fig. 4. Derivation of the (E2) axiom

("If") part: The proof is by showing that the contrapositive statement holds. Given the soundness and completeness of both systems under consideration, this means to show that if $\not\models_{\mathcal{E}_{Ax}} \alpha$ then there exists a ground term λ such that $\mathcal{C}_{\{\}} \not\models_{\mathrm{E}} \alpha:\lambda$. The formal proof is given in [24] and it informally consists of constructing a classical interpretation $\mathcal{M}$ from the Kripke countermodel of α and showing that $\mathcal{M}$ is a model of the $\mathrm{E}_{\mathrm{CLDS}}$ system (i.e. it satisfies the schemas of the extended algebra $\mathcal{A}^+_{\mathrm{E}}$). This implies that there exists a ground term λ, specifically the one corresponding to the Kripke possible world where α is false, such that $\mathcal{M} \not\models_{\mathrm{FOL}} [\alpha]^*(\lambda)$[9]. Hence $\mathcal{C}_\emptyset \not\models_{\mathrm{E}} \alpha:\lambda$. $\Box$

It is easy to show that the above theorem can be generalised to global and local assumptions of the logic of elsewhere using the notation introduced by Fitting in [14] (i.e. $T \models_{\mathcal{E}_{Ax}} U \Rightarrow \alpha$ denote that the formula α is derivable from the global assumptions T and the local assumptions U). This is achieved by considering initial configurations of the form $\mathcal{C}_{TU} = \langle\{\}, \mathcal{F}_{TU}\rangle$, where $\mathcal{F}_{TU}(s_0) = T \cup U$ and for each label $\lambda \neq s_0$, $\mathcal{F}(\lambda) = T$.

The above result effectively provides a translation method from a modal theory $\langle T, U\rangle$ of the logic of elsewhere into an equivalent $\mathrm{E}_{\mathrm{CLDS}}$ configuration, which preserves derivability and semantic entailment. However, it is clear that many initial configurations are not the translation of any modal theory. (For

[9] Note that $[\alpha]^*(\lambda)$ is proved to be false in $\mathcal{M}$ by the construction of $\mathcal{M}$ using the fact that α is false at the Kripke possible world that corresponds to λ.

example, any configuration whose diagram $\mathcal{D}$ is not empty or whose $\mathcal{F}$ differs at more than one label.) Hence, the information that such configurations encode cannot be represented within the standard logic of elsewhere, making the $\mathrm{E}_{\mathrm{CLDS}}$ system strictly more general than the standard Hilbert system.

4 The $\mathbf{L}_{\mathrm{CLDS}}$ System

In this section the $\mathrm{L}_{\mathrm{CLDS}}$ system is defined, again based on the general CLDS approach described in Section 2. For this application of CLDS to resource logics, some additional inference rules are defined for dealing with the $\otimes$ operator and with the interactions between R-literals and declarative units. The extended algebra $\mathcal{A}_{\mathrm{L}}^{+}$ is defined, which, together with the notions of model and semantic entailment given in Definitions 10 and 11, provide a model theoretic semantics for $\mathrm{L}_{\mathrm{CLDS}}$. Soundness and completeness of the $\mathrm{L}_{\mathrm{CLDS}}$ proof system with respect to this semantics are then proved. The correspondence with a standard Hilbert system of linear logic is discussed, showing the $\mathrm{L}_{\mathrm{CLDS}}$ system to be more general than standard presentations of linear logics.

Syntax. The language of the $\mathrm{L}_{\mathrm{CLDS}}$ system is the pair $\langle \mathcal{L}_{\mathcal{P}}, \mathcal{L}_L \rangle$, where $\mathcal{L}_{\mathcal{P}}$ is a standard substructural propositional language restricted to the operators $\{\rightarrow, \otimes, \neg\}$ and including the two propositions $\top$ and $\bot$. The proposition $\top$ is the identity of $\otimes$ and $\bot$ is equivalent[10] to $\neg\top$. The labelling language $\mathcal{L}_L$ is composed of a countable set of symbols $\{a, b, \ldots, f, a_1, b_1, \ldots, f_1, \ldots\}$ called *constants*, a countable set of variables $\{x, y, z \ldots\}$, a binary function symbol $\circ$, called "resource composition" and the R-predicate $\preceq$. The function $\circ$ and the predicate $\preceq$ are both usually written in infix form. The language $\mathcal{L}_L$ is extended into $\mathit{Func}(\mathcal{L}_P, \mathcal{L}_L)$ by adding for each wff α of $\mathcal{L}_{\mathcal{P}}$ different from $\top$ the symbol c_α, and for the wff $\top$, the symbol 1. The symbols c_α and 1 are called *parameters*. All parameters have a special role in the proof theory and semantics, especially $c_\bot$. As mentioned in Section 2 parameters of the form c_α represent the smallest label verifying α. Terms of the semi-extended labelling language $\mathit{Func}(\mathcal{L}_P, \mathcal{L}_L)$ are defined inductively, as consisting of constants, parameters and variables, together with expressions of the form $x \circ y$ where x and y are terms. Labels are ground terms of $\mathit{Func}(\mathcal{L}_P, \mathcal{L}_L)$.

In this system, the R-literals are referred to as *constraints*. $\mathrm{L}_{\mathrm{CLDS}}$ theories are as described in Definition 2. A configuration is therefore given by a set of constraints (a diagram) and a set of declarative units. Pairs of constraints Δ and $\overline{\Delta}$ within a configuration will be denoted by the shorthand notation $\bot$, representing classical inconsistency. This symbol should not be confused with the declarative unit $\bot : \lambda$, in which $\bot$ is just a substructural proposition letter. The labelling algebra $\mathcal{A}_{\mathrm{L}}$ is a set of first-order axioms expressing the

[10] This is easy to show using the $\mathrm{L}_{\mathrm{CLDS}}$ rules.

properties of the function symbol $\circ$ and the property of pre-ordering for the binary relation $\preceq$.

Definition 33 (Labelling algebra $\mathcal{A}_\mathrm{L}$). The labelling algebra $\mathcal{A}_\mathrm{L}$, written in $Func(\mathcal{L}_P, \mathcal{L}_L)$, is the first-order theory given by the following axioms:

$\forall x(x \preceq x)$	(Reflexivity)
$\forall x, y, z((x \preceq y \wedge y \preceq z) \rightarrow x \preceq z)$	(Transitivity)
$\forall x((1 \circ x \preceq x) \wedge (x \preceq 1 \circ x))$	(Identity)
$\forall x, y, z(x \preceq y \rightarrow (x \circ z \preceq y \circ z) \wedge (z \circ x \preceq z \circ y))$	(Order-preserving)
$\forall x, y(x \circ y \preceq y \circ x)$	(Commutativity)
$\forall x, y, z((x \circ y) \circ z \preceq x \circ (y \circ z))$	(Associativity)

A L_CLDS system $\mathcal{L}$ is then defined by the tuple $\mathcal{L} = \langle\langle\mathcal{L}_P, \mathcal{L}_\mathrm{L}\rangle, \mathcal{A}_L, \mathcal{R}_L\rangle$ where the set $\mathcal{R}_L$ of inference rules is described in the following section.

4.1 Proof theory and Semantics of L_CLDS

The set $\mathcal{R}_L$ of inference rules includes (i) elimination and introduction rules for the linear operators and (ii) rules describing the interaction between constraints and declarative units. The first set of rules includes those in Table 2, except $(R\mathcal{I})$, and the rules in Table 5. Each of the operator rules incorporates the idea of combining resources[11] in order to derive new information. For example, the $(\rightarrow\mathcal{E})$ rule expresses how resources λ and λ', verifying $\alpha \rightarrow \beta$ and α respectively, are combined into $\lambda \circ \lambda'$ to verify β. The (ch) rule and (unit) rule respectively reflect the notions that there is a smallest resource verifying a formula and that any increase of resource maintains verifiability. The $(R\mathcal{I})$ rule in Table 5 differs slightly from that given in Table 2 because in the L_CLDS system a contradiction is the proof of the classical proposition $\bot$. An alternative version of the rule would have been to add to the rule in Table 2 the premise $c_\bot \not\preceq \lambda$. (It was considered simpler to use the version given in Table 5.) The (unit) rule often interacts with the (R-A) rule (given in Table 2). For instance, the (R-A) rule can be used to infer the constraint that a resource λ' is greater than another resource λ ($\lambda \preceq \lambda'$) so to allow the (unit) rule to extend a configuration with $\alpha : \lambda'$ if it includes already $\alpha : \lambda$. With respect to the notion of length of inference rules, $(\rightarrow\mathcal{I})$, $(\neg\mathcal{I})$ and $(R\mathcal{I})$ belong to the category $\mathcal{I}^+$ (given in Definition 14), whereas all the other rules belong to the category $\mathcal{I}^0$.

The derivability relation $\vdash_\mathrm{L}$ for the L_CLDS system is identical to that given in Definition 5. Declarative units of the form $\alpha : 1$ derivable from an empty configuration are called *theorems*. Figure 5 illustrates an example derivation of the linear logic theorem $\alpha \otimes \beta \rightarrow \neg(\alpha \rightarrow \neg\beta)$ at the label 1.

[11] Notice that in linear logic two combinations of resources are only related if they comprise exactly the same resources, but possibly combined in different orders.

Table 5. Additional rules for substructural operators and R-literals, in $\mathrm{L_{CLDS}}$.

$$\frac{\begin{array}{c}\mathcal{C}\langle[\alpha : c_\alpha]\rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle\bot : \lambda \circ c_\alpha\rangle\end{array}}{\mathcal{C}'\langle\neg\alpha : \lambda\rangle}\ (\neg\mathcal{I}) \qquad \frac{\begin{array}{c}\mathcal{C}\langle[\alpha : c_\alpha\rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle\beta : \lambda \circ c_\alpha\rangle\end{array}}{\mathcal{C}'\langle\alpha \rightarrow \beta : \lambda\rangle}\ (\rightarrow\mathcal{I})$$

$$\frac{\mathcal{C}\langle\neg\alpha : \lambda_1, \alpha : \lambda_2\rangle}{\mathcal{C}'\langle\bot : \lambda_1 \circ \lambda_2\rangle}\ (\neg\mathcal{E}) \qquad \frac{\mathcal{C}\langle\alpha \rightarrow \beta : \lambda, \alpha : \lambda'\rangle}{\mathcal{C}'\langle\beta : \lambda \circ \lambda'\rangle}\ (\rightarrow\mathcal{E})$$

$$\frac{\mathcal{C}\langle\alpha \otimes \beta : \lambda\rangle}{\mathcal{C}'\langle\alpha : c_\alpha, \beta : c_\beta, c_\alpha \circ c_\beta \preceq \lambda\rangle}\ (\otimes\mathcal{E}) \qquad \frac{\mathcal{C}\langle\alpha : \lambda_1, \beta : \lambda_2\rangle}{\mathcal{C}'\langle\alpha \otimes \beta : \lambda_1 \circ \lambda_2\rangle}\ (\otimes\mathcal{I})$$

$$\frac{\mathcal{C}\langle\alpha : \lambda\rangle}{\mathcal{C}'\langle\alpha : c_\alpha, c_\alpha \preceq \lambda\rangle}\ (\mathrm{ch}) \qquad \frac{\mathcal{C}\langle\alpha : \lambda, \lambda \preceq \lambda'\rangle}{\mathcal{C}'\langle\alpha : \lambda'\rangle}\ (\mathrm{unit})$$

$$\frac{\mathcal{C}}{\mathcal{C}'\langle\bot : c_\bot\rangle}\ (\mathrm{base}) \qquad \frac{\begin{array}{c}\mathcal{C}\langle[\Delta\rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle\bot\rangle\end{array}}{\mathcal{C}'\langle\overline{\Delta}\rangle}\ (R\mathcal{I})$$

$$\begin{array}{lr}
\mathcal{C}_\emptyset\langle\rangle & \\
\quad \mathcal{C}_1\langle[\alpha \otimes \beta : c_{\alpha\otimes\beta)}]\rangle & (\text{assumption}) \\
\qquad \mathcal{C}_2\langle[\alpha \rightarrow \neg\beta : c_{\alpha\rightarrow\neg\beta}]\rangle & (\text{assumption}) \\
\qquad \mathcal{C}_3\langle\alpha : c_\alpha, \beta : c_\beta, c_\alpha \circ c_\beta \preceq c_{\alpha\otimes\beta}\rangle & (\otimes\mathcal{E}) \\
\qquad \mathcal{C}_4\langle\neg\beta : c_{\alpha\rightarrow\neg\beta} \circ c_\alpha\rangle & (\rightarrow\mathcal{E}) \\
\qquad \mathcal{C}_5\langle\bot : c_{\alpha\rightarrow\neg\beta} \circ c_\alpha \circ c_\beta\rangle & (\neg\mathcal{E}) \\
\qquad \mathcal{C}_6\langle c_{\alpha\rightarrow\neg\beta} \circ c_\alpha \circ c_\beta \preceq c_{\alpha\rightarrow\neg\beta} \circ c_{\alpha\otimes\beta}\rangle & (R\text{-A}) \\
\qquad \mathcal{C}_7\langle\bot : c_{\alpha\rightarrow\neg\beta} \circ c_\alpha \otimes c_\beta\rangle & (\text{unit}) \\
\quad \mathcal{C}_8\langle\neg(\alpha \rightarrow \neg\beta) : c_{\alpha\otimes\beta}\rangle & (\neg\mathcal{I}) \\
\mathcal{C}_9\langle\alpha \otimes \beta \rightarrow \neg(\alpha \rightarrow \neg\beta) : 1\rangle & (\rightarrow\mathcal{I})
\end{array}$$

Fig. 5. $\mathrm{L_{CLDS}}$ derivation of $\alpha \otimes \beta \rightarrow \neg(\alpha \rightarrow \neg\beta) : 1$.

Semantics. The language $Mon(\mathcal{L}_P, \mathcal{L}_L)$ is defined by adding to the language $Func(\mathcal{L}_P, \mathcal{L}_L)$ monadic predicates $[\alpha]^*$, for each wff α in $\mathcal{L}_P$. The atomic formula $[\alpha]^*(x)$ can be read as "the resource x verifies α". The extended algebra $\mathcal{A}^+_{\mathrm{L}}$ for $\mathrm{L_{CLDS}}$, written in $Mon(\mathcal{L}_P, \mathcal{L}_L)$, expresses relationships between the monadic predicates and constraints according to the semantic meaning of the substructural operators. This is given by the following definition, where axioms (Ax3a)–(Ax6) characterise the operators $\rightarrow$, $\neg$ and $\otimes$ respectively, whilst axioms (Ax1), (Ax2) and (Ax7) characterise the (unit), (ch) and (base) rules respectively.

Definition 34. Given the extended labelling language $Mon(\mathcal{L}_P, \mathcal{L}_L)$ and the labelling algebra $\mathcal{A}_{\mathrm{L}}$ of the $\mathrm{L}_{\mathrm{CLDS}}$ system, the *extended algebra* $\mathcal{A}_{\mathrm{L}}^{+}$ is the theory in $Mon(\mathcal{L}_P, \mathcal{L}_L)$ given by $\mathcal{A}_{\mathrm{L}}$ extended with the following axiom schemas. For any wffs α and β of $\mathcal{L}_P$:

(Ax1) $\forall x, y(x \preceq y \wedge [\alpha]^*(x) \rightarrow [\alpha]^*(y))$
(Ax2) $\forall x([\alpha]^*(x) \rightarrow [\alpha]^*(c_\alpha) \wedge c_\alpha \preceq x)$
(Ax3a) $\forall x, y([\alpha \rightarrow \beta]^*(x) \wedge [\alpha]^*(y) \rightarrow [\beta]^*(x \circ y))$
(Ax3b) $\forall x(([\alpha]^*(c_\alpha) \rightarrow [\beta]^*(x \circ c_\alpha)) \rightarrow [\alpha \rightarrow \beta]^*(x))$
(Ax4a) $\forall x, y([\neg\alpha]^*(x) \wedge [\alpha]^*(y) \rightarrow [\bot]^*(x \circ y))$
(Ax4b) $\forall x(([\alpha]^*(c_\alpha) \rightarrow [\bot]^*(x \circ c_\alpha)) \rightarrow [\neg\alpha]^*(x))$
(Ax5a) $\forall x([\alpha \otimes \beta]^*(x) \rightarrow ([\alpha]^*(c_\alpha) \wedge [\beta]^*(c_\beta) \wedge c_\alpha \circ c_\beta \preceq x))$
(Ax5b) $\forall x, y([\alpha]^*(x) \wedge [\beta]^*(y) \rightarrow [\alpha \otimes \beta]^*(x \circ y))$
(Ax6) $\forall x([\neg\neg\alpha]^*(x) \rightarrow [\alpha]^*(x))$
(Ax7) $[\bot]^*(c_\bot)$

4.2 Main results about the $\mathrm{L}_{\mathrm{CLDS}}$ system

Soundness. The notions of satisfiability and semantic entailment of the $\mathrm{L}_{\mathrm{CLDS}}$ system, denoted with $\models_{\mathrm{L}}$, are as specified in Definitions 10 and 11, but based on the extended algebra $\mathcal{A}_{\mathrm{L}}^{+}$. Also for this system, the soundness and completeness proofs are based respectively on the two methodologies described in Section 2.3. They take advantage of the soundness and completeness of first-order logic. Most of the theorems, lemmas and propositions used to prove these two properties are similar in nature to those given in [24], for the class of CLDS normal modal logic systems.

Theorem 35 (Soundness of $\mathrm{L}_{\mathrm{CLDS}}$). *Let $\mathcal{L} = \langle\langle \mathcal{L}_P, \mathcal{L}_L\rangle, \mathcal{A}_{\mathrm{L}}^{+}, \mathcal{R}_L\rangle$ be an $\mathrm{L}_{\mathrm{CLDS}}$ system and let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations. If $\mathcal{C} \vdash_{\mathrm{L}} \mathcal{C}'$ then $\mathcal{C} \models_{\mathrm{L}} \mathcal{C}'$.*

The proof of Theorem 35, represented diagrammatically in Figure 1, is mainly based on the proof of the following lemma.

Lemma 36. *Let $\mathcal{A}_{\mathrm{L}}^{+}$ be the extended algebra of the $\mathrm{L}_{\mathrm{CLDS}}$ system, let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations and let $FOT(\mathcal{C})$ and $FOT(\mathcal{C}')$ be their respective first-order translations. If $\mathcal{C} \vdash_{\mathrm{L}} \mathcal{C}'$ then $\mathcal{A}_{\mathrm{L}}^{+}$, $FOT(\mathcal{C}) \vdash_{\mathrm{L}} FOT(\mathcal{C}')$.*

PROOF: The proof is by induction on the smallest size of derivations of the form $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, \overline{m}\rangle$, where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_n = \mathcal{C}'$, following the same

argument as in Lemma 20. The inductive step is proved by cases on the inference rule applied on the last step $\mathcal{C}_{n-1}/\mathcal{C}_n$ of the derivation.
Case 1: (ch).
In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in$ (ch). Then there exists a declarative unit $\alpha : \lambda$ in $\mathcal{C}_{n-1}$. Therefore, $[\alpha]^*(\lambda) \in FOT(\mathcal{C}_{n-1})$. This implies, by applying (Ax2) of $\mathcal{A}_{\mathrm{L}}^+$, that $\mathcal{A}_{\mathrm{L}}^+$, FOT($\mathcal{C}_{n-1}$) $\vdash_{\mathrm{FOL}} [\alpha]^*(c_\alpha)$ and $\mathcal{A}_{\mathrm{L}}^+$, FOT($\mathcal{C}_{n-1}$) $\vdash_{\mathrm{FOL}} c_\alpha \preceq \lambda$. Since $\mathcal{C}_n = \mathcal{C}_{n-1} + [\alpha : c_\alpha, c_\alpha \preceq \lambda]$, $\mathcal{A}_{\mathrm{L}}^+$, $FOT(\mathcal{C}_{n-1}) \vdash_{\mathrm{FOL}} FOT(\mathcal{C}_n)$.
Case 2: $(\otimes\mathcal{E})$.
In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in (\otimes\mathcal{E})$. Then there exist a declarative unit $\alpha \otimes \beta : \lambda$ in $\mathcal{C}_{n-1}$. Therefore, $[\alpha \otimes \beta]^*(\lambda) \in FOT(\mathcal{C}_{n-1})$. Using (Ax5a), each of these three elements $[\alpha]^*(c_\alpha)$, $[\beta]^*(c_\beta)$, $c_\alpha \circ c_\beta \preceq \lambda$ are derivable in first-order logic from $FOT(\mathcal{C}_{n-1})$. Since $\mathcal{C}_n = \mathcal{C}_{n-1} + [\alpha : c_\alpha, \beta : c_\beta, c_\alpha \circ c_\beta]$, $\mathcal{A}_{\mathrm{L}}^+, FOT(\mathcal{C}_{n-1}) \vdash_{\mathrm{FOL}} FOT(\mathcal{C}_n)$.
Case 3: $(\otimes\mathcal{I})$.
In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in (\otimes\mathcal{I})$. The set $\{\alpha : \lambda, \beta : \lambda'\} \subseteq \mathcal{C}_{n-1}$. Therefore, $\{[\alpha]^*(\lambda), [\beta]^*(\lambda')\} \subseteq FOT(\mathcal{C}_{n-1})$. Using (Ax5b), $\mathcal{A}_{\mathrm{L}}^+$, $FOT(\mathcal{C}_{n-1}) \vdash_{\mathrm{FOL}} [\alpha \otimes \beta]^*(\lambda \circ \lambda')$. Since $\mathcal{C}_n = \mathcal{C}_{n-1} + [\alpha \otimes \beta : \lambda \circ \lambda']$, $\mathcal{A}_{\mathrm{L}}^+$, $FOT(\mathcal{C}_{n-1}) \vdash_{\mathrm{FOL}} FOT(\mathcal{C}_n)$.
Case 4: $(\neg\mathcal{I})$.
In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in (\neg\mathcal{I})$. There exists a proof of $\bot : \lambda \circ c_\alpha$ from $\mathcal{C}_{n-1} + [\alpha : c_\alpha]$. By the induction hypothesis $\mathcal{A}_{\mathrm{L}}^+$, $FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(c_\alpha)\} \vdash_{\mathrm{FOL}} [\bot]^*(\lambda \circ c_\alpha)$. Using the axiom (Ax4b) and the deduction theorem of first-order logic, $\mathcal{A}_{\mathrm{L}}^+$, $FOT(\mathcal{C}_{n-1}) \vdash_{\mathrm{FOL}} [\neg\alpha]^*(\lambda)$. Since $\mathcal{C}_n = \mathcal{C}_{n-1} + [\neg\alpha : \lambda]$, $\mathcal{A}_{\mathrm{L}}^+$, $FOT(\mathcal{C}_{n-1}) \vdash_{\mathrm{FOL}} FOT(\mathcal{C}_n)$.
The cases for (unit), (C-R), (R-A), $(\rightarrow\mathcal{E})$, $(\neg\mathcal{E})$, $(\neg\neg)$, $(\bot\mathcal{E})$ and (base) rules follow the same argument as Case 2 above, whereas the cases for $(\rightarrow\mathcal{I})$, $(R\mathcal{I})$ rules are proved in the same way as Case 4. □

Proof of Theorem 35:
By hypothesis $\mathcal{C} \vdash_{\mathrm{L}} \mathcal{C}'$. By Lemma 36 $\mathcal{A}_{\mathrm{L}}^+$, $FOT(\mathcal{C}) \vdash_{\mathrm{FOL}} FOT(\mathcal{C}')$. By soundness of first-order logic $\mathcal{A}_{\mathrm{L}}^+, FOT(\mathcal{C}) \models_{\mathrm{FOL}} FOT(\mathcal{C}')$. Hence, by definition of semantic entailment, $\mathcal{C} \models_{\mathrm{L}} \mathcal{C}'$.

Completeness. The completeness theorem states that, given the $\mathrm{L}_{\mathrm{CLDS}}$ system and two configurations $\mathcal{C}$ and $\mathcal{C}'$, whose difference is finite, if $\mathcal{C}'$ is semantically entailed from $\mathcal{C}$ then $\mathcal{C}'$ is also derived from $\mathcal{C}$. This is formally defined in Theorem 37 below.

Theorem 37 (Completeness of $\mathbf{L}_{\mathrm{CLDS}}$). *Let $L = \langle\langle\mathcal{L}_P, \mathcal{L}_L\rangle, \mathcal{A}_{\mathrm{L}}, \mathcal{R}_{\mathrm{L}}\rangle$ be a L_{CLDS} system and let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations such that $\mathcal{C}' - \mathcal{C}$ is a finite configuration. If $\mathcal{C} \models_{\mathrm{L}} \mathcal{C}'$ then $\mathcal{C} \vdash_{\mathrm{L}} \mathcal{C}'$.*

As discussed in Section 2.3, the Henkin-style methodology for classical logic is here adopted to prove Theorem 37. Basic notions in this proof are those of *consistent* and *maximal consistent* configurations.

Definition 38 (Consistent configuration). Given the $\mathrm{L}_{\mathrm{CLDS}}$ system and a configuration $\mathcal{C}$, $\mathcal{C}$ is *inconsistent* if and only if there exists a configuration

$\mathcal{C}'$ such that $\mathcal{C} \vdash_{\mathrm{L}} \mathcal{C}'$ and $\{\Delta, \overline{\Delta}\} \subseteq \mathcal{C}'$. This is also written in the shorthand notation $\mathcal{C} \vdash_{\mathrm{L}} \bot$. A configuration $\mathcal{C}$ is *consistent* if it is not inconsistent.

The configuration, $\{\alpha : \lambda, \neg\alpha : \lambda', \alpha \otimes \neg\alpha : \lambda \circ \lambda', \bot : \lambda \circ \lambda', c_{\bot} \preceq \lambda \circ \lambda'\}$ is an example of a consistent configuration. This is supposed to be so since in linear logic the formula $\alpha \otimes \neg\alpha$ does not allow a derivation of any arbitrary formula as it is in the case of classical logic (where the linear operator $\otimes$ is replaced by the classical operator $\wedge$). However, the addition of the constraint $c_{\bot} \not\preceq \lambda \circ \lambda'$, which expresses that the resource $\lambda \circ \lambda'$ is a *consistent resource*, to the above configuration would make it inconsistent. The notion of a maximal consistent configuration, denoted with $\mathcal{C}_{\mathrm{mcc}}$, in $\mathrm{L}_{\mathrm{CLDS}}$ is identical to that given for the $\mathrm{E}_{\mathrm{CLDS}}$ system, and it is constructed in the same way as described in Lemma 24.

As for the $\mathrm{E}_{\mathrm{CLDS}}$ case, it is the contrapositive statement of Theorem 37 that is proved, namely, if $\mathcal{C} \nvdash_{\mathrm{L}} \mathcal{C}'$ then $\mathcal{C} \not\models_{\mathrm{L}} \mathcal{C}'$. The proof uses the property of a consistent configuration given in Lemma 39 and the properties of a maximal consistent configuration defined below.

Lemma 39 (Consistency). *Let $\mathcal{C}$ be a consistent configuration. Then*

1. *If $\mathcal{C} \nvdash_{\mathrm{L}} \alpha : \lambda$ then $\mathcal{C} + [c_{\bot} \not\preceq \lambda \circ c_{\neg\alpha}] + [\neg\alpha : c_{\neg\alpha}]$ is consistent.*
2. *If $\mathcal{C} \nvdash_{\mathrm{L}} \Delta$ then $\mathcal{C} + [\overline{\Delta}]$ is consistent.*

Proof: (1) By contradiction, suppose that $\mathcal{C} + [c_{\bot} \not\preceq \lambda \circ c_{\neg\alpha}] + [\neg\alpha : c_{\neg\alpha}]$ is not consistent. Then $\mathcal{C} + [\neg\alpha : c_{\neg\alpha}] + [c_{\bot} \not\preceq \lambda \circ c_{\neg\alpha}] \vdash_{\mathrm{L}} \bot$. By $(R\mathcal{I})$ $\mathcal{C} + [\neg\alpha : c_{\neg\alpha}] \vdash_{\mathrm{L}} c_{\bot} \preceq \lambda \circ c_{\neg\alpha}$. By the (base) rule, $\mathcal{C} + [\neg\alpha : c_{\neg\alpha}] \vdash_{\mathrm{L}} \bot : c_{\bot}$, and then by the (unit) rule $\mathcal{C} + [\neg\alpha : c_{\neg\alpha}] \vdash_{\mathrm{L}} \bot : \lambda \circ c_{\neg\alpha}$. Hence by $(\neg\mathcal{I})$ $\mathcal{C} \vdash_{\mathrm{L}} \alpha : \lambda$, contradicting the initial assumption. (2) By contradiction, suppose that $\mathcal{C} + [\overline{\Delta}] \vdash_{\mathrm{L}} \bot$. Then, by $(R\mathcal{I})$, $\mathcal{C} \vdash_{\mathrm{L}} \Delta$, contradicting the initial assumption. □

The following five lemmas show the properties of a maximal consistent configuration $\mathcal{C}_{\mathrm{mcc}}$. Note that the proof of these lemmas use the following basic results of a general CLDS system, for which the reader is referred to [24]. Firstly, if $\pi \notin \mathcal{C}_{\mathrm{mcc}}$ for some declarative unit or R-literal π, then $\mathcal{C}_{\mathrm{mcc}} + [\pi]$ is not consistent and $\mathcal{C}_{\mathrm{mcc}} + [\pi] \vdash_{\mathrm{L}} \bot$. Therefore whenever it is shown that $\mathcal{C}_{\mathrm{mcc}} \vdash_{\mathrm{L}} \mathcal{C}_{\mathrm{mcc}} + [\pi]$, it can be concluded that $\mathcal{C}_{\mathrm{mcc}} \vdash_{\mathrm{L}} \bot$. Secondly, for a maximal configuration $\mathcal{C}_{\mathrm{mcc}}$, if $\mathcal{C}_{\mathrm{mcc}} \vdash_{\mathrm{L}} \alpha : \lambda$ then there exists a (finite) configuration $\mathcal{C}_1 \subseteq \mathcal{C}_{\mathrm{mcc}}$ such that $\mathcal{C}_1 \vdash_{\mathrm{L}} \alpha : \lambda$. Thirdly, by the monotonicity property of $\vdash_{\mathrm{L}}$, for any configuration $\mathcal{C}_1 \subseteq \mathcal{C}_{\mathrm{mcc}}$, if $\mathcal{C}_1 \vdash_{\mathrm{L}} \alpha : \lambda$ then $\mathcal{C}_{\mathrm{mcc}} \vdash_{\mathrm{L}} \alpha : \lambda$.

Lemma 40 (Properties of $\mathcal{C}_{\mathrm{mcc}}$ with respect to $\mathcal{A}_{\mathrm{L}}$). *Given the L_{CLDS} system and a maximal consistent configuration $\mathcal{C}_{\mathrm{mcc}}$, the following properties hold for any labels λ, λ' and λ'':*

1. *If $\lambda \preceq \lambda' \in \mathcal{C}_{\mathrm{mcc}}$ and $\lambda' \preceq \lambda'' \in \mathcal{C}_{\mathrm{mcc}}$ then $\lambda \preceq \lambda'' \in \mathcal{C}_{\mathrm{mcc}}$.*

2. $\lambda \preceq \lambda \in \mathcal{C}_{\text{mcc}}$.
3. $1 \circ \lambda \preceq \lambda \in \mathcal{C}_{\text{mcc}}$ *and* $\lambda \preceq 1 \circ \lambda \in \mathcal{C}_{\text{mcc}}$.
4. *If* $\lambda \preceq \lambda' \in \mathcal{C}_{\text{mcc}}$ *then* $\lambda \circ \lambda'' \preceq \lambda' \circ \lambda'' \in \mathcal{C}_{\text{mcc}}$ *and* $\lambda'' \circ \lambda \preceq \lambda'' \circ \lambda' \in \mathcal{C}_{\text{mcc}}$.
5. $\lambda \circ \lambda' \preceq \lambda' \circ \lambda \in \mathcal{C}_{\text{mcc}}$.
6. $(\lambda \circ \lambda') \circ \lambda'' \preceq \lambda \circ (\lambda' \circ \lambda'') \in \mathcal{C}_{\text{mcc}}$.

PROOF: Only the first property is proved here. Similar arguments are used to prove the other properties. Assume by contradiction that $\lambda \preceq \lambda'' \notin \mathcal{C}_{\text{mcc}}$. Then $\mathcal{C}_{\text{mcc}} + [\lambda \preceq \lambda''] \vdash_{\text{L}} \bot$. But using the asusmptions it is easy to show that $\mathcal{C}_{\text{mcc}} \vdash_{\text{L}} \mathcal{C}_{\text{mcc}} + [\lambda \preceq \lambda'']$. Therefore $\mathcal{C}_{\text{mcc}}$ is inconsistent, which is a contradiction. □

Lemma 41 (Properties of characteristic labels). *Given the L_{CLDS} system and a maximal consistent configuration $\mathcal{C}_{\text{mcc}}$, the following properties hold, for any wff α and labels λ and λ':*

1. *If* $\alpha : \lambda \in \mathcal{C}_{\text{mcc}}$ *then* $\alpha : c_\alpha \in \mathcal{C}_{\text{mcc}}$ *and* $c_\alpha \preceq \lambda \in \mathcal{C}_{\text{mcc}}$.
2. *If* $\lambda \preceq \lambda' \in \mathcal{C}_{\text{mcc}}$ *and* $\alpha : \lambda \in \mathcal{C}_{\text{mcc}}$ *then* $\alpha : \lambda' \in \mathcal{C}_{\text{mcc}}$.
3. $\bot : c_\bot \in \mathcal{C}_{\text{mcc}}$.
4. $\top : 1 \in \mathcal{C}_{\text{mcc}}$.

PROOF: (1) Assume, by contradiction that $\alpha : c_\alpha \notin \mathcal{C}_{\text{mcc}}$. Then $\mathcal{C}_{\text{mcc}} + [\alpha : c_\alpha]$ is inconsistent. Using the assumptions and (ch) rule $\mathcal{C}_{\text{mcc}} \vdash_{\text{L}} \mathcal{C}_{\text{mcc}} + [\alpha : c_\alpha]$. Thus $\mathcal{C}_{\text{mcc}}$ is inconsistent, which is a contradiction. Similarly, for proving that $c_\alpha \preceq \lambda \in \mathcal{C}_{\text{mcc}}$. (2) Assume by contradiction that $\alpha : \lambda' \notin \mathcal{C}_{\text{mcc}}$. Then $\mathcal{C}_{\text{mcc}} + [\alpha : \lambda']$ is inconsistent. Using the assumptions and the (unit) rule $\mathcal{C}_{\text{mcc}} \vdash_{\text{L}} \mathcal{C}_{\text{mcc}} + [\alpha : \lambda']$. Thus $\mathcal{C}_{\text{mcc}}$ is inconsistent, which is a contradiction. (3) If $\bot : c_\bot$ were not in $\mathcal{C}_{\text{mcc}}$ then $\mathcal{C}_{\text{mcc}} + [\bot : c_\bot] \vdash_{\text{L}} \bot$, but using the (base) rule this would imply $\mathcal{C}_{\text{mcc}}$ was inconsistent, a contradiction. (4) If $\top : 1$ were not in $\mathcal{C}_{\text{mcc}}$ then $\mathcal{C}_{\text{mcc}} + [\top : 1] \vdash_{\text{L}} \bot$. $\top$ is equivalent to $\neg\bot$ and by $(\neg\mathcal{I})$ $\mathcal{C}_{\text{mcc}} \vdash_{\text{L}} \top : 1$, hence $\mathcal{C}_{\text{mcc}}$ is inconsistent, which is a contradiction. □

The above characteristic properties will be used several times in the following lemmas but without explicitly being stated.

Lemma 42 (Consistency properties of $\mathcal{C}_{\text{mcc}}$). *Given the L_{CLDS} system and a maximal consistent configuration $\mathcal{C}_{\text{mcc}}$, the following properties hold, for any wff α:*

1. *Given two labels λ and λ', either* $\lambda \preceq \lambda' \notin \mathcal{C}_{\text{mcc}}$ *or* $\lambda \not\preceq \lambda' \notin \mathcal{C}_{\text{mcc}}$.
2. *If* $c_\bot \not\preceq c_\alpha \circ c_{\neg\alpha} \in \mathcal{C}_{\text{mcc}}$, *then either, for each λ,* $\alpha : \lambda \notin \mathcal{C}_{\text{mcc}}$, *or for each λ,* $\neg\alpha : \lambda \notin \mathcal{C}_{\text{mcc}}$.
3. *For any label λ, if* $c_\alpha \not\preceq \lambda \in \mathcal{C}_{\text{mcc}}$ *then* $\alpha : \lambda \notin \mathcal{C}_{\text{mcc}}$.

PROOF: (1) Assume, by contradiction, that $\lambda \preceq \lambda' \in \mathcal{C}_{\text{mcc}}$ and $\lambda \not\preceq \lambda' \in \mathcal{C}_{\text{mcc}}$, then by the ($R$-A) rule $\mathcal{C}_{\text{mcc}} \vdash_{\text{L}} \bot$, which is a contradiction. (2) Assume, by contradiction, that $\exists\lambda'', \lambda'$ such that $\alpha : \lambda'' \in \mathcal{C}_{\mathrm{mcc}}$ and $\neg\alpha : \lambda' \in \mathcal{C}_{\mathrm{mcc}}$. Then $\alpha : c_\alpha \in \mathcal{C}_{\mathrm{mcc}}$ and $\neg\alpha : c_{\neg\alpha} \in \mathcal{C}_{\mathrm{mcc}}$ (by Lemma 41). Hence by the $(\neg\mathcal{E})$ and (ch) rules $\mathcal{C}_{\mathrm{mcc}} \vdash_{\mathrm{L}} c_\perp \preceq c_\alpha \circ c_{\neg\alpha}$. But then, using the assumptions, by the (R-A) rule $\mathcal{C}_{\mathrm{mcc}} \vdash_{\mathrm{L}} \perp$, which is a contradiction. In particular, $\alpha : c_\alpha \notin \mathcal{C}_{\mathrm{mcc}}$ or $\neg\alpha : c_{\neg\alpha} \notin \mathcal{C}_{\mathrm{mcc}}$ for every α, so if $\alpha : c_\alpha \in \mathcal{C}_{\mathrm{mcc}}$, then $\neg\alpha : c_{\neg\alpha} \notin \mathcal{C}_{\mathrm{mcc}}$ (under the condition $c_\perp \not\preceq c_\alpha \circ c_{\neg\alpha} \in \mathcal{C}_{\mathrm{mcc}}$). (3) Assume, by contradiction, that $\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$, then $c_\alpha \preceq \lambda \in \mathcal{C}_{\mathrm{mcc}}$ and hence by $(\neg\mathcal{E})$ $\mathcal{C}_{\mathrm{mcc}} \vdash_{\mathrm{L}} \perp$, which is a contradiction. □

From statement (3) in the above lemma it follows, in particular, that for a consistent resource λ, $\perp : \lambda \notin \mathcal{C}_{\mathrm{mcc}}$.

Lemma 43 (Maximality of $\mathcal{C}_{\mathrm{mcc}}$). *Given the L_{CLDS} system and a maximal consistent configuration $\mathcal{C}_{\mathrm{mcc}}$, the following properties hold, for any wff α and labels λ and λ':*

1. *Either $\alpha : c_\alpha \in \mathcal{C}_{\mathrm{mcc}}$ or $\neg\alpha : c_{\neg\alpha} \in \mathcal{C}_{\mathrm{mcc}}$.*
2. *Either $\lambda \preceq \lambda' \in \mathcal{C}_{\mathrm{mcc}}$ or $\lambda \not\preceq \lambda' \in \mathcal{C}_{\mathrm{mcc}}$.*

PROOF: (1) Assume, by contradiction, that $\alpha : c_\alpha \notin \mathcal{C}_{\mathrm{mcc}}$ and $\neg\alpha : c_{\neg\alpha} \notin \mathcal{C}_{\mathrm{mcc}}$, then $\mathcal{C}_{\mathrm{mcc}} + [\alpha : c_\alpha] \vdash_{\mathrm{L}} \perp$ and $\mathcal{C}_{\mathrm{mcc}} + [\neg\alpha : c_{\neg\alpha}] \vdash_{\mathrm{L}} \perp$. By the $(\perp\mathcal{E})$ rule $\mathcal{C}_{\mathrm{mcc}} + [\alpha : c_\alpha] \vdash_{\mathrm{L}} \perp : c_\alpha \circ c_{\neg\alpha}$ and by the $(\neg\mathcal{I})$ rule $\mathcal{C}_{\mathrm{mcc}} \vdash_{\mathrm{L}} \neg\alpha : c_{\neg\alpha}$. Hence $\mathcal{C}_{\mathrm{mcc}}$ is inconsistent, which is a contradiction. (2) Similarly for this property. □

As a consequence of statement (1) in the above lemma, if $\alpha : c_\alpha \notin \mathcal{C}_{\mathrm{mcc}}$ then $\neg\alpha : c_{\neg\alpha} \in \mathcal{C}_{\mathrm{mcc}}$. The following properties of a $\mathcal{C}_{\mathrm{mcc}}$ are mainly used in proving the existence of a $\mathrm{L}_{\mathrm{CLDS}}$ model for a given $\mathcal{C}_{\mathrm{mcc}}$.

Lemma 44. *Given a maximal consistent configuration $\mathcal{C}_{\mathrm{mcc}}$, then for any wffs α and β, their characteristic labels c_α and c_β, and labels λ and λ', the following properties hold.*

1. *$\alpha \otimes \beta : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ iff $\alpha : c_\alpha \in \mathcal{C}_{\mathrm{mcc}}$, $\beta : c_\beta \in \mathcal{C}_{\mathrm{mcc}}$ and $c_\alpha \circ c_\beta \preceq \lambda \in \mathcal{C}_{\mathrm{mcc}}$.*
2. *$\alpha \to \beta : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ iff* for all labels *λ' if $\alpha : \lambda' \in \mathcal{C}_{\mathrm{mcc}}$ then $\beta : \lambda \circ \lambda' \in \mathcal{C}_{\mathrm{mcc}}$*
3. *If $\neg\neg\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ then $\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$.*
4. *If $\neg\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ and $\alpha : \lambda' \in \mathcal{C}_{\mathrm{mcc}}$ then $\perp : \lambda \circ \lambda' \in \mathcal{C}_{\mathrm{mcc}}$.*
5. *If $\alpha : c_\alpha \in \mathcal{C}_{\mathrm{mcc}}$ implies that $\perp : c_\alpha \circ \lambda \in \mathcal{C}_{\mathrm{mcc}}$, then $\neg\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$.*

PROOF: The reader is referred to [6] for the proof. □

The above lemmas are now used to prove the Model Existence Lemma for $\mathrm{L}_{\mathrm{CLDS}}$. This consists in showing that there exists a canonical model of $\mathrm{L}_{\mathrm{CLDS}}$ that satisfies a $\mathcal{C}_{\mathrm{mcc}}$. This model is constructed in a way similar to that shown in Definition 26. It is basically given by an Herbrand interpretation $\mathcal{H}_{\mathcal{A}}$, such that $\mathcal{H}_{\mathcal{A}} \mathrel{\|\!-} [\alpha]^*(x)$ iff $\alpha : x \in \mathcal{C}_{\mathrm{mcc}}$. By construction, $\mathcal{H}_{\mathcal{A}}$ satisfies the $\mathcal{C}_{\mathrm{mcc}}$ under consideration and it is a $\mathrm{L}_{\mathrm{CLDS}}$ model since it satisfies the extended algebra $\mathcal{A}^+_{\mathrm{L}}$ as stated in Lemma 45.

Lemma 45. *The Herbrand interpretation $\mathcal{H}_{\mathcal{A}}$ is a model of the extended algebra $\mathcal{A}_{\mathrm{L}}^{+}$.*

PROOF: It is easy to show that all axioms of $\mathcal{A}_{\mathrm{L}}^{+}$ are satisfied by the interpretation $\mathcal{H}_{\mathcal{A}}$, using Lemmas 40 — 44. □

Proof of Theorem 37:
Assume, by contrapositive, that $\mathcal{C} \nvdash_{\mathrm{L}} \mathcal{C}'$. Then there is a π, where π is either a declarative unit or an R-literal, such that $\pi \in \mathcal{C}'$ and $\mathcal{C} \nvdash_{\mathrm{L}} \pi$. There are two cases.
Case 1:
If π is Δ then $\mathcal{C} + [\overline{\Delta}]$ is consistent by Lemma 39 and there is a model that makes $\mathcal{A}_{\mathrm{L}}^{+}$ and $FOT(\mathcal{C})$ true but Δ false.
Case 2:
If π is $\alpha : \lambda$, then $\mathcal{C}+[\neg\alpha : c_{\neg\alpha}, c_{\perp} \not\preceq \lambda \circ c_{\neg\alpha}]$ is consistent and can be expanded into a maximally consistent configuration $\mathcal{C}_{\mathrm{mcc}}$ from which a model $\mathcal{H}_{\mathcal{A}}$ is constructed. $\mathcal{H}_{\mathcal{A}}$ assigns true to both $[\neg\alpha]^*(c_{\neg\alpha})$ and $c_{\perp} \not\preceq \lambda \circ c_{\neg\alpha}$. Assume, by contradiction, that $\mathcal{H}_{\mathcal{A}}$ assigns true to $[\alpha]^*(\lambda)$ also, then by construction $\alpha : \lambda \in \mathcal{C}_{\mathrm{mcc}}$. By (Ax4a) $[\perp]^*(\lambda \circ c_{\neg\alpha})$ and by (Ax2) $c_{\perp} \preceq \lambda \circ c_{\neg\alpha}$ would also be assigned true by $\mathcal{H}_{\mathcal{A}}$, which is a contradiction.

Correspondence of $\mathrm{L}_{\mathrm{CLDS}}$ with a Standard Hilbert System. In a similar way to that given for the $\mathrm{E}_{\mathrm{CLDS}}$ system, it is shown here that, when the initial configuration is empty, the $\mathrm{L}_{\mathrm{CLDS}}$ system corresponds to a standard Hilbert style presentation of linear logic. Specifically, Theorems 47 and 48 show that any linear logic theorem α of a standard Hilbert system for linear logic can be proved to be a theorem of $\mathrm{L}_{\mathrm{CLDS}}$ (i.e. the declarative unit $\alpha : 1$ is derivable from an empty configuration) and vice versa. A definition of the Hilbert system taken into consideration is first given.

Definition 46. Let $\mathcal{L}_P$ be the propositional fragment of linear logic considered in the $\mathrm{L}_{\mathrm{CLDS}}$ system. The Hilbert system for this fragment of linear logic, written $\mathcal{L}_{Ax}$, is given by the following set of schemas (see [1]), together with the Modus Ponens rule (MP).

(L1) $\alpha \to \alpha$
(L2) $(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$
(L3) $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$
(L4) $\neg\neg\alpha \to \alpha$
(L5) $\alpha \to (\neg\alpha \to \perp)$
(L6) $\neg\perp$
(L7) $(\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$
(L8) $\alpha \to (\beta \to (\alpha \otimes \beta))$
(L9) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \otimes \beta) \to \gamma)$

The Hilbert proof of a theorem consists of one or more steps, where each step is either an instance of one of the axioms, or is derived from the results of two previous steps by (MP). Examples of such proofs can be found in Figures 10 and 11. Various other axiom schemas can be derived from (L1) - (L9). In particular, the schema $(\alpha_1 \to \beta_1) \to ((\beta_1 \to \gamma_1) \to (\alpha_1 \to \gamma_1))$ called (L10) can be proved by first instantiating in schema (L3), α with $\beta_1 \to \gamma_1$, β with $\alpha_1 \to \beta_1$, γ with $\alpha_1 \to \gamma_1$, and in schema (L2) α with β_1, β with γ_1 and γ with α_1, and then applying (MP) to the resulting two instantiated schemas.

Theorem 47. *Consider the* $\mathrm{L_{CLDS}}$ *system, the Hilbert system* $\mathcal{L}_{Ax}$ *and the initial empty configuration* $\mathcal{C}_\emptyset = \langle \{\ \}, \mathcal{F} \rangle$*, given by* $\mathcal{F}(\lambda) = \{\ \}$ *for any label* λ*. Let* α *be a wff of* $\mathcal{L}_P$*.*

If $\vdash_{\mathrm{Ax}} \alpha$ then $\mathcal{C}_\emptyset \vdash_{\mathrm{L}} \alpha : 1$

Proof: Let P be a Hilbert proof of α in $\mathcal{L}_{Ax}$. Without loss of generality, it can be assumed that the proof P either consists of a single step, given by an instance of an axiom, or of a sequence of steps involving either instantiation of axioms or (MP), such that only instantiations necessary for a subsequent (MP) step are made. The proof is by induction on the number n ($n \geq 0$) of applications of the (MP) rule in $\mathcal{P}$.

Base case. In case $n = 0$, there are no applications of (MP) and P consists of an instance of one of the axioms (L1) - (L9). Figures 6 to 9 show that axioms (L2), (L7), (L8) and (L9) can be proved with label 1 from the empty configuration in a $\mathrm{L_{CLDS}}$ system. The cases for (L1), (L3), (L4), (L5) and (L6) are easy and are left to the reader.

$\mathcal{C}_\emptyset\langle\rangle$

$\mathcal{C}_1\langle[(\alpha \to \beta) : c_{(\alpha\to\beta)}]\rangle$ (assumption)

$\mathcal{C}_2\langle[\gamma \to \alpha : c_{\gamma\to\alpha}]\rangle$ (assumption)

$\mathcal{C}_3\langle[\gamma : c_\gamma]\rangle$ (assumption)

$\mathcal{C}_4\langle\alpha : c_{\gamma\to\alpha} \circ c_\gamma\rangle$ $(\to\mathcal{E})$

$\mathcal{C}_5\langle\beta : c_{(\alpha\to\beta)} \circ c_{\gamma\to\alpha} \circ c_\gamma\rangle$ $(\to\mathcal{E})$

$\mathcal{C}_6\langle c_{(\alpha\to\beta)} \circ c_{\gamma\to\alpha} \circ c_\gamma \preceq 1 \circ c_{(\alpha\to\beta)} \circ c_{\gamma\to\alpha} \circ c_\gamma\rangle$ (R-A)

$\mathcal{C}_7\langle\beta : 1 \circ c_{(\alpha\to\beta)} \circ c_{\gamma\to\alpha} \circ c_\gamma\rangle$ (unit)

$\mathcal{C}_8\langle\gamma \to \beta : 1 \circ c_{(\alpha\to\beta)} \circ c_{\gamma\to\alpha}\rangle$ $(\to\mathcal{I})$

$\mathcal{C}_9\langle(\gamma \to \alpha) \to (\gamma \to \beta) : 1 \circ c_{(\alpha\to\beta)}\rangle$ $(\to\mathcal{I})$

$\mathcal{C}_{10}\langle(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta)) : 1\rangle$ $(\to\mathcal{I})$

Fig. 6. Derivation of $(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta)) : 1$

$$
\begin{array}{ll}
\mathcal{C}_\emptyset\langle\rangle & \\
\quad \mathcal{C}_1\langle[\alpha \rightarrow \neg\beta : c_{\alpha\rightarrow\neg\beta}]\rangle & \text{(assumption)} \\
\quad\quad \mathcal{C}_2\langle[\beta : c_\beta]\rangle & \text{(assumption)} \\
\quad\quad\quad \mathcal{C}_3\langle[\alpha : c_\alpha]\rangle & \text{(assumption)} \\
\quad\quad\quad \mathcal{C}_4\langle\neg\beta : c_{\alpha\rightarrow\neg\beta} \circ c_\alpha\rangle & (\rightarrow\mathcal{E}) \\
\quad\quad\quad \mathcal{C}_5\langle\bot : c_{\alpha\rightarrow\neg\beta} \circ c_\alpha \circ c_\beta\rangle & (\neg\mathcal{E}) \\
\quad\quad\quad \mathcal{C}_6\langle c_{\alpha\rightarrow\neg\beta} \circ c_\alpha \circ c_\beta \preceq 1 \circ c_{\alpha\rightarrow\neg\beta} \circ c_\beta \circ c_\alpha\rangle & (R\text{-A}) \\
\quad\quad\quad \mathcal{C}_7\langle\bot : c_{1 \circ c_{\alpha\rightarrow\neg\beta} \circ c_\beta \circ c_\alpha}\rangle & \text{(unit)} \\
\quad\quad \mathcal{C}_8\langle\neg\alpha : c_{1 \circ c_{\alpha\rightarrow\neg\beta} \circ c_\beta}\rangle & (\neg\mathcal{I}) \\
\quad \mathcal{C}_9\langle\beta \rightarrow \neg\alpha : 1 \circ c_{\alpha\rightarrow\neg\beta}\rangle & (\rightarrow\mathcal{I}) \\
\mathcal{C}_{10}\langle(\alpha \rightarrow \neg\beta) \rightarrow (\beta \rightarrow \neg\alpha) : 1\rangle & (\rightarrow\mathcal{I})
\end{array}
$$

Fig. 7. Derivation of $(\alpha \rightarrow \neg\beta) \rightarrow (\beta \rightarrow \neg\alpha) : 1$

$$
\begin{array}{ll}
\mathcal{C}_\emptyset\langle\rangle & \\
\quad \mathcal{C}_1\langle[(\alpha : c_\alpha)]\rangle & \text{(assumption)} \\
\quad\quad \mathcal{C}_2\langle[\beta : c_\beta]\rangle & \text{(assumption)} \\
\quad\quad \mathcal{C}_3\langle\alpha \otimes \beta : 1 \circ c_\alpha \circ c_\beta\rangle & (\otimes\mathcal{I}) \\
\quad \mathcal{C}_4\langle\beta \rightarrow \alpha \otimes \beta : c_\alpha\rangle & (\rightarrow\mathcal{I}) \\
\mathcal{C}_5\langle\alpha \rightarrow (\beta \rightarrow (\alpha \otimes \beta)) : 1\rangle & (\rightarrow\mathcal{I})
\end{array}
$$

Fig. 8. Derivation of $\alpha \rightarrow (\beta \rightarrow (\alpha \otimes \beta)) : 1$

Induction Step. Suppose by inductive hypothesis that, for any Hilbert proof of γ using m applications of (MP), $1 \leq m \leq k$, there exists a derivation $\mathcal{C}_\emptyset \vdash_{\mathrm{L}} \gamma : 1$. Suppose now that there is a Hilbert proof P of α using $k+1$ applications of (MP) and that the $k+1$th step in P is an application of (MP). Then there exist $(\beta \rightarrow \alpha)$ and β such that $\vdash_{\mathrm{Ax}} \beta \rightarrow \alpha$ and $\vdash_{\mathrm{Ax}} \beta$, both with a number of (MP) applications strictly less than $k+1$. By inductive hypothesis $\mathcal{C}_\emptyset \vdash_{\mathrm{L}} \beta \rightarrow \alpha : 1$ and $\mathcal{C}_\emptyset \vdash_{\mathrm{L}} \beta : 1$. This implies, using monotonicity, that there exists a configuration $\mathcal{C}'$, such that $\beta \rightarrow \alpha : 1 \in \mathcal{C}'$ and $\beta : 1 \in \mathcal{C}'$ and $\mathcal{C}_\emptyset \vdash_{\mathrm{L}} \mathcal{C}'$. By three subsequent applications of $(\rightarrow\mathcal{E})$, $(R$-A) and (unit) and by transitivity of $\vdash_{\mathrm{L}}$, it is possible to show that $\mathcal{C}_\emptyset \vdash_{\mathrm{L}} \alpha : 1$. □

Theorem 48. *Consider the* $\mathrm{L}_{\mathrm{CLDS}}$ *system, the Hilbert system* $\mathcal{L}_{Ax}$ *and the empty initial configuration* $\mathcal{C}_\emptyset = \langle\{\ \}, \mathcal{F}\rangle$*, given by* $\mathcal{F}(\lambda) = \{\ \}$ *for any label* λ*. Let* α *be a wff of* $\mathcal{L}_P$*.*

If $\mathcal{C}_\emptyset \vdash_{\mathrm{L}} \alpha : 1$ then $\vdash_{\mathrm{Ax}} \alpha$

$\mathcal{C}_\emptyset\langle\rangle$	
$\mathcal{C}_1\langle[\alpha \to (\beta \to \gamma) : c_{\alpha\to(\beta\to\gamma)}]\rangle$	(assumption)
$\mathcal{C}_2\langle[\alpha \otimes \beta : c_{\alpha\otimes\beta}]\rangle$	(assumption)
$\mathcal{C}_3\langle\alpha : c_\alpha\rangle$	$(\otimes\mathcal{E})$
$\mathcal{C}_4\langle\beta : c_\beta\rangle$	$(\otimes\mathcal{E})$
$\mathcal{C}_5\langle c_\alpha \circ c_\beta \preceq c_{\alpha\otimes\beta}\rangle$	$(\otimes\mathcal{E})$
$\mathcal{C}_6\langle\beta \to \gamma : c_{\alpha\to(\beta\to\gamma)} \circ c_\alpha\rangle$	$(\to\mathcal{E})$
$\mathcal{C}_7\langle\gamma : c_{\alpha\to(\beta\to\gamma)} \circ c_\alpha \circ c_\beta\rangle$	$(\to\mathcal{E})$
$\mathcal{C}_8\langle\gamma : 1 \circ c_{\alpha\to(\beta\to\gamma)} \circ c_{\alpha\otimes\beta}\rangle$	(R-A)
$\mathcal{C}_9\langle(\alpha \otimes \beta) \to \gamma : 1 \circ c_{\alpha\to(\beta\to\gamma)}\rangle$	$(\to\mathcal{I})$
$\mathcal{C}_{10}\langle(\alpha \to (\beta \to \gamma)) \to ((\alpha \otimes \beta) \to \gamma) : 1\rangle$	$(\to\mathcal{I})$

Fig. 9. Derivation of $(\alpha \to (\beta \to \gamma)) \to ((\alpha \otimes \beta) \to \gamma) : 1$

PROOF: Suppose that $\mathcal{C}_\emptyset \vdash_{\mathrm{L}} \alpha : 1$. By soundness of $\vdash_{\mathrm{L}}$, $\mathcal{C}_\emptyset \models_{\mathrm{L}} \alpha : 1$, so $\mathcal{A}_{\mathrm{L}}^+$, $\mathrm{FOT}(\mathcal{C}_\emptyset) \models [\alpha]^*(1)$, (or $\mathcal{A}_{\mathrm{L}}^+ \models [\alpha]^*(1)$). Hence any model of $\mathcal{A}_{\mathrm{L}}^+$ is also a model of $[\alpha]^*(1)$. If a model of $\mathcal{A}_{\mathrm{L}}^+$ can be constructed such that it satisfies $[\beta]^*(1)$ iff $\vdash_{\mathrm{Ax}} \beta$, for any wff β, then it can be concluded that $\vdash_{\mathrm{Ax}} \alpha$. Such a model does exist and it is based on the canonical interpretation first introduced in [10] and defined below. Lemma 50 shows that such an interpretation satisfies $\mathcal{A}_{\mathrm{L}}$ and Lemma 51 shows that it also satisfies $\mathcal{A}_{\mathrm{L}}^+$. □

Definition 49 (Canonical Interpretation). Let $\langle\mathcal{L}_P, \mathcal{L}_L\rangle$ be a $\mathrm{L}_{\mathrm{CLDS}}$ language. A *canonical interpretation* $\mathcal{I}$ is an interpretation from $Mon(\mathcal{L}_P, \mathcal{L}_L)$ onto $PW(\mathcal{L}_P)$ defined as follows:

- each characteristic label c_α is interpreted as $\{z | \vdash_{\mathrm{Ax}} \alpha \to z\}$.
- each constant label λ is interpreted as $\emptyset$.
- $||\lambda \circ \lambda'|| = \{z | \vdash_{\mathrm{Ax}} (\alpha \otimes \beta) \to z,$ where $\alpha \in ||\lambda||$ and $\beta \in ||\lambda'||\}$.
- 1 is interpreted as $\{z | \vdash_{\mathrm{Ax}} z\}$.
- $x \preceq y$ is interpreted as $||x|| \subseteq ||y||$.
- $||[\alpha]^*|| = \{||x|| \mid \alpha \in ||x||\}$.

By definition, the canonical interpretation $\mathcal{I}$ satisfies an atomic formula $[\alpha]^*(x)$ if and only if $\alpha \in ||x||$. This means, in particular, that $\mathcal{I}$ satisfies a ground atomic formula $[\alpha]^*(1)$ if and only if $\alpha \in ||1||$, and hence $\vdash_{\mathrm{Ax}} \alpha$. (Note that the interpretation of compound terms including one or more constant labels is also an empty set.) Lemmas 50 and 51 show that the canonical interpretation $\mathcal{I}$, defined above, is a model of $\mathcal{A}_{\mathrm{L}}$ and $\mathcal{A}_{\mathrm{L}}^+$ respectively.

Lemma 50. *Let $\mathcal{I}$ be the canonical interpretation given in Definition 49. Then $\mathcal{I}$ is a model of the labelling algebra $\mathcal{A}_{\mathrm{L}}$.*

Proof: The proof consists of showing that $\mathcal{I}$ satisfies each axiom schema of $\mathcal{A}_{\mathrm{L}}$.

Commutativity and Associativity:

Consider first the (Commutativity) schema. Suppose that $\delta \in ||\lambda \circ \lambda'||$ then there is a Hilbert proof of $(\alpha \otimes \beta) \rightarrow \delta$, where $\alpha \in ||\lambda||$ and $\beta \in ||\lambda'||$. Figure 10 shows that using this Hilbert proof it is possible to construct a proof of $(\beta \otimes \alpha) \rightarrow \delta$. The proof for the (Associativity) schema follows a similar argument.

(1) $(\alpha \otimes \beta) \rightarrow \delta$	(given)
(2) $\alpha \rightarrow (\beta \rightarrow (\alpha \otimes \beta))$	(L8)
(3) $(\alpha \rightarrow (\beta \rightarrow (\alpha \otimes \beta))) \rightarrow (\beta \rightarrow (\alpha \rightarrow (\alpha \otimes \beta)))$	(Instance of (L3))
(4) $\beta \rightarrow (\alpha \rightarrow (\alpha \otimes \beta))$	(MP – (2) and (3))
(5) $(\beta \rightarrow (\alpha \rightarrow (\alpha \otimes \beta))) \rightarrow ((\beta \otimes \alpha) \rightarrow (\alpha \otimes \beta))$	(Instance of (L9))
(6) $(\beta \otimes \alpha) \rightarrow (\alpha \otimes \beta)$	(MP – (4) and (5))
(7) $((\beta \otimes \alpha) \rightarrow (\alpha \otimes \beta)) \rightarrow (((\alpha \otimes \beta) \rightarrow \delta) \rightarrow ((\beta \otimes \alpha) \rightarrow \delta))$	(Instance of (L10))
(8) $((\alpha \otimes \beta) \rightarrow \delta) \rightarrow ((\beta \otimes \alpha) \rightarrow \delta)$	(MP – (6) and (7))
(9) $(\beta \otimes \alpha) \rightarrow \delta$	(MP – (1) and (8))

Fig. 10. Derivation of $(\beta \otimes \alpha) \rightarrow \delta$ from $(\alpha \otimes \beta) \rightarrow \delta$

Identity

To show that $1 \circ \lambda \preceq \lambda$, suppose that $\delta \in ||1 \circ \lambda||$. Then $(\beta \otimes \alpha) \rightarrow \delta$, where $\vdash_{\mathrm{Ax}} \beta$ and $\vdash_{\mathrm{Ax}} \gamma \rightarrow \alpha$ for $\gamma \in ||\lambda||$. It is required to show that there exists a Hilbert proof of $\gamma \rightarrow \delta$. This is illustrated in Figure 11. To show that $\lambda \preceq 1 \circ \lambda$, suppose that $\delta \in ||\lambda||$. Then $\vdash_{\mathrm{Ax}} \gamma \rightarrow \delta$ for some $\gamma \in |||\lambda||$. It is required to show $\delta \in ||1 \circ \lambda||$. It is sufficient to show that $((\gamma \rightarrow \gamma) \otimes \gamma) \rightarrow \delta$, since $\gamma \rightarrow \gamma \in ||1||$. The proof of $((\gamma \rightarrow \gamma) \otimes \gamma) \rightarrow \delta$ is illustrated in Figure 12.

Order-preserving

Suppose that $||\lambda|| \subseteq ||\lambda'||$ and $\delta \in ||\lambda \circ \lambda''||$. Then $\vdash_{\mathrm{Ax}} (\alpha \otimes \beta) \rightarrow \delta$, where $\alpha \in ||\lambda||$ and $\beta \in ||\lambda''||$. Hence $\alpha \in ||\lambda'||$ and $\vdash_{\mathrm{Ax}} (\alpha \otimes \beta) \rightarrow \delta$ and $\delta \in ||\lambda' \circ \lambda''||$.

Reflexivity and Transitivity

The satisfiability of these two schemas follows immediately from the reflexive and transitive properties of the $\subseteq$ relation. □

Lemma 51. *Let $\mathcal{I}$ be the canonical interpretation given in Definition 49. $\mathcal{I}$ is a model of the extended algebra $\mathcal{A}_{\mathrm{L}}^{+}$.*

Proof: The proof follows a similar argument as that used to prove Lemma 50. Only the satisfiability of schemas (Ax3a) and (Ax3b) are given here

(1) $\underline{\beta}$	(given)
(2) $\underline{\gamma \to \alpha}$	(given)
(3) $\underline{(\beta \otimes \alpha) \to \delta}$	(given)
(4) $\underline{\beta \to (\alpha \to (\beta \otimes \alpha))}$	(Instance of (L8))
(5) $\underline{\alpha \to (\beta \otimes \alpha)}$	(MP – (1) and (4))
(6) $\underline{(\alpha \to (\beta \otimes \alpha)) \to ((\gamma \to \alpha) \to (\gamma \to (\beta \otimes \alpha)))}$	(Instance of (L2))
(7) $\underline{(\gamma \to \alpha) \to (\gamma \to (\beta \otimes \alpha))}$	(MP – (5) and (6))
(8) $\underline{\gamma \to (\beta \otimes \alpha)}$	(MP – (2) and (7))
(9) $\underline{(\gamma \to (\beta \otimes \alpha)) \to (((\beta \otimes \alpha) \to \delta) \to (\gamma \to \delta))}$	(Instance of (L10))
(10) $\underline{((\beta \otimes \alpha) \to \delta) \to (\gamma \to \delta)}$	(MP – (8) and (9))
(11) $\gamma \to \delta$	(MP – (3) and (10))

Fig. 11. Derivation of $\gamma \to \delta$ from $\beta, (\gamma \to \alpha)$ and $(\beta \otimes \alpha) \to \delta$

(1) $\underline{\gamma \to \delta}$	(given)
(2) $\underline{(\gamma \to \delta) \to ((\gamma \to \gamma) \to (\gamma \to \delta))}$	(Instance (L2))
(3) $\underline{(\gamma \to \gamma) \to (\gamma \to \delta)}$	(MP – (1) and (2))
(4) $\underline{((\gamma \to \gamma) \to (\gamma \to \delta)) \to ((\gamma \to \gamma) \otimes \gamma) \to \delta}$	(Instance of (L9))
(5) $((\gamma \to \gamma) \otimes \gamma) \to \delta$	(MP – (3) and (4))

Fig. 12. Derivation of $((\gamma \to \gamma) \otimes \gamma) \to \delta$ from $\gamma \to \delta$

as the other cases are similar. The definitions of satisfiability of atomic formulae of the form $[\alpha]^*(x)$ (i.e. they are satisfied if and only if $\alpha \in ||x||$) and of atomic formulae of the form $x \preceq y$ (i.e. $x \preceq y$ is satisfied if and only if $||x|| \subseteq ||y||$) are used implicitly.

(Ax3a)
Suppose that $\alpha \to \beta \in ||\lambda||$ and $\alpha \in ||\lambda'||$, hence $\vdash_{\mathrm{Ax}} \gamma \to (\alpha \to \beta)$ and $\vdash_{\mathrm{Ax}} \delta \to \alpha$ for some $\gamma \in ||\lambda||$ and $\delta \in ||\lambda'||$. To show $\beta \in ||\lambda \circ \lambda'||$ it is sufficient to show $\vdash_{\mathrm{Ax}} (\gamma \otimes \delta) \to \beta$. A proof of this is in Figure 13.

(Ax3b)
Since, by the construction of the canonical interpretation, $\alpha \in ||c_\alpha||$, suppose that $\beta \in ||\lambda \circ c_\alpha||$. Then $\vdash_{\mathrm{Ax}} (\gamma \otimes \alpha) \to \beta$, where $\gamma \in ||\lambda||$. To show $\alpha \to \beta \in ||\lambda||$ it suffices to show $\vdash_{\mathrm{Ax}} \gamma \to (\alpha \to \beta)$. A proof of this is given in Figure 14.

$\square$

(1) $\underline{\gamma \to (\alpha \to \beta)}$	(given)
(2) $\underline{\delta \to \alpha}$	(given)
(3) $\underline{(\delta \to \alpha) \to ((\alpha \to \beta) \to (\delta \to \beta))}$	(Instance of (L10))
(4) $\underline{(\alpha \to \beta) \to (\delta \to \beta)}$	(MP – (2) and (3))
(5) $\underline{((\alpha \to \beta) \to (\delta \to \beta)) \to ((\gamma \to (\alpha \to \beta)) \to (\gamma \to (\delta \to \beta)))}$	(Instance of (L2))
(6) $\underline{(\gamma \to (\alpha \to \beta)) \to (\gamma \to (\delta \to \beta))}$	(MP – (4) and (5))
(7) $\underline{\gamma \to (\delta \to \beta)}$	(MP – (1) and (6))
(8) $\underline{(\gamma \to (\delta \to \beta)) \to ((\gamma \otimes \delta) \to \beta)}$	(Instance of (L9))
(9) $(\gamma \otimes \delta) \to \beta$	(MP – (7) and (8))

Fig. 13. Derivation of $(\gamma \otimes \delta) \to \beta$ from $\gamma \to (\alpha \to \beta)$ and $(\delta \to \alpha)$

(1) $\underline{(\gamma \otimes \alpha) \to \beta}$	(given)
(2) $\underline{\gamma \to (\alpha \to (\gamma \otimes \alpha))}$	(Instance of (L8))
(3) $\underline{((\gamma \otimes \alpha) \to \beta) \to ((\alpha \to (\gamma \otimes \alpha)) \to (\alpha \to \beta))}$	(Instance of (L2))
(4) $\underline{(\alpha \to (\gamma \otimes \alpha)) \to (\alpha \to \beta)}$	(MP – (1) and (3))
(5) $\underline{(\gamma \to (\alpha \to (\gamma \otimes \alpha))) \to (((\alpha \to (\gamma \otimes \alpha)) \to (\alpha \to \beta)) \to (\gamma \to (\alpha \to \beta)))}$	(Instance of (L10))
(6) $\underline{(((\alpha \to (\gamma \otimes \alpha)) \to (\alpha \to \beta)) \to (\gamma \to (\alpha \to \beta))}$	(MP – (2) and (5))
(7) $\gamma \to (\alpha \to \beta)$	(MP – (4) and (6))

Fig. 14. Derivation of $\gamma \to (\alpha \to \beta)$ from $(\gamma \otimes \alpha) \to \beta$

5 The $\mathbf{F}_{\mathrm{CLDS}}$ System

In this section, the $\mathrm{F}_{\mathrm{CLDS}}$ system for Łukasiewicz fuzzy logic is defined. In particular, it is shown how Łukasiewicz fuzzy logic can be formalised as a variation of the system $\mathrm{L}_{\mathrm{CLDS}}$.

Syntax. The language of $\mathrm{F}_{\mathrm{CLDS}}$ system is given by the pair $\langle \mathcal{L}_P, \mathcal{L}_L \rangle$, where $\mathcal{L}_P$ is the standard propositional language of $\mathrm{L}_{\mathrm{CLDS}}$, but without the operator $\otimes$, and $\mathcal{L}_L$ is defined as the labelling language $\mathcal{L}_L$ of $\mathrm{L}_{\mathrm{CLDS}}$. The ground terms of $\mathcal{L}_L$ denote, in this system, degrees of truth, the binary function $\circ$ denotes an arithmetic combination of degrees of truth, whereas the binary R-predicate R[12] represents a total-ordering relation $\geq$ on the degrees of truth.

[12] The symbol $\preceq$ of $\mathrm{L}_{\mathrm{CLDS}}$ is replaced again by R to avoid confusion with its later interpretation as $\geq$.

[13] The language $\mathcal{L}_L$ is also, in this case, extended to the semi-extended labelling language $Func(\mathcal{L}_P, \mathcal{L}_L)$, in the same way as defined in Section 4 for the $\mathrm{L_{CLDS}}$ system, but with the following minor notational differences. The special parameter $c_\perp$ is denoted with 0 in the $\mathrm{F_{CLDS}}$ system. The parameter 1 corresponds to the semantic notion of "truth" in Łukasiewicz fuzzy logic.

Definition 52 (Labelling algebra $\mathcal{A}_\mathrm{F}$). The labelling algebra $\mathcal{A}_\mathrm{F}$, written in $Func(\mathcal{L}_P, \mathcal{L}_L)$, is given by the labelling algebra $\mathcal{A}_\mathrm{L}$ of Definition 33 extended with the following axiom schemas:

$$\forall x, y(R(x,y) \vee R(y,x)) \qquad \text{(Totality)}$$
$$\forall x, y.R(x, x \circ y) \qquad \text{(Monotonicity)}$$

Note that $\forall x.R(1,x)$ is a derived axiom of $\mathcal{A}_\mathrm{F}$, which can be proved using the (Transitivity) axiom on the R-literals $R(1, 1 \circ x)$, given by (Monotonicity), and $R(1 \circ x, x)$, given by (Identity). A $\mathrm{F_{CLDS}}$ system is then defined by the tuple $\mathcal{L} = \langle\langle\mathcal{L}_P, \mathcal{L}_L\rangle, \mathcal{A}_\mathrm{F}, \mathcal{R}_F\rangle$, where the set $\mathcal{R}_F$ of inference rules is defined in the following section.

5.1 Proof theory and Semantics of $\mathrm{F_{CLDS}}$

The set $\mathcal{R}_F$ of inference rules includes the set of inference rules $\mathcal{R}_L$ of the $\mathrm{L_{CLDS}}$ system, with the exception of $(\otimes\mathcal{E})$ and $(\otimes\mathcal{I})$, together with the additional rules given in Table 6. These rules allow the derivation of the ex-falsum property (i.e. conclude $\gamma : \lambda'$ from $\perp : \lambda$) as shown in Figure 15. Sample derivations of $((\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \alpha)) \rightarrow (\beta \rightarrow \alpha) : 1$ and $((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow ((\beta \rightarrow \alpha) \rightarrow \alpha) : 1$ are shown in Figure 16 and 17 respectively.

Semantics. The extended labelling language $Mon(\mathcal{L}_P, \mathcal{L}_L)$ is defined as in Section 4. An atomic formula of the form $[\alpha]^*(x)$ denotes, in the $\mathrm{F_{CLDS}}$ system, that "the formula α has a degree of truth x". An extended algebra $\mathcal{A}_\mathrm{F}^+$ is defined below as a variation of the extended algebra $\mathcal{A}_\mathrm{L}^+$ of the $\mathrm{L_{CLDS}}$ system.

Definition 53. Given the extended labelling language $Mon(\mathcal{L}_P, \mathcal{L}_L)$ and the labelling algebra $\mathcal{A}_\mathrm{F}$, the *extended algebra* $\mathcal{A}_\mathrm{F}^+$ is the theory, written in $Mon(\mathcal{L}_P, \mathcal{L}_L)$, given by axioms (Ax1)–(Ax4b) and axiom (Ax6) of the extended algebra $\mathcal{A}_\mathrm{L}^+$, given in Definition 34, together with the following four additional axioms:

(Ax7) $[\alpha]^*(0)$

[13] The symbol $\preceq$ of $\mathrm{L_{CLDS}}$ is replaced again by R to avoid confusion with its later interpretation as $\geq$.

Table 6. Additional rules for fuzzy operators.

$$\frac{\mathcal{C}\langle(\alpha \to \beta) \to \beta : \lambda\rangle \quad \begin{matrix}\mathcal{C}\langle[\alpha : c_\alpha]\rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle\gamma : \lambda' \circ c_\alpha\rangle\end{matrix} \quad \begin{matrix}\mathcal{C}\langle[\beta : c_\beta]\rangle \\ \vdots \\ \overline{\mathcal{C}}\langle\gamma : \lambda' \circ c_\beta\rangle\end{matrix}}{\mathcal{C}'\langle\gamma : \lambda \circ \lambda'\rangle} \quad \text{(cond-split)}$$

$$\frac{\begin{matrix}\mathcal{C}\langle[R(c_\alpha, c_\beta)]\rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle\gamma : \lambda\rangle\end{matrix} \quad \begin{matrix}\mathcal{C}\langle[R(c_\beta, c_\alpha)]\rangle \\ \vdots \\ \overline{\mathcal{C}}\langle\gamma : \lambda\rangle\end{matrix}}{\mathcal{C}'\langle\gamma : \lambda\rangle} \quad \text{(split)} \qquad \frac{\mathcal{C}\langle R(c_\alpha, c_\beta)\rangle}{\mathcal{C}'\langle\beta \to \alpha : 1\rangle} \quad \text{(universal 1)}$$

$$\frac{\mathcal{C}}{\mathcal{C}'\langle\alpha : 0\rangle} \quad \text{(base)} \qquad \frac{\mathcal{C}\langle\beta \to \alpha : 1\rangle}{\mathcal{C}'\langle R(c_\alpha, c_\beta)\rangle} \quad \text{(universal 2)}$$

$\mathcal{C}_\emptyset\langle\rangle$	
$\mathcal{C}_1\langle[\bot : \lambda]\rangle$	(assumption)
$\mathcal{C}_2\langle[\neg\alpha : c_{\neg\alpha}]\rangle$	(assumption)
$\mathcal{C}_3\langle\alpha : 0\rangle$	(base)
$\mathcal{C}_4\langle\bot : 0 \circ c_{\neg\alpha}\rangle$	$(\neg\mathcal{E})$
$\mathcal{C}_5\langle R(0, \lambda)\rangle$	(ch)
$\mathcal{C}_6\langle R(0 \circ c_{\neg\alpha}, \lambda \circ c_{\neg\alpha})\rangle$	(R-A)
$\mathcal{C}_7\langle\bot : \lambda \circ c_{\neg\alpha}\rangle$	(unit)
$\mathcal{C}_8\langle\neg\neg\alpha : \lambda\rangle$	$(\neg\mathcal{I})$
$\mathcal{C}_9\langle\alpha : \lambda\rangle$	$(\neg\neg)$

Fig. 15. Derivation of the ex-falsum property

(Ax8) $R(c_\alpha, c_\beta) \leftrightarrow [\beta \to \alpha]^*(1)$

(Ax9) $\forall x, y([(\alpha \to \beta) \to \beta]^*(x) \wedge [\alpha \to \gamma]^*(y) \wedge [\beta \to \gamma]^*(y)) \to [\gamma]^*(x \circ y)$

(Ax10) $[\alpha \to \beta]^*(1) \vee [\beta \to \alpha]^*(1)$

Axiom (Ax7) is a generalisation of axiom (Ax7) given in the extended algebra $\mathcal{A}_{\mathrm{L}}^{+}$ for the $\mathrm{L_{CLDS}}$ system. A special theorem of $\mathcal{A}_{\mathrm{F}}^{+}$ is the schema $[\bot \to \alpha]^*(1)$, which can be proved using axiom (Ax7) of $\mathcal{A}_{\mathrm{F}}^{+}$.[14]

[14] Axiom (Ax10) is required since there is no explicit disjunction in $\mathcal{L}_P$.

$\mathcal{C}_\emptyset\langle\rangle$

$\mathcal{C}_1\langle[(\alpha \to \beta) \to (\beta \to \alpha) : c_{(\alpha\to\beta)\to(\beta\to\alpha)}]\rangle$ (assumption)

$\mathcal{C}_2\langle[R(c_\beta, c_\alpha)]\rangle$ (assumption)

$\mathcal{C}_3\langle\alpha \to \beta : 1\rangle$ (universal 2)

$\mathcal{C}_4\langle\beta \to \alpha : c_{(\alpha\to\beta)\to(\beta\to\alpha)} \circ 1\rangle$ $(\to\mathcal{E})$

$\mathcal{C}_5\langle R(c_{(\alpha\to\beta)\to(\beta\to\alpha)}, c_{(\alpha\to\beta)\to(\beta\to\alpha)} \circ 1)\rangle$ (R-A)

$\mathcal{C}_6\langle\beta \to \alpha : c_{(\alpha\to\beta)\to(\beta\to\alpha)}\rangle$ (unit)

$\mathcal{C}_7\langle[R(c_\alpha, c_\beta)]\rangle$ (assumption)

$\mathcal{C}_8\langle\beta \to \alpha : 1\rangle$ (universal 2)

$\mathcal{C}_9\langle R(1, c_{(\alpha\to\beta)\to(\beta\to\alpha)})\rangle$ (R-A)

$\mathcal{C}_{10}\langle\beta \to \alpha : c_{(\alpha\to\beta)\to(\beta\to\alpha)}\rangle$ (unit)

$\mathcal{C}_{11}\langle\beta \to \alpha : c_{(\alpha\to\beta)\to(\beta\to\alpha)}\rangle$ (split)

$\mathcal{C}_{12}\langle R(1 \circ c_{(\alpha\to\beta)\to(\beta\to\alpha)}, c_{(\alpha\to\beta)\to(\beta\to\alpha)})\rangle$ (R-A)

$\mathcal{C}_{13}\langle\beta \to \alpha : 1 \circ c_{(\alpha\to\beta)\to(\beta\to\alpha)}\rangle$ (unit)

$\mathcal{C}_{14}\langle((\alpha \to \beta) \to (\beta \to \alpha)) \to (\beta \to \alpha) : 1\rangle$ $(\to\mathcal{I})$

Fig. 16. Derivation of $((\alpha \to \beta) \to (\beta \to \alpha)) \to (\beta \to \alpha) : 1$

$\mathcal{C}_\emptyset\langle\rangle$

$\mathcal{C}_1\langle[(\alpha \to \beta) \to \beta : c_{(\alpha\to\beta)\to\beta}]\rangle$ (assumption)

$\mathcal{C}_2\langle[\beta \to \alpha : c_{\beta\to\alpha}]\rangle$ (assumption)

$\mathcal{C}_3\langle[\beta : c_\beta]\rangle$ (assumption)

$\mathcal{C}_4\langle\alpha : c_{\beta\to\alpha} \circ c_\beta\rangle$ $(\to\mathcal{E})$

$\mathcal{C}_5\langle[\alpha : c_\alpha]\rangle$ (assumption)

$\mathcal{C}_6\langle R(c_\alpha, c_{\beta\to\alpha} \circ c_\alpha)\rangle$ (R-A)

$\mathcal{C}_7\langle\alpha : c_{\beta\to\alpha} \circ c_\alpha\rangle$ (unit)

$\mathcal{C}_8\langle\alpha : c_{(\alpha\to\beta)\to\beta} \circ c_{\beta\to\alpha}\rangle$ (cond-split)

$\mathcal{C}_9\langle R(c_{(\alpha\to\beta)\to\beta} \circ c_{\beta\to\alpha}, 1 \circ c_{(\alpha\to\beta)\to\beta} \circ c_{\beta\to\alpha})\rangle$ (R-A)

$\mathcal{C}_{10}\langle\alpha : 1 \circ c_{(\alpha\to\beta)\to\beta} \circ c_{\beta\to\alpha}\rangle$ (unit)

$\mathcal{C}_{11}\langle(\beta \to \alpha) \to \alpha : 1 \circ c_{(\alpha\to\beta)\to\beta}\rangle$ $(\to\mathcal{I})$

$\mathcal{C}_{12}\langle((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha) : 1\rangle$ $(\to\mathcal{I})$

Fig. 17. Derivation of $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha) : 1$

The notion of satisfiability and semantic entailment of the $\mathrm{F_{CLDS}}$ systems, denoted with $\models_{\mathrm{F}}$, are as specified in Definitions 10 and 11, but based on the extended algebra $\mathcal{A}^{+}_{\mathrm{F}}$.

5.2 Main results about the $\mathbf{F}_{\mathrm{CLDS}}$ system

Soundness. Also for the $\mathrm{F_{CLDS}}$, the soundness and completeness of the proof system with respect to the notion of semantic entailment $\models_{\mathrm{F}}$, are proved using the same methodology described in Section 2.3. Most of the theorems, lemmas and propositions used to prove these two properties are similar to those used in the $\mathrm{L_{CLDS}}$ system (see [6]).

Theorem 54 (Soundness of $\mathbf{F}_{\mathrm{CLDS}}$). *Let $\mathcal{F} = \langle\langle\mathcal{L}_P, \mathcal{L}_L\rangle, \mathcal{A}_{\mathrm{F}}, \mathcal{R}_F\rangle$ be a $\mathrm{F_{CLDS}}$ system and let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations. If $\mathcal{C} \vdash_{\mathrm{F}} \mathcal{C}'$ then $\mathcal{C} \models_{\mathrm{F}} \mathcal{C}'$.*

Proof: The proof follows the same argument as given in Theorem 35. The proof is by induction on the smallest size of derivations of the form $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, \overline{m}\rangle$, where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_n = \mathcal{C}'$. The inductive step is proved by cases on the inference rule applied on the last step $\mathcal{C}_{n-1}/\mathcal{C}_n$ of the derivation. The cases for the (universal 1), (split) and (cond-split) are given here. The cases for the (base) and (universal 2) rules are similar.
Case 1: (universal 1).
In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in$ (universal 1). Then there exists an R-literal $R(c_\alpha, c_\beta)$ in $\mathcal{C}_{n-1}$ and in $\mathrm{FOT}(\mathcal{C}_{n-1})$.Using (Ax8), $[\beta \rightarrow \alpha]^*(1)$ is derivable in first-order logic from $FOT(\mathcal{C}_{n-1})$. Since $\mathcal{C}_n = \mathcal{C}_{n-1} + [\beta \rightarrow \alpha : 1]$, $\mathcal{A}^{+}_{\mathrm{F}}, FOT(\mathcal{C}_{n-1}) \vdash_{\mathrm{FOL}} FOT(\mathcal{C}_n)$.
Case 3: (split).
In this case $\mathcal{C}_{n-1}/\mathcal{C}_n$ belongs to (split). There exist two proofs, respectively, of $\gamma : \lambda$ from $\mathcal{C}_{n-1} + [R(c_\alpha, c_\beta)]$ and of $\gamma : \lambda$ from $\mathcal{C}_{n-1} + [R(c_\beta, c_\alpha)]$. By the induction hypothesis $\mathcal{A}^{+}_{\mathrm{F}}$, $FOT(\mathcal{C}_{n-1}) \cup \{R(c_\alpha, c_\beta)\} \vdash_{\mathrm{FOL}} [\gamma]^*(\lambda)$ and $\mathcal{A}^{+}_{\mathrm{F}}$, $FOT(\mathcal{C}_{n-1}) \cup \{R(c_\beta, c_\alpha)\} \vdash_{\mathrm{FOL}} [\gamma]^*(\lambda)$. Using the axiom (Ax10) and the $(\vee\mathcal{E})$ rule of first-order logic, $\mathcal{A}^{+}_{\mathrm{F}}$, $FOT(\mathcal{C}_{n-1}) \vdash_{\mathrm{FOL}} [\gamma]^*(\lambda)$. Since $\mathcal{C}_n = \mathcal{C}_{n-1} + [\gamma : \lambda]$, $\mathcal{A}^{+}_{\mathrm{F}}$, $FOT(\mathcal{C}_{n-1}) \vdash_{\mathrm{FOL}} FOT(\mathcal{C}_n)$.
Case 4: (cond-split).
In this case $\mathcal{C}_{n-1}/\mathcal{C}_n$ belongs to (cond-split). Then there exists a declarative unit $(\alpha \rightarrow \beta) \rightarrow \beta : \lambda$ in $\mathcal{C}_{n-1}$ and $[(\alpha \rightarrow \beta) \rightarrow \beta]^*(\lambda) \in FOT(\mathcal{C}_{n-1})$. There exist two proofs, respectively, of $\gamma : \lambda' \circ c_\alpha$ from $\mathcal{C}_{n-1} + [\alpha : c_\alpha]$ and of $\gamma : \lambda' \circ c_\beta$ from $\mathcal{C}_{n-1} + [\beta : c_\beta]$. By the induction hypothesis $\mathcal{A}^{+}_{\mathrm{F}}$, $FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(c_\alpha)\} \vdash_{\mathrm{FOL}} [\gamma]^*(\lambda' \circ c_\alpha)$ and $\mathcal{A}^{+}_{\mathrm{F}}$, $FOT(\mathcal{C}_{n-1}) \cup \{[\beta]^*(c_\beta)\} \vdash_{\mathrm{FOL}} [\gamma]^*(\lambda' \circ c_\beta)$. Using the axiom (Ax9) and the deduction theorem of first-order logic, $\mathcal{A}^{+}_{\mathrm{F}}$, $FOT(\mathcal{C}_{n-1}) \vdash_{\mathrm{FOL}} [\gamma]^*(\lambda \circ \lambda')$. Since $\mathcal{C}_n = \mathcal{C}_{n-1} + [\gamma : \lambda \circ \lambda']$, $\mathcal{A}^{+}_{\mathrm{F}}$, $FOT(\mathcal{C}_{n-1}) \vdash_{\mathrm{FOL}} FOT(\mathcal{C}_n)$.

□

Completeness. The completeness property is also in this system proved using standard Henkin-style methodology for classical logic. Notions of consistent and maximal consistent configurations are identical to those given for the $\mathrm{L_{CLDS}}$ system in Section 4.2. But some additional cases and properties need to be included to cover the additional rules of the $\mathrm{F_{CLDS}}$ proof system. These extra cases are considered in the following lemma.

Lemma 55 (Properties of $\mathcal{C}_{\mathrm{mcc}}$). *Let $\mathcal{C}_{\mathrm{mcc}}$ be a maximal consistent configuration of the F_{CLDS} system. $\mathcal{C}_{\mathrm{mcc}}$ satisfies the following properties, for any wffs α, β and γ and labels λ, λ':*

1. $\alpha : 0 \in \mathcal{C}_{\mathrm{mcc}}$.
2. $\top : c_\alpha$.
3. $\lambda \preceq \lambda \circ \lambda' \in \mathcal{C}_{\mathrm{mcc}}$.
4. *Either* $R(c_\beta, c_\alpha) \in \mathcal{C}_{\mathrm{mcc}}$ *or* $R(c_\alpha, c_\beta) \in \mathcal{C}_{\mathrm{mcc}}$.
5. $R(c_\alpha, c_\beta) \in \mathcal{C}_{\mathrm{mcc}}$ *iff* $\beta \to \alpha : 1 \in \mathcal{C}_{\mathrm{mcc}}$.
6. *If* $(\alpha \to \beta) \to \beta : \lambda \in \mathcal{C}_{\mathrm{mcc}}$ *and* $\alpha \to \gamma : \lambda' \in \mathcal{C}_{\mathrm{mcc}}$ *and* $\beta \to \gamma : \lambda' \in \mathcal{C}_{\mathrm{mcc}}$, *then* $\gamma : \lambda \circ \lambda' \in \mathcal{C}_{\mathrm{mcc}}$.
7. *Either* $\beta \to \alpha : 1 \in \mathcal{C}_{\mathrm{mcc}}$ *or* $\alpha \to \beta : 1 \in \mathcal{C}_{\mathrm{mcc}}$.

Proof: For each property, the proof follows a similar argument to that used in Lemma 40. □

Together with the above properties, a $\mathcal{C}_{\mathrm{mcc}}$ also satisfies the properties in Lemmas 39, 40, 42, 43, properties (1) and (2) of Lemma 41 and properties (2) – (4) of Lemma 44.

Theorem 56 (Completeness of $\mathbf{F}_{\mathrm{CLDS}}$). *Let $\mathcal{F} = \langle\langle \mathcal{L}_P, \mathcal{L}_L \rangle, \mathcal{A}_{\mathrm{F}}, \mathcal{R}_{\mathrm{F}} \rangle$ be a F_{CLDS} system, and let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations such that $\mathcal{C}' - \mathcal{C}$ is finite. If $\mathcal{C} \models_{\mathrm{F}} \mathcal{C}'$ then $\mathcal{C} \vdash_{\mathrm{F}} \mathcal{C}'$.*

Proof: The proof is similar to the proof of Theorem 37. □

Correspondence. The $\mathrm{F_{CLDS}}$ is now compared with a standard Hilbert axiomatisation of Łukasiewicz fuzzy logic [15]. This axiomatisation is a variation of the Hilbert system $\mathcal{L}_{\mathrm{Ax}}$ for linear logic given in Definition 46 as follows.

Definition 57. Let $\mathcal{L}_P$ be the propositional language of the $\mathrm{F_{CLDS}}$ system. The Hilbert system $\mathcal{F}_{\mathrm{Ax}}$ for Łukasiewicz fuzzy logic is given by axioms (L1)–(L7) of the Hilbert system $\mathcal{L}_{\mathrm{Ax}}$ together with the following specific axioms and the Modus Ponens (MP) rule:

(F1) $\alpha \to (\beta \to \alpha)$
(F2) $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$
(F3) $((\alpha \to \beta) \to (\beta \to \alpha)) \to (\beta \to \alpha)$
(F4) $\bot \to \alpha$

(F5) $\alpha \to \top$

The following theorem captures the correspondence between the proof system of $\mathrm{F}_{\mathrm{CLDS}}$ and the Hilbert system $\mathcal{F}_{\mathrm{Ax}}$. Each proof uses the notion of a canonical interpretation for $\mathrm{F}_{\mathrm{CLDS}}$. This interpretation maps ground terms of $\mathrm{F}_{\mathrm{CLDS}}$ onto a real number belonging to the interval $[0,1]$, in the following way.

- For each characteristic label c_α, $\| c_\alpha \| \in [0,1]$.
- For each constant label λ, $\| \lambda \| = 0$.
- $\| \lambda \circ \lambda' \| = max\{0, (\| \lambda \| + \| \lambda' \|) - 1\}$
- $\| 1 \| = 1$.
- $\| 0 \| = 0$.
- $\| R \| = \{\langle \| \lambda \|, \| \lambda' \| \rangle \mid \| \lambda \| \geq \| \lambda' \|\}$.
- For each monadic predicate $[\alpha]^*$, $\| [\alpha]^* \| = \{\| \lambda \| \mid \| \lambda \| \in [0, \| c_\alpha \|]\}$

Note that $\| c_{\alpha\to\beta} \| = min\{1, 1-(\| c_\alpha \| - \| c_\beta \|)\}$, which also implies $\| c_{\neg\alpha} \| = 1 - \| c_\alpha \|$. On the basis of this interpretation, an atomic formula $[\alpha]^*(x)$ is satisfied by $\mathcal{I}$ if and only if $\| x \| \in [0, c_\alpha]$. This means that if $[\alpha]^*(1)$ is satisfied, then $1 \in [0, \| c_\alpha \|]$, which implies $\| c_\alpha \| = 1$. This implies that $\alpha \equiv \top$ and therefore that α is a theorem of Łukasiewicz fuzzy logic.

Theorem 58 (Correspondence). *Consider the F_{CLDS} system, the Hilbert System $\mathcal{F}_{\mathrm{Ax}}$ and the initial empty configuration $\mathcal{C}_{\{\}} = \langle \{\}, \mathcal{F} \rangle$ given by $\mathcal{F}(\lambda) = \{\}$ for any label λ. Let α be a formula of $\mathcal{L}_P$. Then $\vdash_{\mathcal{F}_{\mathrm{Ax}}} \alpha$ if and only if $\mathcal{C}_{\{\}} \vdash_{\mathrm{L}} \alpha{:}1$.*

PROOF: The "only if" part of the theorem can be easily proved by induction on the Hilbert proof $\vdash_{\mathcal{F}_{\mathrm{Ax}}} \alpha$, following the same argument used in Theorem 47. The proof of axioms (F3) and (F2), as part of the base case are already shown in Figures 16 and 17, respectively. The proofs of Axioms (F1), (F4) and (F5) are omitted as they are quite simple. The "if part" of the statement can be proved using an argument similar to that shown in Theorem 48, but with respect to the canonical interpretation of $\mathrm{F}_{\mathrm{CLDS}}$ given above. □

The following lemma shows that the above canonical interpretation is a model of $\mathrm{F}_{\mathrm{CLDS}}$ system.

Lemma 59. *Given the F_{CLDS} system, and a canonical interpretation $\mathcal{I}$ for F_{CLDS}, $\mathcal{I}$ is a model of F_{CLDS}.*

PROOF: The proof requires to show that $\mathcal{I}$ satisfies all the axioms of the extended algebra $\mathcal{A}_{\mathrm{F}}^{+}$. Only axioms (Ax8), (Ax9), (Monotonicity) and (Order-preserving) are considered here[15]. The proof of satisfiability of the remaining axioms is left to the reader.

[15] Note that, for simplicity, the interpretation symbols $\| \;\|$ have often been omitted.

Monotonicity:
To show that for any labels λ and λ', $\langle \lambda, \lambda \circ \lambda' \rangle \in \parallel R \parallel$, it is needed to show that $\lambda \geq max\{0, (\lambda + \lambda') - 1\}$. There are two cases, (i) $\lambda + \lambda' \geq 1$ and (ii) $\lambda + \lambda' < 1$. (i) $max\{0, (\lambda + \lambda') - 1\} = (\lambda + \lambda') - 1$. Now, since $\lambda' \leq 1$, $(\lambda + \lambda') - 1 \leq \lambda$ as required. (ii) $max\{0, (\lambda + \lambda') - 1\} = 0 \leq \lambda$, which is obviously true by the construction of the interpretation $\mathcal{I}$.
Order-preserving:
This requires to show that for any labels λ, λ' and λ'', if $\langle \lambda, \lambda' \rangle \in \parallel R \parallel$ then $\langle \lambda \circ \lambda'', \lambda' \circ \lambda'' \rangle \in \parallel R \parallel$. The hypothesis implies that $\lambda \geq \lambda'$. Then it is needed to show that $max\{0, (\lambda + \lambda'') - 1\} \geq max\{0, (\lambda' + \lambda'') - 1\}$. There are three cases: (i) $\lambda + \lambda'' \geq 1$ and $\lambda' + \lambda'' \geq 1$, (ii) $\lambda + \lambda'' \geq 1$ and $\lambda' + \lambda'' < 1$ and (iii) $\lambda + \lambda'' < 1$ and $\lambda' + \lambda'' < 1$. For case (i), $(\lambda + \lambda'') - 1 \geq (\lambda' + \lambda'')$ since $\lambda \geq \lambda'$. The other two cases are simpler.
Ax8: This requires to show that if $\langle c_\alpha, c_\beta \rangle \in \parallel R \parallel$ then $1 \in \parallel [\beta \to \alpha]^* \parallel$. The hypothesis implies that $c_\alpha \geq c_\beta$. It is needed to show that $c_{\beta \to \alpha} = 1$. $c_{\beta \to \alpha} = min\{1, 1 - (c_\beta - c_\alpha)\}$, Hence, if $c_\alpha \geq c_\beta$ then $c_{\beta \to \alpha} = 1$. The converse direction follows as easily.
Ax9:
It is required to show that, if $x \in \parallel [(\alpha \to \beta) \to \beta]^* \parallel$ and $y \in \parallel [\alpha \to \gamma]^* \parallel$ and $y \in \parallel [\beta \to \gamma]^* \parallel$, then $x \circ y \in \parallel [\gamma]^* \parallel$. There are six cases in all: (i) $c_\gamma \geq c_\alpha \geq c_\beta$, (ii) $c_\alpha \geq c_\gamma \geq c_\beta$ and (iii) $c_\alpha \geq c_\beta \geq c_\gamma$, together with another three cases for $c_\beta \geq c_\alpha$. Just the first three cases are considered here, as the others are similar. (i) $c_{\alpha \to \beta} = 1 - (c_\alpha - c_\beta)$. Hence $c_{(\alpha \to \beta) \to \beta} = min\{1, 1 - (1 - (c_\alpha - c_\beta) - c_\beta)\} = c_\alpha$. The first assumption yields $x \leq c_\alpha$. The second and third assumptions give $y \leq 1$. Then $x \circ y = max\{0, x + y - 1\} \leq c_\alpha \leq c_\gamma$ as required. (ii) The second assumption gives $y \leq 1 - (c_\alpha - c_\gamma)$ and the third assumption gives $y \leq 1$, which together imply $y \leq 1 - (c_\alpha - c_\gamma)$. Then $max\{0, x + y - 1\} \leq c_\gamma$ again. (iii) $y \leq 1 - (c_\alpha - c_\gamma)$ and $y \leq 1 - (c_\beta - c_\gamma)$. Since $c_\alpha \geq c_\beta$, $c_\alpha - c_\gamma \geq c_\beta - c_\gamma$. Hence $y \leq 1 - (c_\alpha - c_\gamma)$. The rest of the proof is as in case (ii). □

6 Conclusions

This paper illustrates a new method, based on Labelled Deductive Systems [15] for providing logics belonging to different families with a uniform presentation of their derivability relations and semantic entailments.

The presentations of the three systems $\mathrm{E_{CLDS}}$, $\mathrm{L_{CLDS}}$ and $\mathrm{F_{CLDS}}$ are extensions and refinements of the general CLDS system given in Section 2. New inference rules are mainly included for the specific logical operators. Rules for reasoning about the structures of configurations are instead common to all the three logics, with some additional rules in the $\mathrm{L_{CLDS}}$ and $\mathrm{F_{CLDS}}$ systems. The different standard semantics of the logic of elsewhere, linear logic and Łukasiewicz fuzzy logic are captured by appropriately refining the axiomatisation of the extended algebra to the specific meaning of their associated logical operators. The general notion of semantical entailment of a CLDS system is instead equally applied to $\mathrm{E_{CLDS}}$, $\mathrm{L_{CLDS}}$ and $\mathrm{F_{CLDS}}$ systems and the same methodology for proving the soundness and completeness of each of the three CLDS proof systems has been deployed. This uniformity makes the CLDS framework an ideal framework not only for facilitating technical studies of existing logics and their combinations, but mainly for providing a technical methodology for the development and investigation of new logics.

As for the logic of elsewhere, Hilbert-style proof systems have already been developed [12,25] and a first tableaux system has been described in [9]. The $\mathrm{E_{CLDS}}$ is a first example of a natural-deduction proof system for this type of enriched classical modal logics. This natural deduction proof system is uniform also with respect to the natural deduction systems developed for the standard family of modal logics [24]. The set of rules for the elsewhere modal operators is identical to the set of rules for the standard normal modal operators. No additional modal rules need to be included to capture the specific semantic meaning of the elsewhere modalities. This is entirely due to the explicit syntactic formalisation of the properties of the accessibility relation by means of the labelling algebra, and to its use in reasoning with possible worlds and with relations between possible worlds as part of the modal system. This differs from the tableaux system for the logic of elsewhere described in [9] where specific tableaux rules are introduced for the elsewhere modal operators. Other related work are [17,16,14,26] where labelled proof sytems for normal modal logics have been developed, different however from the logic of elsewhere. The combined approach of the $\mathrm{E_{CLDS}}$ system (syntactic representation of the possible worlds and accessibility relation) facilitates also an easy extension of the $\mathrm{E_{CLDS}}$ system to systems which combine the elsewhere operator with other modal operators, such as the "universal" modal operator. Such systems could be achieved by extended the labelling language with binary relations R_i for each modality $\Box_i$ (and $\Diamond_i$) and extending the labelling algebra $\mathcal{A}_{\mathrm{E}}$ with sets of schemas which respectively axiomatise the properties of the added accessibility relations R_i and then duplicating modal and structural inference rules for each added modality. This would allow the formalisation of logics such as the logic of *inequality* described in [12].

As for linear logic, the $\mathrm{L_{CLDS}}$ is a proper generalisation of the standard approaches to this logic in that it facilitates explicit assumptions, and reasoning, about relationships between resources. Rules for linear operators described in Section 4 are similar, apart from the labels, to the standard sequent calculus rules for linear logics. The $\mathrm{L_{CLDS}}$ system described in this paper could be extended to include the additive operators & and $\vee$ as well as the exponential operators. To do so, the semantic meaning of the additive operator could, for example, be captured by including the following axiom schemas in the extended algebra $\mathcal{A}_{\mathrm{L}}^{+}$.

$$\forall x([\alpha \& \beta]^*(x) \leftrightarrow ([\alpha]^*(x) \wedge [\beta]^*(x)))$$
$$\forall x([\alpha \vee \beta]^*(x) \leftarrow [\alpha]^*(x))$$
$$\forall x([\alpha \vee \beta]^*(x) \leftarrow [\beta]^*(x))$$
$$\forall x, y([\alpha \vee \beta]^*(x) \rightarrow (([\alpha]^*(c_\alpha) \rightarrow [\gamma]^*(y \circ c_\alpha)) \wedge ([\beta]^*(c_\beta) \rightarrow [\gamma]^*(y \circ c_\beta)) \rightarrow [\gamma]^*(x \circ y)))$$

One of the benefits of the LDS approach to substructural logic is its uniformity, in that the same set of inference rules can be defined for any other substructural logic. Only the case of linear logic has been considered here, but other different substructural logics could be equally defined by considering appropriate labelling algebrae. For example, labelling algebrae for Lambek, Relevance and Intuitionistic logics could be defined by incrementally adding the axioms shown in Table 7 to the basic labelling algebra of the $\mathrm{L_{CLDS}}$ system. The (R-A) rule would use in this case the appropriate labelling algebra to differentiate one logic from another. (See [4] and [10] for a full discussion of this issue.) Note that in the specific case of intuitionistic logic, the ($\neg\neg$)

1	$x \circ y \preceq y \circ x$	commutativity	Lambek Calculus = { }
2	$x \circ x \preceq x$	contraction	Linear Logic = { 1 }
3	$x \preceq x \circ x$	expansion	Relevance Logic = { 1,2}
4	$x \preceq x \circ y$	monotonicity	Intuitionistic Logic = {1,2,4,6}
5	$x \preceq y \vee y \preceq x$	totality	Łukasiewicz fuzzy logic = {1,4,5,6}

Table 7.

Properties of $\circ$ in different logics

rule of the $\mathrm{L_{CLDS}}$ proof system would need to be removed (see [5]) and, as shown in the case of Łukasiewicz fuzzy logic, a new rule would need to be included, which allows the deduction of any declarative unit of the form $\alpha : \lambda$ whenever the antecedent configuration includes declarative unit of the form $\perp : \lambda$. The addition of the monotonicity and totality property to the $\mathrm{L_{CLDS}}$ labelling algebra, has allowed to capture Łukasiewicz fuzzy logic reasoning, as shown in Section 5. Related work on Łukasiewicz fuzzy logic which also adopt forms of labelling mechanisms are [22,21,19]. Other uniform proof systems based on the LDS methodology have been developed for substructural logics. Examples are the natural deduction system described in [4] and the LKE system described in [10], where complete lattices are used as labelling algebra. The $\mathrm{L_{CLDS}}$ system differs from the LKE system in several ways. One example is the introduction in the LKE labelling algebra of a special operator for handling the $\neg$ connective. This operator, called the "star" operator [10], satisfies the basic property that, for any label λ, $\lambda \circ \lambda^* \preceq 1^*$, or equivalently, that $\bot$ is false at $\lambda \circ \lambda^*$. This operator could have also been introduced in the L_{CLDS} system, by adding to the extended algebra the axiom $\forall x(c_\bot \not\preceq x \circ x^* \wedge \forall z(c_\bot \not\preceq x \circ z \rightarrow z \preceq x^*))$. This states that the star operator exists and is the largest label "consistent" with x.

Finally, the translation method in Section 2 facilitates the use of first-order theorem provers for deriving theorems of the underlying logic. The first-order axioms of a CLDS extended algebra $\mathcal{A}_S^+$ can be translated into clausal form, and so any clausal theorem proving method can be used to automate the CLDS proof system. The clauses resulting from instantiating the extended algebra schemas with respect to the translation of a particular configuration, represent a "partial coding" of the data. An example derivation using Otter [20] is given in Figure 18 which illustrates the proof of the logic of elsewhere axiom $(\Box p \wedge p) \rightarrow \Box\Box p$. In this example, the translation into standard clauses uses the $holds(\alpha, x)$ predicate, where α is a wff and x is a label, instead of the monadic predicate $[\alpha]^*(\alpha)$. The functors i, a and b denote the connectives $\rightarrow$, $\wedge$ and $\Box$ respectively. The functor $box(\alpha, x)$ represents the term $box_\alpha(x)$. The proof is made with the *unit-resulting* and *binary resolution* options and it is easy to see that it very closely mirrors the proof given in Figure 3. In the case of substructural logics, an alternative theorem prover

```
list(usable).
  1    holds(A,X) — holds(i(A,B),X).
  2    -holds(B,X) — holds(i(A,B),X).
  3    r(X,box(A,X)) — holds(b(A),X).
  4    -holds(A,box(A,X)) — holds(b(A),X).
  5    -holds(a(A,B),X) — holds(A,X).
  6    -holds(a(A,B),X) — holds(B,X).
  7    r(X,Y) — (X = Y).
  8    -holds(b(A),X) — -r(X,Z) — holds(A,Z).
  list(sos).
  9    -holds(i(a(b(p),p),b(b(p))),s).

  Proof.
  12 [ur,9,2] -holds(b(b(p)),s).
  13 [ur,9,1] holds(a(b(p),p),s).
  17 [ur,12,4] -holds(b(p),box(b(p),s)).
  20 [ur,13,6] holds(p,s).
  21 [ur,13,5] holds(b(p),s).
  37 [ur,17,4] -holds(p,box(p,box(b(p),s))).
  45 [ur,37,8,21] -r(s,box(p,box(b(p),s))).
  49 [ur,45,7] (box(p,box(b(p),s)) = s).
  52 [parafrom,49,37] -holds(p,s).
  53 [binary,52,20] .
```

Fig. 18. An automated proof using Otter

approach could be adopted [7]. This is based on the fact that the clauses of the extended algebra are nearly all Horn clauses (one positive literal at most). The only exceptions are disjunctions with exactly two positive literals. But it is always the case that one of these disjuncts has the form $[\alpha]^*(c_\alpha)$, for some wff α, whereas the other has the form $[\gamma]^*(x)$. A theorem prover which uses an adaptation of the Davis Putnam method [8] has been built in Prolog for the subcase of wffs using just the $\rightarrow$ and $\neg$ operators, where wffs $A \otimes B$ involving $\otimes$ are rewritten as $\neg(A \rightarrow \neg B)$ [5]. Further investigation is however necessary on the automated theorem proving aspect of the CLDS approach. The results obtained from this initial investigation makes this line of research very promising.

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